Metrics for assessing the impact of observations in NWP: a theoretical study. Part II: suboptimal systems

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John Eyre

Abstract

Two methods are widely used to assess the impact of observations in global numerical weather prediction (NWP): data denial experiments (DDEs) and the forecast sensitivity-based observation impact (FSOI) method. A DDE measures the impact on forecast accuracy of removing an observation type from the system, whereas FSOI measures the amount by which an observation type reduces the short-range forecast error within a system containing all observation types. This paper describes the second part of a two-part study. In the first part, the theory behind DDE and FSOI metrics was presented and then applied to a simple model with two state variables, all in the context of optimal data assimilation (DA), for which the error covariances used in the DA system match reality. In this second part, we extend the theory to suboptimal systems and then apply it to a very simple model, in this case with one state variable. As expected, DDE impacts are reduced when the system is suboptimal. By contrast, FSOI impacts increase for an observation type for which the errors are underestimated. This gives the erroneous impression that the change in assumed errors has led to an improvement, whereas the opposite is the case. These results provide some insights into the interpretation of FSOI results from a suboptimal DA system.

1. Introduction

This paper describes the second part of a two-part study. In the first part, Eyre (2021) hereafter referred to as "Part I", we considered the role of observation impact metrics for assessing the value of observations used in numerical weather prediction (NWP). We presented the theory underlying two commonly used metrics: those derived from observing system experiments (OSEs), particularly in a mode known as data denial experiments (DDEs), and those obtained using the forecast sensitivity-based observation impact (FSOI) method. We noted that, although those observing systems that provided the most impact as measured by DDE also tended to have high impact as measured by FSOI, the absolute magnitudes of impacts given by the two methods were very different. We attempted to shed some light on why this might be: on what these two approaches really measure and how they are related. We explored this problem using a very simple assimilation and forecast system, in which the forecast model has only two variables. We also restricted attention to optimal data assimilation (DA) systems, in which the values assumed for the error covariances of the observations and of the background equated to their true values. For optimal systems we found that, in order to explain the commonly found result of FSOI impact greater than DDE impact, the following system properties were important: mixing of information (and error) by the forecast model between state variables, the rate of forecast error growth, the presence of forecast model error, the way in which observational information is distributed between state variables and denied from them, and the presence of error in the data used for forecast verification.

In this paper we extend the study to suboptimal DA systems and we explore, again with a very simple model, how the DDE and FSOI metrics respond to suboptimality, and specifically to suboptimal assumptions concerning observation errors. One of the motivations for this work was the finding in other studies, e.g. Lupu et al. (2015), that underestimation of observation errors used in the DA can show increased impact of the these observations as measured by FSOI but lead to degraded forecasts as measured by DDEs.

In section 2 we present the theoretical basis of the study, and we prepare the ground for applying it to a very simple assimilation and forecast system – in this case to a forecast model with only one variable. Appendix A presents the theory for the general, multivariate case. In section 3 we present the experimental design, and in section 4 the results of the experiments. In section 5 we summarise and discuss these results, and in section 6 present some conclusions.

2. Theory

2.1 NWP error analysis for suboptimal data assimilation

The basic equation for analysis error has the same form as Part I, eq.(2.1.3):

$$\boldsymbol{\varepsilon}^a = \boldsymbol{\varepsilon}^b + \mathbf{K}_s.(\boldsymbol{\varepsilon}^o - \mathbf{H}\boldsymbol{\varepsilon}^b) \ , \qquad (2.1.1)$$

where $\boldsymbol{\varepsilon}^a$ is the error in the analysis,
$\boldsymbol{\varepsilon}^b$ is the error in the background, with covariance $\mathbf{B}$,
$\boldsymbol{\varepsilon}^o$ is the error in the observations, with covariance $\mathbf{R}$,
$\mathbf{K}_s$ is the analysis operator (Kalman gain matrix), which maps observation increments (“innovations”) into analysis increments,
and $\mathbf{H}$ is the Jacobian of the observation operator, $H[\dots]$, i.e. the gradient of the observation operator with respect to the state space variables.

We retain here the assumption of unbiased observation and background errors, and the assumption (at least for the purposes of error analysis) that the system can be treated as quasi-linear, but we now depart from Part I: we remove the assumption that the DA system is optimal. Therefore, $\mathbf{K}_s$ is the observation weight matrix assumed by the DA system and, in general, will not take its optimal value. (We will use symbols with subscript s to refer to the generally suboptimal values assumed within the DA system and symbols without this subscript to refer to true or optimal values.)

Taking the statistical expectations of (2.1.1) and assuming the observation errors are uncorrelated with background errors, we find that the mean analysis error is zero (by definition) and the covariance of analysis error, $\mathbf{A}$, is given by

$$\mathbf{A} = (\mathbf{I} - \mathbf{K}_s\mathbf{H})\mathbf{B}(\mathbf{I} - \mathbf{K}_s\mathbf{H})^T + \mathbf{K}_s\mathbf{R}\mathbf{K}_s{}^T \ , \qquad (2.1.2)$$

We assume that $\mathbf{K}_s$ takes the form:

$$\mathbf{K}_s = \mathbf{B}_s\mathbf{H}^T(\mathbf{H}\mathbf{B}_s\mathbf{H}^T + \mathbf{R}_s)^{-1} = (\mathbf{B}_s{}^{-1} + \mathbf{H}^T\mathbf{R}_s{}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{R}_s{}^{-1} \ , \qquad (2.1.3)$$

where $\mathbf{B}_s$ and $\mathbf{R}_s$ are, respectively, the assumed values of background and observation error covariance which, in general, are not the true values.

When $\mathbf{K}_s = \mathbf{K}$, its optimal value, $\mathbf{A}$ takes the forms derived in Part I:

$$\mathbf{A}^{-1} = \mathbf{B}^{-1} + \textstyle\sum_j \mathbf{H}_j{}^T\mathbf{R}_j{}^{-1}\,\mathbf{H}_j = \mathbf{B}^{-1} + \sum_j \mathbf{Z}_j = \mathbf{B}^{-1} + \mathbf{Z} \ , \qquad (2.1.4)$$

where we define the observation precision matrix:

$$\mathbf{Z} = \textstyle\sum_j \mathbf{Z}_j = \sum_j \mathbf{H}_j{}^T\mathbf{R}_j{}^{-1}\,\mathbf{H}_j. \qquad (2.1.5)$$

j denotes the jth observation or group of observations, and $\mathbf{H}_j$, $\mathbf{Z}_j$ and $\mathbf{R}_j$ represent the associated sub-matrices of $\mathbf{H}$, $\mathbf{Z}$ and $\mathbf{R}$ respectively. We make the assumption that observation errors may be correlated within group j but they are not correlated with the errors of observations in other groups.

In this study we will consider in particular the effect of $\mathbf{R}_s \neq \mathbf{R}$, where $\mathbf{R}$ is the true observation error covariance.

We make the same assumptions concerning the propagation of error by the forecast model as in Part I: the forecast error covariance, $\mathbf{P}_n$, is given by

$$\mathbf{P}_1 = \mathbf{MAM}^T + \mathbf{Q} = \mathbf{B} \; . \tag{2.1.6}$$

$$\mathbf{P}_{n+1} = \mathbf{MP}_n\mathbf{M}^T + \mathbf{Q} \; , \tag{2.1.7}$$

where $\mathbf{Q}$ is the model error covariance,
$\mathbf{M}$ is the Jacobian of forecast model,
and n is the index of forecast-assimilation cycle.

2.2 The system used in this study

The analysis error covariance for the system in equilibrium is calculated by solving (2.1.2) and (2.1.6) for $\mathbf{A}$:

$$\mathbf{A} = (\mathbf{I} - \mathbf{K}_s\mathbf{H})(\mathbf{MAM}^T + \mathbf{Q})(\mathbf{I} - \mathbf{K}_s\mathbf{H})^T + \mathbf{K}_s\mathbf{R}\mathbf{K}_s{}^T \; . \tag{2.2.1}$$

As in Part I, we assume that the system is stationary from one assimilation cycle to the next. Therefore, $\mathbf{H}$, $\mathbf{R}$, $\mathbf{M}$ and $\mathbf{Q}$ are constants, and hence also are $\mathbf{Z}$ and $\mathbf{K}_s$.

When the system is optimal, we recall from Part I (eq.2.3.4) that this simplifies to:

$$\mathbf{A}^{-1} = (\mathbf{MAM}^T + \mathbf{Q})^{-1} + \mathbf{Z} \; , \tag{2.2.2}$$

In either the optimal or suboptimal case, we can then evaluate $\mathbf{B}$ from (2.1.6) or (2.1.4).

In contrast to Part I, where we used a simple two-variable forecast model, we choose here an even simpler, one-variable model, which is adequate to illustrate the main effects of suboptimality. It also allows us to manipulate the values of the observation error covariances directly (rather than through the projection of their inverses into state space, as in Part I), in order to investigate the effects of their suboptimal specification. However, it introduces some additional limitations, which are discussed in section 5.

With this simplification, $\mathbf{M}$ becomes the forecast error growth parameter a. For a system with one state variable but multiple observations, $\mathbf{K}_s$ becomes a row vector and $\mathbf{H}$ becomes a vector with all components equal to 1. Eq.(2.2.1) for the suboptimal case then simplifies to:

$$A = (1 - \mathbf{K}_s\mathbf{H})(a^2A + Q)(1 - \mathbf{K}_s\mathbf{H})^T + \mathbf{K}_s\mathbf{R}\mathbf{K}_s{}^T \; , \tag{2.2.3}$$

or

$$A = \left\{(1 - \mathbf{K}_s\mathbf{H})^2 Q + \mathbf{K}_s\mathbf{R}\mathbf{K}_s{}^T\right\} / (1 - (1 - \mathbf{K}_s\mathbf{H})^2 a^2) \; . \tag{2.2.4}$$

Note that A is linear in $\mathbf{R}$ for a given $\mathbf{K}_s$ (i.e. for given $\mathbf{R}_s$ and $\mathbf{B}_s$).

Eq.(2.2.2) for the optimal case simplifies to:

$$A^{-1} = (a^2 A + Q)^{-1} + Z \ , \tag{2.2.5}$$

and, assuming that $\mathbf{R}$ is diagonal, eq.(2.1.5) simplifies to

$$Z = \sum_j Z_j = \sum_j R_j{}^{-1}. \tag{2.2.6}$$

Eq.(2.2.5) can be written:

$$a^2 Z\, A^2 \ + (1 - a^2 + QZ)\, A - Q \ = 0 \ , \tag{2.2.7}$$

which can be solved for A.

Note that when $Q = 0$, A becomes:

$$A^{-1} = (1 - a^{-2})^{-1} Z \ . \tag{2.2.8}$$

This is the scalar system as described in Part I (eq.(2.3.5) and section 2.6).

We will explore a system containing two observation types, one of which has its assumed observation error covariance specified correctly and the other incorrectly. We specify the difference in the true relative information contents of the two observation types through a ratio ρ:

$$\rho = Z_1 / Z_2 = R_2 / R_1 \ , \tag{2.2.9}$$

Concerning the other system parameters: the total observational information is fixed through $Z = \sum_j Z_j = 1$; a usually takes the same nominal value as used in Part I (a=1.2), but some other values have been examined; Q takes either the nominal value used in Part I (Q=0.02) or is set to zero.

2.3 The observation impact metrics

We retain the DDE metric used in Part I (eq.2.4.1) with the simplification that there is now only one forecast variable:

$$\%DDE_n = 100 \frac{P_n(deg) - P_n(full)}{P_n(full)} \ , \tag{2.3.1}$$

where $full$ denotes the full observing system and deg the degraded system, and we again focus on the DDE score for the 24h forecast (n=4), defining $\%DDE = \%DDE_4$.

We also compute, for reasons explained below, an alternative metric based on the forecast precision (i.e. the inverse of the error variance):

$$\%DDE_n^* = 100 \frac{1/P_n(full) - 1/P_n(deg)}{1/P_n(full)} \ . \tag{2.3.2}$$

The FSOI metric for the suboptimal case is discussed in Appendix A. It is shown that it no longer measures accurately the reduction in forecast error variance attributable to a subset of the observations. Instead it measures a quantity that is linear in the assumed observation weights, $\mathbf{K}_s$. The FSOI method still allows a difference in forecast error variance to be propagated backwards in time, through the adjoint of the forecast model, to give a difference in analysis error variance. This is then propagated back to individual observations through the observation weights (derived using the assumed observation error covariances, not their true values). In a one-variable system, the difference in analysis error variance is a single scalar value; the final step of the FSOI calculation then partitions this between the different observation types according to their observation weights. Therefore, the FSOI metric simplifies considerably to:

$$\%FSOI = 100\,\delta e_j/\delta e = 100\,\delta e_j/\sum_j \delta e_j = 100\,K_{s_j}/\sum_j K_{s_j}\ , \tag{2.3.3}$$

where δe_j is the contribution of the jth observation type to the FSOI energy metric and δe is the sum of contributions from all observation types, as discussed in Part I.

We also recall, from the discussion of the scalar case in Part I, our expectation of the results for an optimal scalar system: when Z is reduced by 1%, we expect a DDE impact of 1%, and also that the observations denied will contribute 1% of the total FSOI in the full system. The purpose of this study is to investigate what happens to these values when the observation error covariances are not specified optimally.

3. Experimental design

The calculation proceeds as follows:

(a) Set the values of the system parameters: a, Q and Z.
(b) Set a value for $\rho = Z_1/Z_2$. Compute the true values of Z_1 and Z_2, and hence R_1 and R_2.
(c) Choose one of the observation types to be the one that will be denied in the DDE. (We choose observation type 2, for which the true observation error variance is R_2.)
(d) Evaluate the optimal value of A through eq.(2.2.7), for the full observing system and for the degraded system.
(e) Evaluate B using eq.(2.1.6) for both cases.
(f) Vary the assumed value of observation error variance for observation type 2. For each value, compute $\mathbf{K}_s$ through eq.(2.1.3) using B_s calculated for the optimal system, step (e). Then compute the suboptimal A through eq.(2.2.4).
(g) For each value of A, calculate the equivalent values of P_n, using eq.(2.1.7).
(h) For each set of P_n, calculate $\%DDE_n$, $\%DDE_n^*$ and $\%FSOI$. We consider values equivalent to the 24h forecast error (n=4).

We use an assumed value of background error variance, B_s, that is the same as the true value for the optimal system. (The effects of mis-specifying B have been discussed by Eyre and Hilton (2013)).

4. Results

We first conduct three experiments in each of which we set the observation error for the suboptimally-specified observation type to a range of values, from $R_s = 0.25R$ to $R_s = 4R$. Note that, for simplicity, we denote here R_2 by R and R_{2_s} by R_s. (R_{1_s} is constant and equal to its true value R_1.) In all experiments we set $a = 1.2$ and $Z = 1$. The experiments are summarised in Table 1.

Experiment	Q	ρ
1	0	99
2	0.02	99
3	0	9

Table 1. Summary of experiments.

In Experiment 1, $\rho = 99$. This corresponds to 1% of the information in the observation type for which errors are incorrectly specified and 99% in observation types for which the errors are correctly specified. The model error, Q, is set to zero. Results are shown in Fig.1.

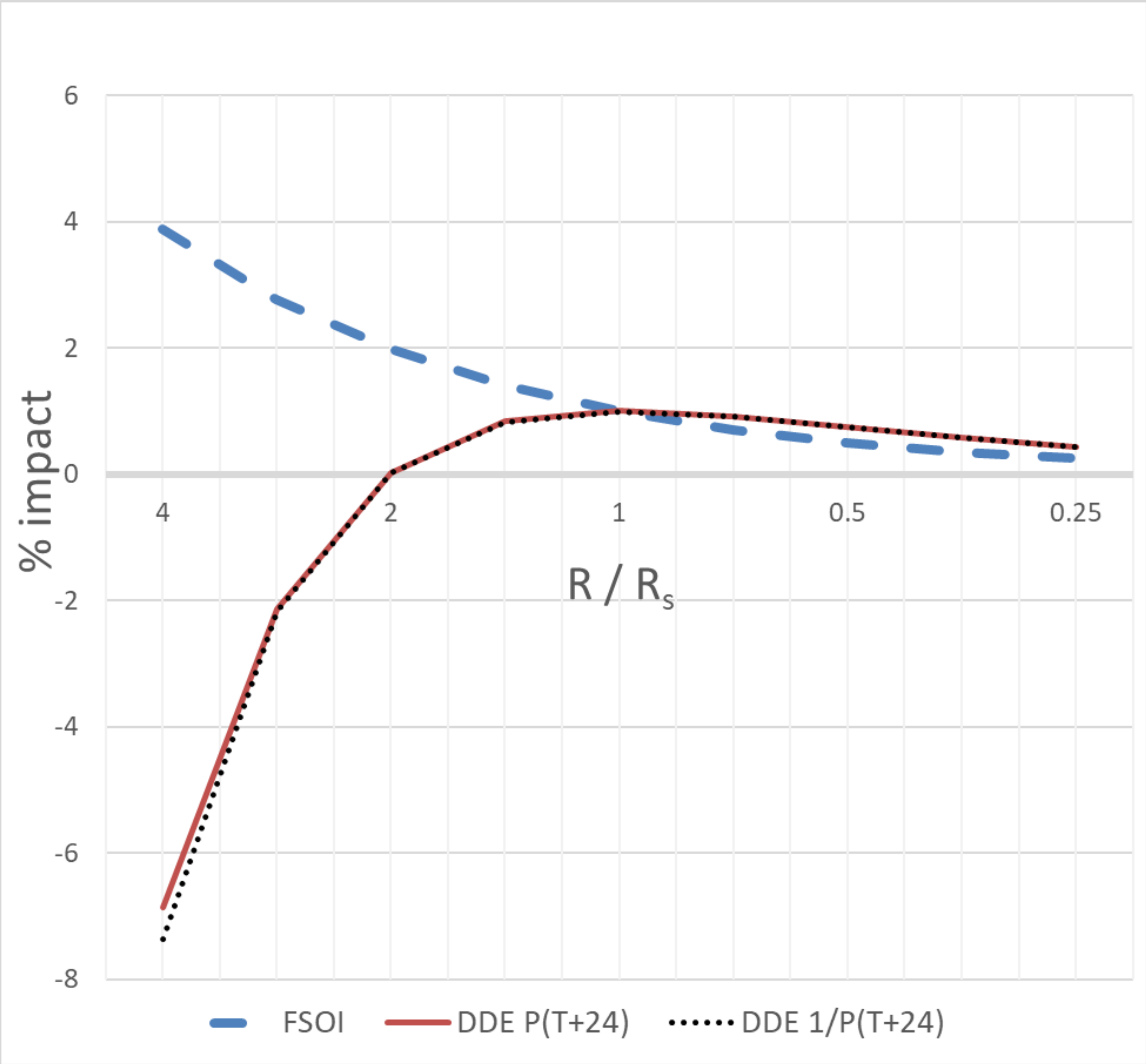

Figure 1. Results of $\%FSOI$ (dashed/blue), $\%DDE$ (solid/red) and $\%DDE^*$ for (dotted/black) for Experiment 1 (Q=0, ρ=99).

It can be seen that, when $R_s = R$, all three metrics are in agreement at an impact of 1%. (In fact, $\%DDE^*$ and $\%FSOI$ are exactly 1% and $\%DDE$ is very close to 1%. This is consistent with the problem being exactly linear in precision as discussed in Part I.) When $R_s \neq R$, $\%DDE$ (or $\%DDE^*$) is reduced, as is expected for a suboptimal system. When $R_s > R$, the DDE impact decreases towards zero, corresponding to the observations being given decreasing weight, relative to their optimal weight. However, when $R_s < R$, the positive impact decreases more rapidly. This corresponds to the observations being given too much weight relative to their optimal value. In fact, the impact of these observations becomes negative for $R/R_s \gtrsim 2$. This is analogous to the "danger zone" problem discussed by Eyre and Hilton (2013) for the equivalent problem with the suboptimal specification of background error covariance.

The behaviour of $\%FSOI$ is seen to be very different; it increases linearly with R/R_s. This shows that, in the suboptimal case, $\%FSOI$ is responding primarily to the observation weights.

In Experiment 2, Q is set to 0.02 whilst retaining $\rho = 99$. Results are shown in Fig.2. Compared with Expt.1 this change in Q results in a small reduction in the DDE impacts, both the positive and the negative impacts. However, the FSOI impacts are unchanged, because the observation weights are unchanged. This is consistent with the effects of changing Q found in Part I.

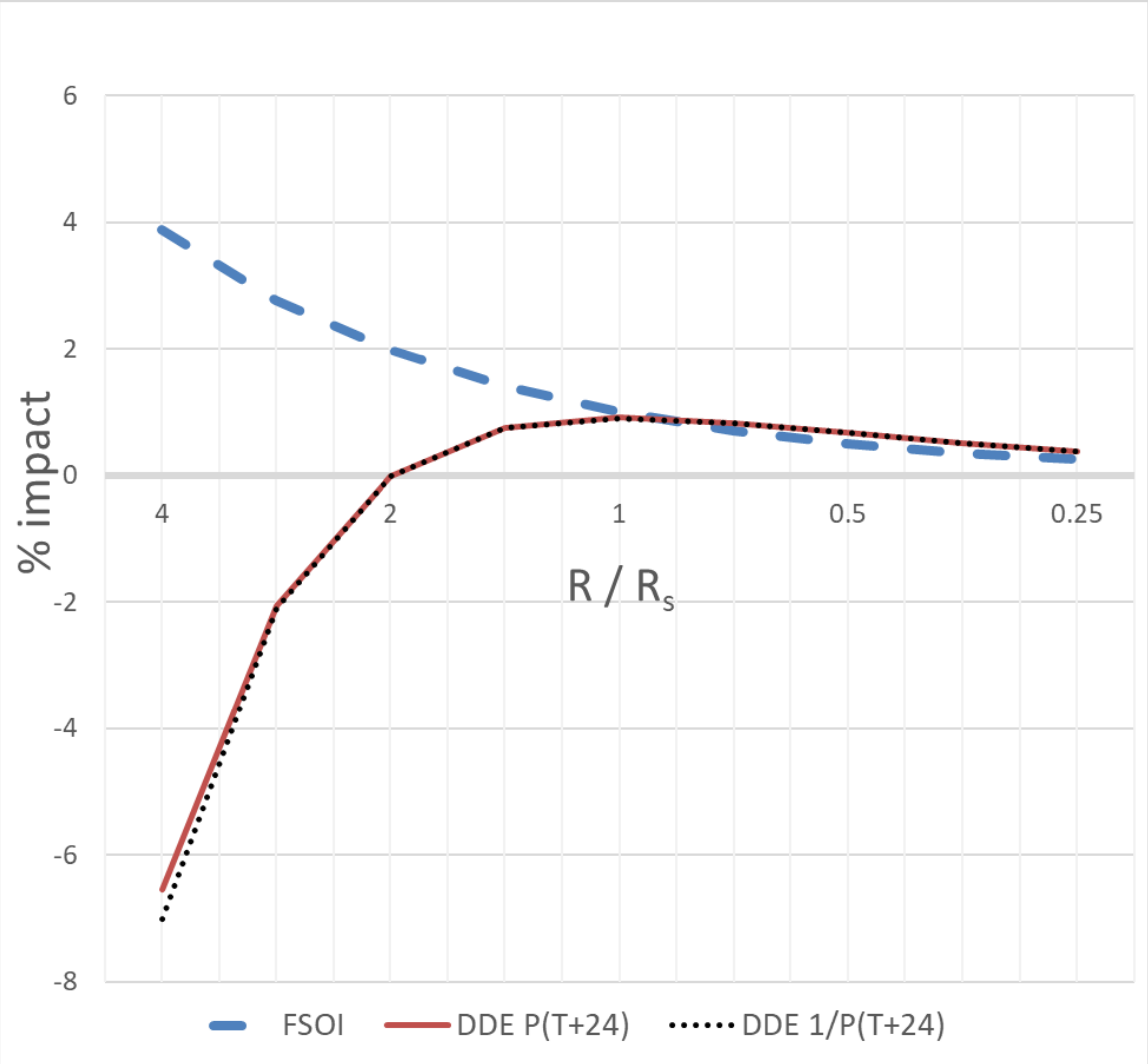

Figure 2. Results of $\%FSOI$ (dashed/blue), $\%DDE$ (solid/red) and $\%DDE^*$ for (dotted/black) for Experiment 2 (Q=0.02, ρ=99).

In Experiment 3, we set $\rho = 9$. This corresponds to 10% of the information in the observation type for which the errors are incorrectly specified and 90% in observation types for which the errors are correctly specified. Q is set to zero. Results are shown in Fig.3. The shapes of all the curves are very similar to Experiment 1, but the impacts are equal or close to 10% for the optimal system. There is also now a clearer separation between $\%DDE$ and $\%DDE^*$, because of the linear relation between A^{-1} and Z, and the nonlinear relation between A and Z.

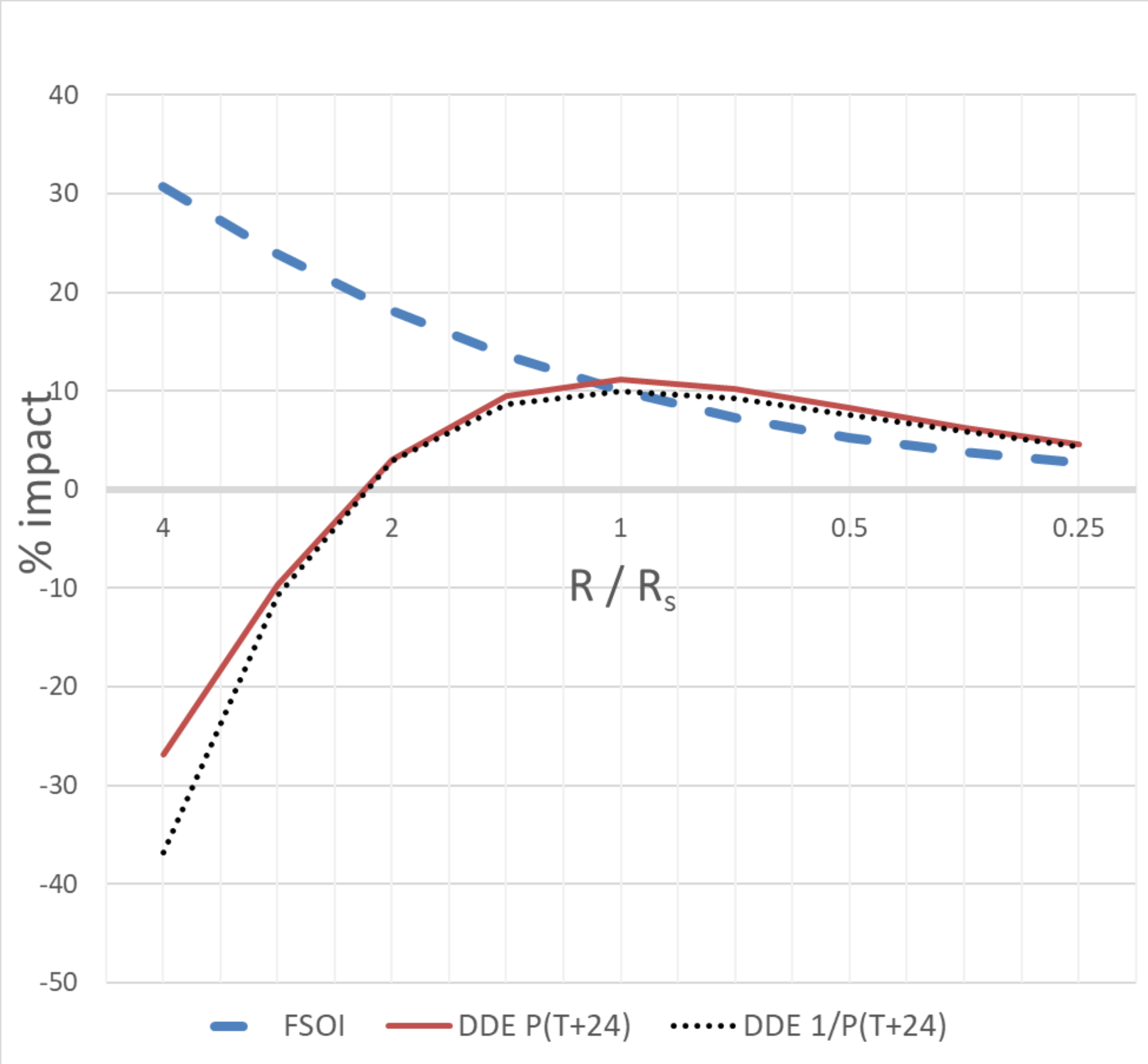

Figure 3. Results of $\%FSOI$ (dashed/blue), $\%DDE$ (solid/red) and $\%DDE^*$ for (dotted/black) for Experiment 3 (Q=0, ρ=9).

Further experiments (results not shown) have been conducted to look at the effect of varying the forecast error growth parameter, a. It is found that, when $Q = 0$, the results of Experiments 1 and 3 are replicated exactly. When $Q \neq 0$, the results are similar to those of Experiment 2 but modified, with larger differences from Experiment 1 for smaller values of a.

5. Discussion

The key feature of all these results is the opposite behaviour of $\%DDE$ and $\%FSOI$ when $R > R_s$: $\%DDE$ is reduced, as it is for all suboptimal values of R_s, but $\%FSOI$ is increased, because of the increased observation weight. This result is independent of the value of ρ. As shown in Appendix A, $\%FSOI$ is no longer measuring how much an observation type contributes to the reduction of forecast error, resulting from the reduction in analysis error on each assimilation cycle. However, $\%FSOI$ is showing the “correct” tendency in one sense; it is showing the increased impact of the observation type on the analysis as a result of the increased observation weight. But this is also a misleading result, as it strongly conflicts with the $\%DDE$ result; it gives the erroneous impression that the observations are more beneficial to the analysis than in the optimal case, when in fact they are not.

The inclusion of model error is found to be important in reducing $\%DDE$ scores, as in Part I. This can be understood by considering model error as one way in which the NWP system progressively “forgets” observational information. $\%FSOI$ is not affected in this very simple system, because the observation weights (relative to each other) are unaffected by the model error.

Compared with the effects explored in Part I, those examined in this study are limited; they have all the limitations of the single-variable problem discussed in Part I. Nevertheless, this helps us to focus on the key difference between $\%DDE$ and $\%FSOI$ for suboptimal systems.

In a one-variable system with static error variances, $\%FSOI$ must always be positive, because analysis error must be lower than background error (for a stable system) and analysis weights must be positive. It is well known from real-world systems that FSOI scores can be negative. This is because, although the analysis must be more accurate than the background overall, some aspects of the analysis can be degraded by observations used in an erroneous way. In these circumstances, the FSOI statistics can highlight problems with observations and/or their assimilation.

With this very simple system it is only possible to study suboptimality as represented by the mis-specification of observation error variances. In real-world systems, observation errors can also be mis-specified in other ways, such as their correlations and their biases. Both of these aspects of suboptimality will lead to degraded DDE scores, but their effect of FSOI scores may be different.

These results show the danger of using FSOI statistics alone to tune an NWP DA system. It is possible for overweighted observations to have misleading high FSOI scores. This shows that other methods (such as OSEs) are needed to tune the observation error covariances themselves. However, once these are tuned close their optimal values, FSOI statistics can be expected to give helpful and informative results, as is found in practice.

6. Conclusions

In Part I, we developed the theory relating the DDE and FSOI impact metrics for an optimal DA system, and we applied this theory to a very simple, two-variable system. In this study, Part II, we have extended the theory to the suboptimal case. We have applied this theory to an even simpler, one-variable system, in order to illustrate a key property of the DDE and FSOI metrics in response to suboptimality.

Within a DA system, when the observation error variance of any observation type is assumed to have a value that differs from its true value, then the analysis error variances and subsequent forecast error variances will be higher than they would be for an optimal system. This is expected, and it is indeed what we mean by an optimal system, i.e. one with minimum error variances. The observation impacts as measured by DDE will tend to confirm this result; DDE impacts will be lower for a system in which the observation errors are specified in a suboptimal way.

The behaviour of FSOI impacts, in response to incorrectly specified observation errors, is very different; if the observation errors assumed by the system for a given observation type are underestimated, then the FSOI impact for that type will increase. It does so because the observation weight is increased. This represents the increased impact of this observation type within the system. However, this can give the misleading impression that the benefit derived from this observation type has increased, which is not the case, as the DDE experiments confirm.

These results, although derived using a highly simplified DA system, provide a warning concerning the interpretation of FSOI results from real-world systems.

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Appendix A. The FSOI metric for suboptimal data assimilation

In section 2.5 of Part I, we derived the FSOI metric for an optimal DA system. Using the expression for the FSOI percentage error reduction attributable to jth observation subset:

$$\%FSOI = 100\ \delta e_j/\delta e \ , \tag{A.1}$$

where δe is the FSOI energy metric and δe_j is its contribution from the jth observation subset, we showed that:

$$\delta e_j = trace[\mathbf{M}^n(\mathbf{B}-\mathbf{A})_j\mathbf{M}^{n\boldsymbol{T}}] = trace[\mathbf{M}^n\left(\mathbf{A}\mathbf{Z}_j\mathbf{B}\right)\mathbf{M}^{n\boldsymbol{T}}] \ , \tag{A.2}$$

and $$\delta e = \textstyle\sum_j \delta e_j = \sum_j trace[\mathbf{M}^n\left(\mathbf{A}\mathbf{Z}_j\mathbf{B}\right)\mathbf{M}^{n\boldsymbol{T}}] \ . \tag{A.3}$$

This derivation relies on the following expressions for optimal DA:

$$\mathbf{B}-\mathbf{A} = \mathbf{KHB} = \mathbf{AZB} = \mathbf{A}\textstyle\sum_j \mathbf{Z}_j\,\mathbf{B} \ . \tag{A.4}$$

and $$(\mathbf{B}-\mathbf{A})_j = \mathbf{A}\mathbf{Z}_j\mathbf{B} \ , \tag{A.5}$$

where $(\mathbf{B}-\mathbf{A})_j$ is the change in analysis error covariance (relative to the background) attributable to the jth observation subset.

In the suboptimal case, we can derive equivalent expressions starting from equations (2.1.2) and (2.1.3):

$$\mathbf{A} = (\mathbf{I}-\mathbf{K}_s\mathbf{H})\mathbf{B}(\mathbf{I}-\mathbf{K}_s\mathbf{H})^T + \mathbf{K}_s\mathbf{R}\mathbf{K}_s{}^T \ , \tag{A.6}$$

$$\mathbf{K}_s = \mathbf{B}_s\mathbf{H}^T(\mathbf{H}\mathbf{B}_s\mathbf{H}^T + \mathbf{R}_s)^{-1} \ . \tag{A.7}$$

Using (A.7) and with $\mathbf{B}_s = \mathbf{B}$, (A.6) becomes

$$\mathbf{B}-\mathbf{A} = \mathbf{K}_s\mathbf{HB} + \mathbf{K}_s(\mathbf{R}_s-\mathbf{R})\mathbf{K}_s{}^T \ , \tag{A.8}$$

This reduces to (A.4) when $\mathbf{R}_s = \mathbf{R}$, and hence $\mathbf{K}_s = \mathbf{K}$.

Denoting $\mathbf{K}_{s_j}$ as the jth block of rows of $\mathbf{K}_s$ and $\mathbf{H}_j$ as the jth block of columns of $\mathbf{H}$, and assuming that $\mathbf{R}_s$ and $\mathbf{R}$ are block diagonal, with diagonal blocks $\mathbf{R}_{s_j}$ and $\mathbf{R}_j$ respectively, we obtain:

$$\mathbf{B} - \mathbf{A} = \sum_j \mathbf{K}_{s_j} \mathbf{H}_j \mathbf{B} + \sum_j \mathbf{K}_{s_j} (\mathbf{R}_{s_j} - \mathbf{R}_j) \mathbf{K}_{s_j}^{T} , \tag{A.9}$$

and $$(\mathbf{B} - \mathbf{A})_j = \mathbf{K}_{s_j} \mathbf{H}_j \mathbf{B} + \mathbf{K}_{s_j} (\mathbf{R}_{s_j} - \mathbf{R}_j) \mathbf{K}_{s_j}^{T} . \tag{A.10}$$

In the case of a scalar (one-variable) system, (A.10) simplifies to:

$$(B - A)_j = K_{s_j} B + (K_{s_j})^2 (R_{s_j} - R_j) . \tag{A.11}$$

Therefore, for the suboptimal case, equations (A.9) and (A.10) take the place of equations (A.4) and (A.5) in the optimal case. Equation (A.9) is the equation we would need to evaluate in order to find the change in analysis error covariance attributable to the jth observation subset. However, this is not what the standard FSOI technique evaluates. Instead, it computes a quantity that is linear in $\mathbf{K}_s$, and in so doing it misallocates the assessment of impact between the different observation types. Note also that, in practice, equations (A.9) and (A.10) cannot be evaluated because the true values of $\mathbf{R}_j$ are unknown.

This result can be understood most easily by considering the case of a one-variable system. The FSOI method effectively allows a difference in forecast error variance to be propagated backwards in time, through the adjoint of the forecast model, to give a difference in analysis error variance. In a one-variable system this is a single scalar value. The final step of the FSOI calculation then partitions this analysis error variance difference between the different observation types according to their observation weights. Therefore, eq.(A.1) becomes:

$$\%FSOI = 100\, \delta e_j / \delta e = 100\, \delta e_j / \sum_j \delta e_j = 100\, K_{s_j} \Big/ \sum_j K_{s_j} \tag{A.12}$$

Note that this partitioning uses the assumed values of weights, $\mathbf{K}_s$, and not their optimal values. The relationship to the FSOI expression for the optimal case (equations A.2 and A.3) can be further appreciated by noting that the K_{s_j} is proportional to ${R_{s_j}}^{-1}$, through eq.(2.1.3), and hence to the assumed value of Z_j.

Acronyms

DA	data assimilation
DDE	data denial experiment
FSOI	forecast sensitivity-based observation impact
NWP	numerical weather prediction
OSE	observing system experiment

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