MET O 11 TECHNICAL NOTE NO 62

CAT AND $\frac{d}{dt}$ Ri

by R Dixon

1. Introduction

This Note is a continuation of Note No 56. It establishes some fairly precise expressions for the rate of change of the Richardson number following an isentropic quasi-geostrophic flow. It shows the role played by the deformation field and also the twisting term and also that there are some special effects if the Richardson number is either very high or very low.

A summary of results is provided in Section 8.

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Note: This paper has not been published. Permission to quote from it must be obtained from the Assistant Director of the above Meteorological Office branch.

-. Notation

f, g, h, p, ω, θ, T, ζ, ρ, α and R have their usual meanings.

$\kappa = c_p/c_v$ the ratio of the specific heats

Ri is the Richardson number $Ri = -\frac{\alpha}{\theta}\frac{\partial\theta}{\partial p} \Big/ \left(\frac{\partial V_H}{\partial p}\right)^2$

s^2 is minus the inverse of Ri, ie $s^2 = +\frac{1}{Ri}$

$\underline{s}$ is the inverse Richardson vector $\underline{s} = \left(\frac{-\theta}{\alpha\frac{\partial\theta}{\partial p}}\right)^{\frac{1}{2}} \frac{\partial V_H}{\partial p}$

Note that $s^2 = \underline{s}\cdot\underline{s} = +\frac{1}{Ri}$

$\sigma = -\frac{\alpha}{\theta}\frac{\partial\theta}{\partial p}$

$\underline{V}$ is the 3-d wind vector $\underline{V} = u\underline{i} + v\underline{j} + w\underline{k}$

$\underline{V_H}$ is the 2-d wind vector $\underline{V_H} = u\underline{i} + v\underline{j}$

Note that this differs from Tech Note No 56 slightly as there $\underline{V}$ was used for the 2-d horizontal wind as the 3-d wind vector did not enter into the discussion.

∇ is the 3-d grad operator $\nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$

∇_p is the 2-d isobaric grad operator $\nabla_p = \underline{i}\left(\frac{\partial}{\partial x}\right)_p + \underline{j}\left(\frac{\partial}{\partial y}\right)_p$

I_H is the 2-d Idemfactor $\underline{i}\,\underline{i} + \underline{j}\,\underline{j}$

$\cdot$ indicates the scalar product

$\times$ indicates the vector product

$\underline{i}$, $\underline{j}$, $\underline{k}$ are cartesian unit vectors, $\underline{i}$ in the x-direction, $\underline{j}$ in the y-direction and $\underline{k}$ in the z direction

$\nabla\underline{V}$ is the dyadic

$$\nabla\underline{V} = \begin{pmatrix} \frac{\partial u}{\partial x}\underline{i}\,\underline{i} & \frac{\partial v}{\partial x}\underline{i}\,\underline{j} & \frac{\partial w}{\partial x}\underline{i}\,\underline{k} \\ \frac{\partial u}{\partial y}\underline{j}\,\underline{i} & \frac{\partial v}{\partial y}\underline{j}\,\underline{j} & \frac{\partial w}{\partial y}\underline{j}\,\underline{k} \\ \frac{\partial u}{\partial z}\underline{k}\,\underline{i} & \frac{\partial v}{\partial z}\underline{k}\,\underline{j} & \frac{\partial w}{\partial z}\underline{k}\,\underline{k} \end{pmatrix}$$

$\nabla_p \underline{V_H}$ is the dyadic

$$\nabla_p\underline{V_H} = \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right)_p\underline{i}\,\underline{i} & \left(\frac{\partial v}{\partial x}\right)_p\underline{i}\,\underline{j} \\ \left(\frac{\partial u}{\partial y}\right)_p\underline{j}\,\underline{i} & \left(\frac{\partial v}{\partial y}\right)_p\underline{j}\,\underline{j} \end{bmatrix}$$

F_H is the deformation dyadic

$$F_H = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

where A is the stretching deformation $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$

and B is the shearing deformation $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$.

Note that although $(A^2 + B^2)$ is invariant with respect to a rotation of the axes A and B are not individually invariant. The dyadics $\nabla_p \underline{V_H}$ and F_H are related by the identity

$$\nabla_p \underline{V_H} = \frac{1}{2}\left[F_H + (\mathrm{div}_p \underline{V_H}) I_H - \zeta_p I_H \times \underline{k}\right]$$

A and B are related to divergence and vorticity by Hamel's identity

$$A^2 + B^2 = \zeta_p^{\,2} + (\mathrm{div}_p \underline{V_H})^2 - 4J(u, v)$$

where J(u, v) is the usual Jacobian $\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

C is the dyadic

$$C = \begin{bmatrix} (A - \mathrm{div}_p \underline{V_H})\underline{i}\,\underline{i} & [B - (\zeta_p + 2f)]\underline{i}\,\underline{j} \\ [B + (\zeta_p + 2f)]\underline{j}\,\underline{i} & -(A + \mathrm{div}_p \underline{V_H})\underline{j}\,\underline{j} \end{bmatrix}$$

i.e.

$$C = F_H - (\mathrm{div}_p \underline{V_H}) I_H - \zeta_p (I_H \times \underline{k}) - 2f(I_H \times \underline{k})$$

Take note that in a product like $\underline{s} \cdot C \cdot \underline{s}$ the contributions from the ntisymmetric elements of C will exactly cancel out. Thus

$$\underline{s} \cdot C \cdot \underline{s} = \underline{s} \cdot F_H \cdot \underline{s} - (\operatorname{div}_p \underline{V}_H) s^2$$

In intrinsic coordinates the $\underline{t}$, $\underline{n}$, $\underline{k}$, system of unit vectors is used where

$\underline{t}$ specifies the direction of the wind, and in the text it is the geostrophic wind: $\underline{t}$ is tangential to the contours.

$\underline{n}$ specifies the direction orthogonal to the wind

$\underline{k}$ is the vertical unit vector, counted positive upwards. It is a right-handed system so that $\underline{t} \times \underline{n} = \underline{k}$

$\frac{\partial}{\partial s}$ is differentiation along the streamlines (contours)

$\frac{\partial}{\partial n}$ is differentiation along the curves orthogonal to the streamlines (contours)

ψ_g is the angle between the direction of the geostrophic wind and the x-axis and is counted positive going anticlockwise from the x-axis.

In this system

$$\underline{V_g} = V_g \underline{t} \quad , \qquad \frac{\partial \underline{V_g}}{\partial p} = \frac{\partial V_g}{\partial p} \underline{t} + V_g \frac{\partial \psi_g}{\partial p} \underline{n}$$

which follows because $\frac{\partial \underline{t}}{\partial p} = \frac{\partial \psi_g}{\partial p} \underline{n}$

The deformation components are different in this system, and are given by

$$A_g' = \frac{\partial V_g}{\partial s} + V_g K_n \qquad (= 2 V_g K_n)$$

$$B_g' = \frac{\partial V_g}{\partial n} + V_g K_s$$

where K_s is the streamline (contour) curvature ($= \frac{\partial \psi_g}{\partial s}$)

and K_n is the orthogonal curvature ($= -\frac{\partial \psi_g}{\partial n}$)

Note that Kn is positive in a confluent region.

The formulae required for computing Ks and Kn are

Ks =

$$K_s = \frac{\left(\frac{\partial h}{\partial x}\right)^2 \frac{\partial^2 h}{\partial y^2} + \left(\frac{\partial h}{\partial y}\right)^2 \frac{\partial^2 h}{\partial x^2} - 2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\frac{\partial^2 h}{\partial x \partial y}}{\left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right]^{3/2}}$$

$$K_n = \frac{\frac{\partial^2 h}{\partial x \partial y}\left[\left(\frac{\partial h}{\partial x}\right)^2 - \left(\frac{\partial h}{\partial y}\right)^2\right] + \frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\left[\frac{\partial^2 h}{\partial y^2} - \frac{\partial^2 h}{\partial x^2}\right]}{\left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right]^{3/2}}$$

and in terms of the nine grid point values

$$\begin{array}{ccc} \dot{6} & \dot{2} & \dot{5} \\ \\ \dot{3} & \dot{0} & \dot{1} \\ \\ \dot{7} & \dot{4} & \dot{8} \end{array}$$

they have the finite difference analogues

$$K_s = \frac{\frac{\beta}{a}\left\{ \begin{array}{c} 2(h_1-h_3)^2(h_2+h_4-2h_0) + 2(h_2-h_4)^2(h_1+h_3-2h_0) \\ -(h_1-h_3)(h_2-h_4)(h_5-h_6+h_7-h_8) \end{array} \right\}}{\left[(h_1-h_3)^2 + (h_2-h_4)^2\right]^{3/2}}$$

$$K_n = \frac{\frac{\beta}{a}\left\{ \begin{array}{c} \frac{1}{2}(h_5-h_6+h_7-h_8)\left[(h_1-h_3)^2-(h_2-h_4)^2\right] \\ +2(h_1-h_3)(h_2-h_4)\left[(h_2+h_4-2h_0)-(h_1+h_3-2h_0)\right] \end{array} \right\}}{\left[(h_1-h_3)^2 + (h_2-h_4)^2\right]^{3/2}}$$

where a is the North Pole grid length.

and β is the map factor.

3. The Problem

Meteorologists are accustomed to dealing with problems on which there is only scanty and possibly erroneous data and for which the verification of theories is rendered difficult by a lack of resolution in observed and forecast fields. Even in Meteorology the CAT phenomenon is outstanding in this respect. It is probably this aspect of it which has caused progress to be so slow since Bannon first began to systematize our knowledge of it in 1952. For this reason the relatively few well-documented cases of CAT are of great importance. Any theory must account for the evidence provided by these cases.

CAT occurs in a variety of synoptic situations. A typical one is depected in Figure 1.

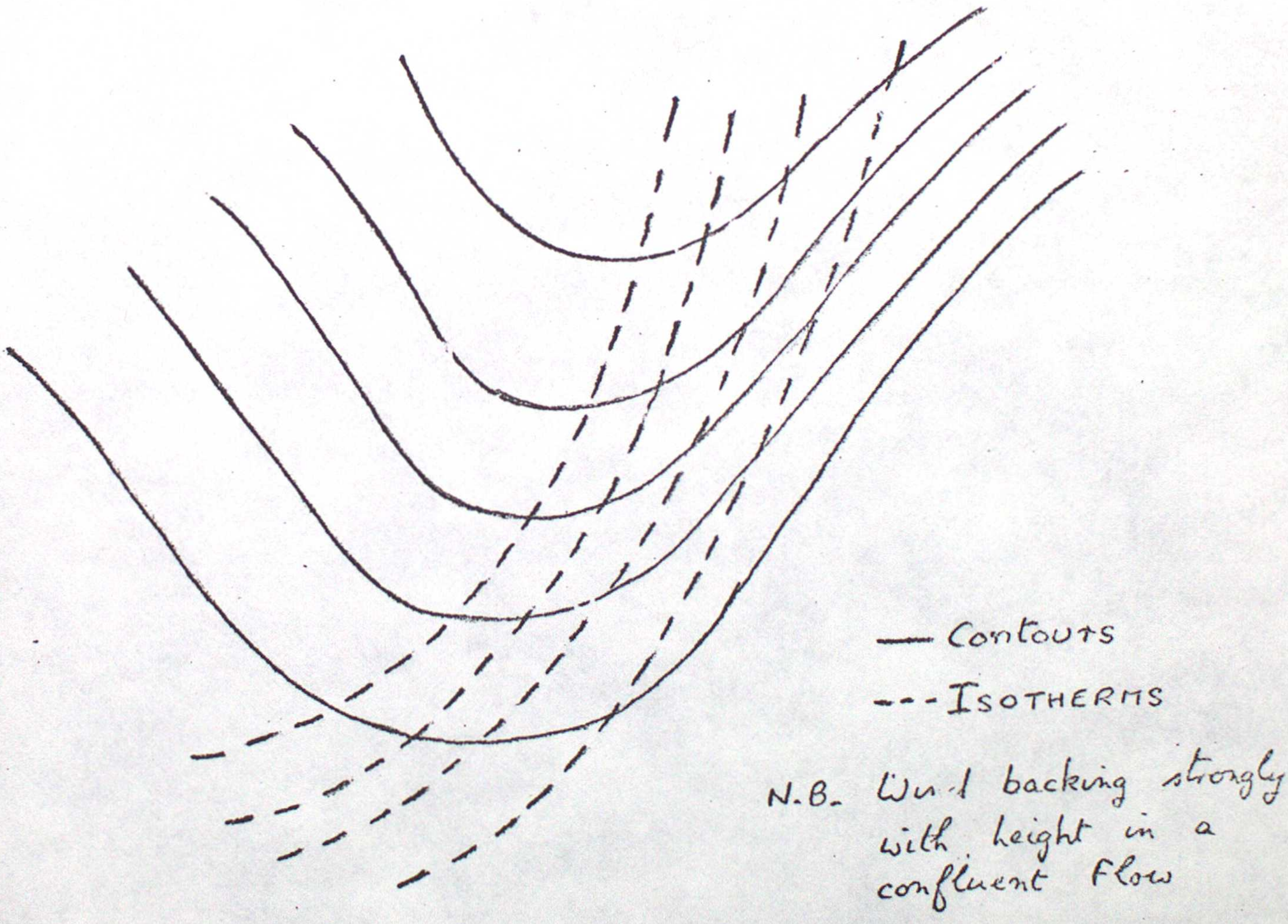

Figure 1. A typical CAT situation

Figure 1 does not represent any actual situation and it does not refer to any particular level. Although CAT mainly occurs at jet-stream levels it can occur at any level. An example of the Figure 1 situation with moderate and severe CAT at 250 mb occurred on 13 April 1962, over the Eastern USA. Another example with CAT at 750 mb occurred in the same general area on 19 February 1970.

When the Richardson Number (Ri) at a locality falls below a critical value and turbulence occurs it seems intuitively likely that the turbulence relieves the condition which causes it and that the turbulence is therefore very transient locally. On the other hand the painstaking work of several authors has placed it beyond reasonable doubt that areas of roughly meso-scale size come into existence within which transient local CAT occurs and that these areas persist over several hours, moving with the associated synoptic system. It is natural therefore to seek some effect, measurable on the small synoptic scale, which will account for the persistent lowering of Ri over an area despite the tendency of transient local turbulence to relieve the condition.

Thus it seems to me that a theoretical basis for CAT occurrences must meet the following requirements at least -

a. It must provide a mechanism to account for the lowering of Ri over an area, and for practical purposes the mechanism must be in terms of parameters which can be measured on a synoptic map or computed from numerically forecast grid point fields.

b. It must provide some reasonable grounds for believing that having become small Ri can stay small, for the indirect evidence is that, when small, Ri is a quasi-conservative quantity.

c. It must be free from ambiguities as to the sign of $\frac{d\,Ri}{dt}$, and especially it must be proof against counter examples in the matter of sign. Thus Oard, in a recent J.A.M. paper, comments "For instance in strong confluent zones aloft, which in many cases are observed to contain widespread CAT the stretching deformation gives a positive Richardson number tendency. This is opposite to what one would expect." The principle of only having to see one tiger to know that you are in a jungle operates strongly here. It requires only a small proportion of well authenticated cases of an index having a significant value but the wrong sign to undermine faith in the theory. And this has been the fate of $\frac{d\,Ri}{dt}$ indices so far.

d. It must account for the turning of the wind with height being as important as the change of wind speed with height. Thus Reiter commenting on the 13 April 1962 event says "The CAT observations are concentrated in a layer which is characterized by strong turning of the wind with height."

e. It must do all this within the context of a flow which on the small synoptic scale does not depart too sensibly from the geostrophic state because although ageostrophic effects doubtless have a role there is abundant evidence to suggest that the atmosphere does not need ageostrophic effects to create the conditions necessary for CAT to occur. The role of ageostrophic effects may well be mainly to provide a local triggering perturbation energy once the geostrophic flow structure is favourable.

The following treatment appears to resolve some of the current theoretical difficulties. Its practical utility depends on the-proof-of-the-pudding and it may have to wait upon technological developments.

4. Ertel's Conservation Theorem and the Richardson Number

In three-dimensional flow in x, y, z, coordinates if $\underline{Q}$ is the absolute vorticity ctor and ∇ is the 3-d grad operator then

$$\frac{d\underline{Q}}{dt} - \underline{Q}\cdot\nabla\underline{V} + (\mathrm{Div}\,\underline{V})\underline{Q} = -\nabla\alpha\times\nabla p \tag{1}$$

There is also the purely kinematical relationship

$$\frac{d\nabla\lambda}{dt} = \nabla\frac{d\lambda}{dt} - \nabla\underline{V}\cdot\nabla\lambda \tag{2}$$

where λ is any scalar point function. Taking the scalar product operation $\cdot\nabla\lambda$ post-factorially through (1) and the operation $\underline{Q}\cdot$ pre-factorially through (2) the term $-\underline{Q}\cdot\nabla\underline{V}\cdot\nabla\lambda$ can be eliminated between the two equations to give

$$\frac{d\underline{Q}}{dt}\cdot\nabla\lambda + \underline{Q}\cdot\frac{d}{dt}\nabla\lambda + (\mathrm{Div}\,\underline{V})\underline{Q}\cdot\nabla\lambda - \underline{Q}\cdot\nabla\frac{d\lambda}{dt} = -\nabla\alpha\times\nabla p\cdot\nabla\lambda \tag{3}$$

which by virtue of the continuity equation

$$\mathrm{Div}\,\underline{V} = \frac{1}{\alpha}\frac{d\alpha}{dt} \tag{4}$$

reduces to

$$\frac{d}{dt}\left[\alpha\underline{Q}\cdot\nabla\lambda\right] - \alpha\underline{Q}\cdot\nabla\frac{d\lambda}{dt} = -\alpha\nabla\alpha\times\nabla p\cdot\nabla\lambda \tag{5}$$

and if $\lambda = \theta$ the potential temperature, and the motion is isentropic

$$\frac{d}{dt}\left[\alpha\underline{Q}\cdot\nabla\theta\right] = 0 \tag{6}$$

which is Hans Ertel's celebrated conservation theorem.

Although Ertel's theorem is rightly described in the literature as a central result in fluid dynamical theory it has found remarkably little use. In the meteorological context this is probably because the corresponding result in x, y, p coordinates is not so simple. However, it may be shown that the isobaric coordinate analogue of (6) is

$$\frac{d}{dt}\left[\frac{\partial\theta}{\partial p}(\zeta_p + f) + \underline{k} \times \frac{\partial \underline{V}_H}{\partial p} \cdot \nabla_p \theta\right] = 0 \tag{7}$$

and that this expression is closely connected with the Richardson Number. First make the geostrophic assumption in the form

$$\frac{\partial \underline{V}_g}{\partial p} = \frac{\alpha}{f\theta} \nabla_p \theta \times \underline{k} \tag{8}$$

which brings (7) to the form

$$\frac{d}{dt}\left[\frac{\partial\theta}{\partial p}(\zeta_p + f) + \frac{f\theta}{\alpha}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2\right] = 0 \tag{9}$$

The Inverse Richardson Number s^2 as defined in Met O 11. Tech. Note No 56 is then

$$s^2 = -\frac{\frac{\theta}{\alpha}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2}{\frac{\partial\theta}{\partial p}} \tag{10}$$

and the connection is obvious. The first term in (9) is the denominator of (10) multiplied by $(\zeta_p + f)$ and the second term in (9) is the negative of the numerator of (10) multiplied by f. There follows

$$\frac{f s^2}{(\zeta_p + f)} = -\frac{\frac{f\theta}{\alpha}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2}{(\zeta_p + f)\frac{\partial\theta}{\partial p}}$$

subtracting 1 from both sides

$$\frac{f s^2}{(\zeta_p + f)} - 1 = -\frac{\frac{f\theta}{\alpha}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2}{(\zeta_p + f)\frac{\partial\theta}{\partial p}} - 1$$

whence

$$\left[\frac{fs^2}{(\zeta_p+f)}-1\right](\zeta_p+f)\frac{\partial\theta}{\partial p}=-\left[\frac{\partial\theta}{\partial p}(\zeta_p+f)+\frac{f\theta}{\alpha}\left(\frac{\partial V_g}{\partial p}\right)^2\right]$$

and now, since the expression on the R.H.S. is the same as in (9) we have

$$\frac{d}{dt}\left\{\left[\frac{fs^2}{(\zeta_p+f)}-1\right](\zeta_p+f)\frac{\partial\theta}{\partial p}\right\}=0 \tag{11}$$

and from (11) <u>if s^2 is small</u> compared with $\dfrac{(\zeta_p+f)}{f}$ then

$$\frac{d}{dt}\left[(\zeta_p+f)\frac{\partial\theta}{\partial p}\right]=0 \tag{12}$$

It then follows from (9) that also

$$\frac{d}{dt}\left[\frac{f\theta}{\alpha}\left(\frac{\partial V_g}{\partial p}\right)^2\right]=0 \tag{13}$$

Inverting (10) and following the same sequence of algebraical manipulations as used to obtain (11) one obtains

$$\frac{d}{dt}\left\{\left[\frac{(\zeta_p+f)}{fs^2}-1\right]\frac{f\theta}{\alpha}\left(\frac{\partial V_g}{\partial p}\right)^2\right\}=0 \tag{14}$$

and from (14) <u>if s^2 is large</u> compared with $\dfrac{(\zeta_p+f)}{f}$ then (12) and (13) again follow. Thus we have established the result -

> If $Ri \ll f/(\zeta_p+f)$ or if $Ri \gg f/(\zeta_p+f)$ then the quantities $(\zeta_p+f)\dfrac{\partial\theta}{\partial p}$ and $\dfrac{f\theta}{\alpha}\left(\dfrac{\partial V_g}{\partial p}\right)^2$ are both approximately conserved.

(See the Appendix at this point). To the extent that the motion is isentropic and the variation of f negligible it also follows that under the stated conditions $\dfrac{1}{\alpha}\left(\dfrac{\partial V_g}{\partial p}\right)^2$ is approximately conserved.

5. The rate of change of Ri following a quasi-geostrophic flow

From (10) we have

$$\frac{\partial\theta}{\partial p}s^2 = -\frac{\theta}{\alpha}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2$$

Differentiating by $\frac{d}{dt}$, multiplying both sides by f, and adding $-\frac{\alpha}{\theta}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2\frac{df}{dt}$ to both sides there follows

$$f\frac{\partial\theta}{\partial p}\frac{ds^2}{dt} + fs^2\frac{d}{dt}\frac{\partial\theta}{\partial p} - \frac{\alpha}{\theta}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2\frac{df}{dt} = -\frac{d}{dt}\left[\frac{f\theta}{\alpha}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2\right]$$

But if s^2 is large (Ri small) in the sense of the previous Section then from (13) the R.H.S is zero. In this case, dividing through by $\frac{\partial\theta}{\partial p}$ we obtain, using (10) again

$$f\frac{ds^2}{dt} + s^2\frac{df}{dt} = -\frac{1}{\frac{\partial\theta}{\partial p}}\frac{d}{dt}\frac{\partial\theta}{\partial p}fs^2 \qquad (15)$$

But it may be shown that for isentropic motion

$$\frac{1}{\frac{\partial\theta}{\partial p}}\frac{d}{dt}\frac{\partial\theta}{\partial p} = \mathrm{div}_\theta\,\underline{V}_H \qquad (16)$$

and it may also be shown that the relationship between the (x,y, θ) divergence and the (x.y,p) divergence is given by

$$\mathrm{div}_\theta\,\underline{V}_H = \mathrm{div}_p\,\underline{V}_H - \frac{1}{\frac{\partial\theta}{\partial p}}\nabla_p\theta\cdot\frac{\partial \underline{V}_H}{\partial p} \qquad (17)$$

If the motion is geostrophic it follows from (8) that the last term in (17) is zero. Consequently in the isentropic geostrophic case (15) reduces to

$$f\frac{ds^2}{dt} + s^2\frac{df}{dt} = -\left(\mathrm{div}_p\,\underline{V}_g\right)fs^2 \qquad (18)$$

But

$$f \operatorname{div}_p \underline{V}_g = -\underline{V}_g \cdot \nabla f \qquad (19)$$

and since both $\frac{\partial f}{\partial t}$ and $\omega \frac{\partial f}{\partial p}$ are zero they may be added to the R.H.S. of (19) to get

$$f \operatorname{div}_p \underline{V}_g = -\frac{df}{dt} \qquad (20)$$

whence (18) reduces to

$$\frac{ds^2}{dt} = 0 \qquad (21)$$

Thus in the case of isentropic geostrophic motion if $Ri \ll \frac{f}{(\zeta_p + f)}$ or equally if Ri $\gg \frac{f}{(\zeta_p + f)}$ it is a conserved quantity, following the flow. This is important for it means that once the Richardson Number has become low it will have a tendency to stay low.

We now examine the more general case, when Ri is neither very low nor very high to see what factors can act to bring it low so that (21) applies. From (11)

$$\frac{d}{dt}\left\{\frac{\partial \theta}{\partial p}\left[fs^2 - (\zeta_p + f)\right]\right\} = 0$$

whence

$$\left[fs^2 - (\zeta_p + f)\right]\frac{d}{dt}\frac{\partial \theta}{\partial p} + \frac{\partial \theta}{\partial p}\frac{d}{dt}\left[fs^2 - (\zeta_p + f)\right] = 0$$

and thus, using (16) and (17) with the isentropic geostrophic assumption we have

$$\frac{d}{dt}\left[fs^2 - (\zeta_p + f)\right] = -\operatorname{div}_p \underline{V}_g \left[fs^2 - (\zeta_p + f)\right] \qquad (22)$$

To rid (22) of the unwanted term in $\frac{d}{dt}(\zeta_p + f)$ the isobaric vorticity equation

$$-\frac{d}{dt}(\zeta_p + f) = (\operatorname{div}_p \underline{V}_g)(\zeta_p + f) - \left(\frac{\partial V_g}{\partial p} \times \nabla_p \omega\right) \cdot \underline{k} \qquad (23)$$

is subtracted from it to give

$$\frac{d}{dt}(fs^2) = -(\operatorname{div}_p \underline{V_g}) fs^2 + \left(\frac{\partial \underline{V_g}}{\partial p} \times \nabla_p \omega\right) \cdot \underline{k} \qquad (24)$$

It is apparent that (24) is simply (18) plus the twisting term. It appears that the twisting term is a primary cause in changing Ri following the flow. It also accounts for the observational evidence that $\nabla_p \omega$ is important. From (24) it is seen that the effect of the twisting term is such that s^2 increases (Ri decreases) following the flow if ω becomes relatively more positive towards the cold air. Expanding the L.H.S., taking note of (20) and dividing through by f gives

$$\frac{ds^2}{dt} = \frac{1}{f}\left(\frac{\partial \underline{V_g}}{\partial p} \times \nabla_p \omega\right) \cdot \underline{k} \qquad (25)$$

It may then be shown directly that the twisting term becomes zero if s^2 is large in isentropic geostrophic flow. Multiply through the vorticity equation (23) by $\frac{\partial \theta}{\partial p}$ and change the L.H.S. a little to get

$$-\frac{d}{dt}\left\{\frac{\partial \theta}{\partial p}(\zeta_p + f)\right\} + (\zeta_p + f)\frac{d}{dt}\frac{\partial \theta}{\partial p} = (\operatorname{div}_p \underline{V_g})\frac{\partial \theta}{\partial p}(\zeta_p + f) - \frac{\partial \theta}{\partial p}\frac{\partial \underline{V_g}}{\partial p} \times \nabla_p \omega \cdot \underline{k}$$

But under the stated conditions, from (12) the first term on the L.H.S is zero, whilst from (16) and (17) the next two terms cancel out leaving only the twisting term . Thus for geostrophic isentropic motion with small Ri

$$\frac{\partial \underline{V_g}}{\partial p} \times \nabla_p \omega \cdot \underline{k} = 0 \qquad (26)$$

and thus in this case (25) reduces to (21).
It seems then that points (a) and (b) of section 3 have been partially met.

However, the twisting term is not readily measurable: so some alternative expressio[n] must be sought. This can be provided by eqn (16) of Met O 11 Tech. Note No. 56, repeated here for convenience

$$\frac{ds^2}{dt} + s^2\frac{d}{dt}\ln\sigma + \underline{s} \cdot C \cdot \underline{s} = \frac{2R}{p\sqrt{\sigma}}\underline{s} \cdot \nabla_p T \qquad (27)$$

σ and C are defined in Section 2. In (27) it may be shown that if the flow is isentropic

$$\frac{d}{dt}\ln\sigma = -\frac{\omega}{\kappa p} + \operatorname{div}_\theta \underline{V_H} - \frac{1}{\frac{\partial \theta}{\partial p}}\nabla_p \theta \cdot \frac{\partial \underline{V_H}}{\partial p} \qquad (28)$$

Also the antisymmetric components of $\underline{s} \cdot C \cdot \underline{s}$ will make no contribution since they cancel each other out and in fact

$$\underline{s} \cdot C \cdot \underline{s} = \underline{s} \cdot F_H \cdot \underline{s} - (\operatorname{div}_p \underline{V_H}) s^2 \qquad (29)$$

where F_H is the deformation dyadic.
Putting (28) and (29) in (27), invoking the geostrophic condition and using (17) and (8) then brings (27) to

$$\frac{ds^2}{dt} = \frac{\omega}{\kappa p} s^2 - \underline{s} \cdot F_g \cdot \underline{s} \qquad (30)$$

We need to show that if the sign of the R.H.S. of (30) is such that $\frac{ds^2}{dt}$ is positive and s^2 becomes large (ie, Ri small) following the flow then in this case the R.H.S. tends to vanish. For this take equation (10) of Met O 11 Tech. Note No 56, repeated here

$$\frac{d}{dt}\frac{\partial \underline{V_H}}{\partial p} + \frac{1}{2} C \cdot \frac{\partial \underline{V_H}}{\partial p} = \frac{R}{p} \nabla_p T \qquad (31)$$

Take $\frac{\partial \underline{V_H}}{\partial p} \cdot$ through and invoke the geostrophic condition to get

$$\frac{d}{dt}\left(\frac{\partial \underline{V_g}}{\partial p}\right)^2 + \frac{\partial \underline{V_g}}{\partial p} \cdot C \cdot \frac{\partial \underline{V_g}}{\partial p} = 0 \qquad (32)$$

The antisymmetric components of (32) again play no part and (32) becomes, (see Section 2)

$$\frac{d}{dt}\left(\frac{\partial \underline{V_g}}{\partial p}\right)^2 + \frac{\partial \underline{V_g}}{\partial p} \cdot F_g \cdot \frac{\partial \underline{V_g}}{\partial p} - (\operatorname{div}_p \underline{V_g})\left(\frac{\partial \underline{V_g}}{\partial p}\right)^2 = 0 \qquad (33)$$

Multiplying through by $\frac{f\theta}{\alpha}$ and expressing the first term differently

$$\frac{d}{dt}\left[\frac{f\theta}{\alpha}\left(\frac{\partial \underline{V_g}}{\partial p}\right)^2\right] - \left(\frac{\partial \underline{V_g}}{\partial p}\right)^2 \frac{d}{dt}\left(\frac{f\theta}{\alpha}\right) + \frac{f\theta}{\alpha}\frac{\partial \underline{V_g}}{\partial p} \cdot F_g \cdot \frac{\partial \underline{V_g}}{\partial p} - (\operatorname{div}_p \underline{V_g})\frac{f\theta}{\alpha}\left(\frac{\partial \underline{V_g}}{\partial p}\right)^2 = 0 \qquad (34)$$

But in the case that the flow is isentropic and Ri is small the first term is zero by (13) and using (20) the equation reduces to

$$\frac{\theta}{\alpha}\frac{\partial \underline{V_g}}{\partial p} \cdot F_g \cdot \frac{\partial \underline{V_g}}{\partial p} - \frac{\omega\theta}{\kappa p \alpha}\left(\frac{\partial \underline{V_g}}{\partial p}\right)^2 = 0 \qquad (35)$$

where $-\frac{1}{\alpha}\frac{d\alpha}{dt} = \operatorname{div}_5 \underline{V} = \frac{\omega}{\kappa p}$ (in the isentropic case) has been used.
Dividing through (35) by $1/\frac{\partial\theta}{\partial p}$ we have

$$\underline{s}\cdot F_g\cdot\underline{s} - \frac{\omega}{\kappa p}s^2 = 0 \qquad (36)$$

Thus, although it is far from obvious, in the geostrophic isentropic case once s^2 has become large (Ri small) the R.H.S. of (30) becomes zero and it reduces to (21) again, just as it did in the twisting term representation, If Ri is neither particularly large nor particularly small then (30) holds as it is and the R.H.S. can be estimated. When Ri is well away from its extremes then the deformation term probably dominates. As Ri becomes smaller this obviously no longer holds, but in any case numerical forecasting routines can supply values for both the R.H.S. terms of (30). More will be written on the use of (30) in the next section. Points (a) & (b) of Section 3 have now been met. However there is one loose end to tie up. Equations (25) and (30) are both expressions for $\frac{ds^2}{dt}$. It has been shown that they both reduce to $\frac{ds^2}{dt} = 0$ if Ri is small (and also if Ri large). They must also be equivalent in the general case. That is, (25) and (30) imply that

$$\frac{1}{f}\left(\frac{\partial \underline{V}_s}{\partial p}\times\nabla_p\omega\right)\cdot\underline{k} = \frac{\omega}{\kappa p}s^2 - \underline{s}\cdot F_g\cdot\underline{s} \qquad (37)$$

if the flow is geostrophic and isentropic. This is not obviously true, but may be proved as follows -

Kinematically we have

$$\frac{d}{dt}\nabla_p\theta = \nabla_p\frac{d\theta}{dt} - \nabla_p\theta\cdot\nabla_p\underline{V}_H - \nabla_p\theta\times\zeta_p\underline{k} - \frac{\partial\theta}{\partial p}\nabla_p\omega \qquad (38)$$

Since θ is the potential temperature the first term on the R.H.S. is zero. If we take $\cdot\nabla_p\theta$ through (36) postfactorially the term involving ζ_p disappears and we have

$$\frac{1}{2}\frac{d}{dt}(\nabla_p\theta)^2 = -\nabla_p\theta\cdot\nabla_p\underline{V}_H\cdot\nabla_p\theta - \frac{\partial\theta}{\partial p}\nabla_p\omega\cdot\nabla_p\theta \qquad (39)$$

Then using the identity

$$\nabla_p\underline{V}_H = \frac{1}{2}\left[F_H + (\operatorname{div}_p\underline{V}_H)I_H - \zeta_p I_H\times\underline{k}\right] \qquad (40)$$

we bring (37) to the form

$$\frac{1}{2}\frac{d}{dt}(\nabla_p\theta)^2 = -\frac{1}{2}\nabla_p\theta\cdot F_H\cdot\nabla_p\theta - \frac{1}{2}(\operatorname{div}_p\underline{V}_H)(\nabla_p\theta)^2 - \frac{\partial\theta}{\partial p}\nabla_p\omega\cdot\nabla_p\theta \qquad (41)$$

the antisymmetric term $\zeta_p \mathrm{I}_H \times \underline{k}$ having cancelled out. Now introducing the geostrophic assumption, using (8) and (34) the terms in (41) can be expressed in turn as

$$\frac{1}{2}\frac{d}{dt}(\nabla_p\theta)^2 = \frac{f\theta}{\alpha}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2 \frac{d}{dt}\left(\frac{f\theta}{\alpha}\right) - \frac{1}{2}\left(\frac{f\theta}{\alpha}\right)^2 \frac{\partial \underline{V}_g}{\partial p}\cdot F_g \cdot \frac{\partial \underline{V}_g}{\partial p} + \frac{1}{2}(\mathrm{div}_p \underline{V}_g)\left(\frac{f\theta}{\alpha}\right)^2\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2 \quad (42)$$

$$-\frac{1}{2}\nabla_p\theta\cdot F_g\cdot\nabla_p\theta = -\frac{1}{2}\left(\frac{f\theta}{\alpha}\right)^2\left(\underline{k}\times\frac{\partial \underline{V}_g}{\partial p}\right)\cdot F_g\cdot\left(\underline{k}\times\frac{\partial \underline{V}_g}{\partial p}\right) = \frac{1}{2}\left(\frac{f\theta}{\alpha}\right)^2\left(\frac{\partial \underline{V}_g}{\partial p}\cdot F_g\cdot\frac{\partial \underline{V}_g}{\partial p}\right) \quad (43)$$

$$-\frac{\partial\theta}{\partial p}\nabla_p\omega\cdot\nabla_p\theta = -\frac{\partial\theta}{\partial p}\frac{f\theta}{\alpha}\nabla_p\omega\cdot\underline{k}\times\frac{\partial \underline{V}_g}{\partial p} \quad (44)$$

$$-\frac{1}{2}(\mathrm{div}_p \underline{V}_g)(\nabla_p\theta)^2 = -\frac{1}{2}\left(\frac{f\theta}{\alpha}\right)^2(\mathrm{div}_p \underline{V}_g)\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2 \quad (45)$$

Noting that in (42).

$$\frac{f\theta}{\alpha}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2\frac{d}{dt}\left(\frac{f\theta}{\alpha}\right) = \left(\frac{f\theta}{\alpha}\right)^2\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2\left[\frac{\omega}{\kappa p} - \mathrm{div}_p \underline{V}_g\right] \quad (46)$$

on substituting (42), (43), (44) and (45) into (41) there results

$$\frac{f\theta}{\alpha}\left(\frac{\partial \underline{V}_g}{\partial p}\right)^2\frac{\omega}{\kappa p} - \frac{f\theta}{\alpha}\frac{\partial \underline{V}_g}{\partial p}\cdot F_g\cdot\frac{\partial \underline{V}_g}{\partial p} = -\frac{\partial\theta}{\partial p}\frac{\partial \underline{V}_g}{\partial p}\times\nabla_p\omega\cdot\underline{k}$$

and then taking account of the definition of $\underline{s}$ and s^2 this is exactly (37) which is thus verified.

6. Comments

Equation (30) may be computed directly as it stands from numerically analysed and initialized fields, and from numerical forecast fields. If this is done then it is advisable to compute it in its entirety. There has been an inclination to single out the stretching deformation component of F_g as being of special importance, but in a cartesian coordinate system this results[8] in a basic ambiguity. From

$$\frac{\partial \underline{V}_g}{\partial p} = \frac{\partial u_g}{\partial p}\underline{i} + \frac{\partial v_g}{\partial p}\underline{j} \quad \text{and} \quad F_g = \begin{pmatrix} A_g\underline{i}\underline{i} & B_g\underline{i}\underline{j} \\ B_g\underline{j}\underline{i} & -A_g\underline{j}\underline{j} \end{pmatrix} \quad (47)$$

where $\underline{i}$, $\underline{j}$, and $\underline{k}$ are the unit vectors in the cartesian system we have explicitly

$$\frac{\partial \underline{V}_g}{\partial p}\cdot F_g\cdot\frac{\partial \underline{V}_g}{\partial p} = \left[\left(\frac{\partial u_g}{\partial p}\right)^2 - \left(\frac{\partial v_g}{\partial p}\right)^2\right]A_g + 2B_g\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial p} \quad (48)$$

but there is the complication that in F_g the components A_g and B_g are not individually invariant with respect to a rotation of axes. The quantity

$$A_g^2 + B_g^2 = \mathcal{F}_g^2 \qquad (49)$$

is invariant, but not A_g or B_g individually. As the cartesian axes are rotated the values of A_g and B_g in F_g will vary over the range $-\mathcal{F}_g$ through zero to $+\mathcal{F}_g$. Provided the whole of (48) is computed this does not matter because the corresponding changes in $\partial u_g/\partial p$ and $\partial v_g/\partial p$ throughout the expression exactly compensate for the changes in A_g and B_g as the axes are rotated. This is not true if one of the terms in (48) is discarded. If the term involving B_g is discarded as unimportant then the value and sign of the term involving A_g depends upon the choice of cartesian axes. Now it does so happen that, over the UK - Atlantic - Eastern USA from which area most of the evidence on CAT which is not too obviously linked to topographical influences has been culled, the direction of the x axis (the i vector) is such that it is broadly the same as the direction of the axis of most jet streams and in the case of a confluent jet A_g will then have the right sign. However the orientation of confluent jets is sufficiently varied that a steady percentage can be expected to turn up having A_g the "wrong" sign. This, and another point concerning the relative importance of changes of speed with height and changes of direction with height, is more clearly brought out by considering (48) in intrinsic coordinates. We choose a right-handed system of unit vectors $\underline{t}$, $\underline{n}$, $\underline{k}$ and we then have in place of (47)

$$\frac{\partial \underline{V}_g}{\partial p} = \frac{\partial V_g}{\partial p}\underline{t} + V_g \frac{\partial \psi_g}{\partial p}\underline{n} \quad , \quad F_g = \begin{pmatrix} A_g'\underline{t}\,\underline{t} & B_g'\underline{t}\,\underline{n} \\ B_g'\underline{n}\,\underline{t} & -A_g'\underline{n}\,\underline{n} \end{pmatrix} \qquad (50)$$

See Section 2 for notation. We now have explicitly

$$\frac{\partial \underline{V}_g}{\partial p} \cdot F_g \cdot \frac{\partial \underline{V}_g}{\partial p} = \left[\left(\frac{\partial V_g}{\partial p}\right)^2 - V_g^2\left(\frac{\partial \psi_g}{\partial p}\right)^2\right] A_g' + 2V_g \frac{\partial V_g}{\partial p}\frac{\partial \psi_g}{\partial p} B_g' \qquad (51)$$

From (51) we may draw certain conclusions -

(a) At around the jet-core level it is likely that either or both of $\frac{\partial V_g}{\partial p}$ and $\frac{\partial \psi_g}{\partial p}$ will be very small and therefore it may be reasonable to neglect the term in B_g'. But this is only so in the unique $\underline{t}$, $\underline{n}$, $\underline{k}$ system. It does not necessarily carry over to the $\underline{i}$, $\underline{j}$, $\underline{k}$ system.

(b) In the event the B_g' term can be neglected then since

$$A_g' = \frac{\partial V_g}{\partial s} + V_g K_n \quad (= 2V_g K_n) \qquad (52)$$

A'_g itself is unambiguously positive in a confluent region, because both of the terms in (52) are then positive. However the sign of the contribution of the first term in (51) involving A'_g will depend on the relative sizes of $\left(\frac{\partial V_g}{\partial p}\right)^2$ and $V_g^2\left(\frac{\partial \psi_g}{\partial p}\right)^2$. Any computations based upon algebra or physical reasoning which leads to the neglect of the term $V_g^2\left(\frac{\partial \psi_g}{\partial p}\right)^2$ are liable to give a wrong answer in a situation such as depicted in Figure 1.

It is clear from (51) that it is dangerous to argue too much from special cases. If neither $\frac{\partial V_g}{\partial p}$ nor $\frac{\partial \psi_g}{\partial p}$ are particularly small it is quite possible that the term in B'_g in (51) could make a substantial or even dominant contribution. Since we have

$$B_g' = \frac{\partial V_g}{\partial n} + V_g K_s \tag{53}$$

there is then the additional complication that it matters whether the flow is cyclonic (Ks + ve) or anticyclonic (Ks - ve) and whether the horizontal shear is positive or negative.

7. Practical suggestions

From (30) we have

$$\frac{ds^2}{dt} = -\frac{\rho\theta}{\frac{\partial\theta}{\partial p}}\left[\frac{\omega}{Kp}\left(\frac{\partial \mathbf{V}_g}{\partial p}\right)^2 + \frac{\partial \mathbf{V}_g}{\partial p}\cdot F_g\cdot\frac{\partial \mathbf{V}_g}{\partial p}\right] \tag{54}$$

Because of the various uncertainties introduced by data and resolution difficulties I suggest a systematic check of the individual intrinsic contributions involved in (51) against CAT occurrences. Intrinsic formulae required are given in Section 2. Hopefully the whole formula (54) should give the best result, but this may not be the case in view of the practical difficulties. Alternatively initial and forecast estimates of the twisting term can be used if the quality of the numerical output is good enough.

8. Summary of Results

In isentropic quasi-geostrophic flow

(1) $$\frac{ds^2}{dt} = \frac{1}{f}\left(\frac{\partial V_g}{\partial p} \times \nabla_p \omega\right)\cdot \underline{k}$$

(2) $$\frac{ds^2}{dt} = \frac{\omega}{\kappa p} s^2 - \underline{s}\cdot F_g \cdot \underline{s}$$

(3) $$\frac{1}{f}\left(\frac{\partial V_g}{\partial p} \times \nabla_p \omega\right)\cdot \underline{k} = \frac{\omega}{\kappa p} s^2 - \underline{s}\cdot F_g \cdot \underline{s}$$

(4) If Ri is very large or very small compared with $f/(\zeta_p + f)$ then

$$\frac{d}{dt}\left[(\zeta_p + f)\frac{\partial \theta}{\partial p}\right] = 0 \quad , \quad \frac{d}{dt}\left[\frac{f}{\alpha}\left(\frac{\partial V_g}{\partial p}\right)^2\right] = 0 \quad , \quad \frac{d}{dt} Ri = 0$$

(5) It is shown in Section 6 that in the evaluation of $\frac{ds^2}{dt}$ from (2) the turning of the geostrophic wind with height is just as important as the increase of speed with height.

APPENDIX

At first sight this is a curious result; from experience it is the sort of oddity that might well arise from an ill-judged use of the geostrophic approximation, but I do not think that this is the case here. Given quantities A, B, C, D and that $S = -\frac{B}{A}$ then if

$$\frac{d}{dt}(AC + DB) = 0 \qquad (1)$$

it follows that if $\frac{d}{dt}(CA) = 0$ then also $\frac{d}{dt}(DB) = 0$ and vice versa. Also from

$$S = -\frac{B}{A} \qquad (2)$$

we have

$$\frac{DS}{C} = -\frac{DB}{CA}$$

whence

$$\left(\frac{DS}{C} - 1\right)CA = -(DB + CA) \qquad (3)$$

and thus using (1) yields

$$\frac{d}{dt}\left[\left(\frac{DS}{C} - 1\right)CA\right] = 0 \qquad (4)$$

from which if $S \ll \frac{C}{D}$ we have $\frac{d}{dt}(CA) = 0$ and consequently $\frac{d}{dt}(DB) = 0$

Then starting again from

$$\frac{1}{S} = -\frac{A}{B} \quad \text{gives} \quad \frac{C}{DS} = -\frac{CA}{DB} \qquad (5)$$

whence

$$\left(\frac{C}{DS} - 1\right)DB = -\left(CA + DB\right) \tag{6}$$

from which, using (1)

$$\frac{d}{dt}\left\{\left(\frac{C}{DS} - 1\right)DB\right\} = 0 \tag{7}$$

and now if $S \gg C/D$ we have $\frac{d}{dt}(DB) = 0$ and consequently $\frac{d}{dt}(CA) = 0$ the same two results as before.

Thus the result is purely a matter of algebra since the quantities A, B, C, D can be anything at all.