A special version of the FFT

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1. Introduction

The Fourier Transform may be represented as a matrix transformation of a given data vector, ie the process may be represented by a matrix/vector equation. If the rows of the matrix are suitably interchanged the resulting matrix factorizes into a form which gives rise to the Gentleman-Sande version of the Fast Fourier Transform (FFT). If the transpose of this factorized matrix equation is taken the result is the Cooley-Tukey version of the FFT. The well-nigh innumerable variations of these two main versions have dominated the literature over the past fifteen years.

A third group of possibilities dealt with in this note concerns a factorization which leads to a version of the FFT which is remarkably constant from stage to stage. This version may have some special interest for scientists working on large problems in the office.

This note is largely an explicit exposition of an article by M C Pease. Readers who turn to Pease's work for further enlightment should beware that his paper contains a few textual and algebraical infelicities. In particular his definition of the Kronecker product of two matrices is non-standard. This note uses the standard definition.

M C PEASE:- "An adaptation of the Fast Fourier Transform for Parallel Processing" J Assocn for Comput Machinery, 15, 2, Apr 1968, pp 252-264

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2. Matrix formulation of a standard FFT

If two mx1 vectors $\underset{m1}{\underline{h}}$ and $\underset{m1}{\underline{a}}$ are related by

$$\underset{m1}{\underline{a}} = \frac{1}{m}\left(\underset{mm}{W} \cdot \underset{m1}{\underline{h}}\right) \tag{1}$$

where

$$\underset{mm}{Wrs} = w^{rs} = \exp 2\pi irs/m \quad r, s=o, --,(m-1) \tag{2}$$

then $\underline{a}$ is said to be the Fourier Transform of $\underline{h}$. The Fast Fourier Transform arises because a certain reordering of the rows of W gives a matrix W^1 which can be advantageously factored. The process is best illustrated by working through a specific case, with m=8 (ie 2^3). Bearing in mind that in (2) w is an m-th root of unity, in the case m=8 the exponents rs can be evaluated mod 8 and $\underset{8}{W}$ is given by

$$\underset{8}{W} = \begin{array}{cc} & \text{Bit Order} \\ \begin{pmatrix} w^o & w^o & w^o & w^o & w^o & w^o & w^o & w^o \\ w^o & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ w^o & w^2 & w^4 & w^6 & w^o & w^2 & w^4 & w^6 \\ w^o & w^3 & w^6 & w^1 & w^4 & w^7 & w^2 & w^5 \\ w^o & w^4 & w^o & w^4 & w^o & w^4 & w^o & w^4 \\ w^o & w^5 & w^2 & w^7 & w^4 & w^1 & w^6 & w^3 \\ w^o & w^6 & w^4 & w^2 & w^o & w^6 & w^4 & w^2 \\ w^o & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w^1 \end{pmatrix} & \begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} \end{array} \tag{3}$$

The numbers to the right of the matrix are the row numbers in binary. The rows are now interchanged to get

$$\underset{8}{\overset{1}{W}} = \begin{array}{cc} \begin{pmatrix} w^o & w^o & w^o & w^o & w^o & w^o & w^o & w^o \\ w^o & w^4 & w^o & w^4 & w^o & w^4 & w^o & w^4 \\ w^o & w^2 & w^4 & w^6 & w^o & w^2 & w^4 & w^6 \\ w^o & w^6 & w^4 & w^2 & w^o & w^6 & w^4 & w^2 \\ w^o & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ w^o & w^5 & w^2 & w^7 & w^4 & w^1 & w^6 & w^3 \\ w^o & w^3 & w^6 & w^1 & w^4 & w^7 & w^2 & w^5 \\ w^o & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w^1 \end{pmatrix} & \begin{array}{c} 000 \\ 100 \\ 010 \\ 110 \\ 001 \\ 101 \\ 011 \\ 111 \end{array} \end{array} \tag{4}$$

The original binary row numbers are listed at the side and it is seen that the new row ordering has been obtained by simply reversing the bits in the original binary row numbers of (3). If W^1 is used to do the transform there results

$$\underline{a}^1 = \frac{1}{m}\left(W^1 \cdot \underline{h}\right) \tag{5}$$

and $\underline{a}^1$ is the bit-reversed form of $\underline{a}$. In matrix terms the connection between W^1 and W is that

$$\underset{8}{W^1} = \underset{8}{P} \cdot \underset{8}{W} \tag{6}$$

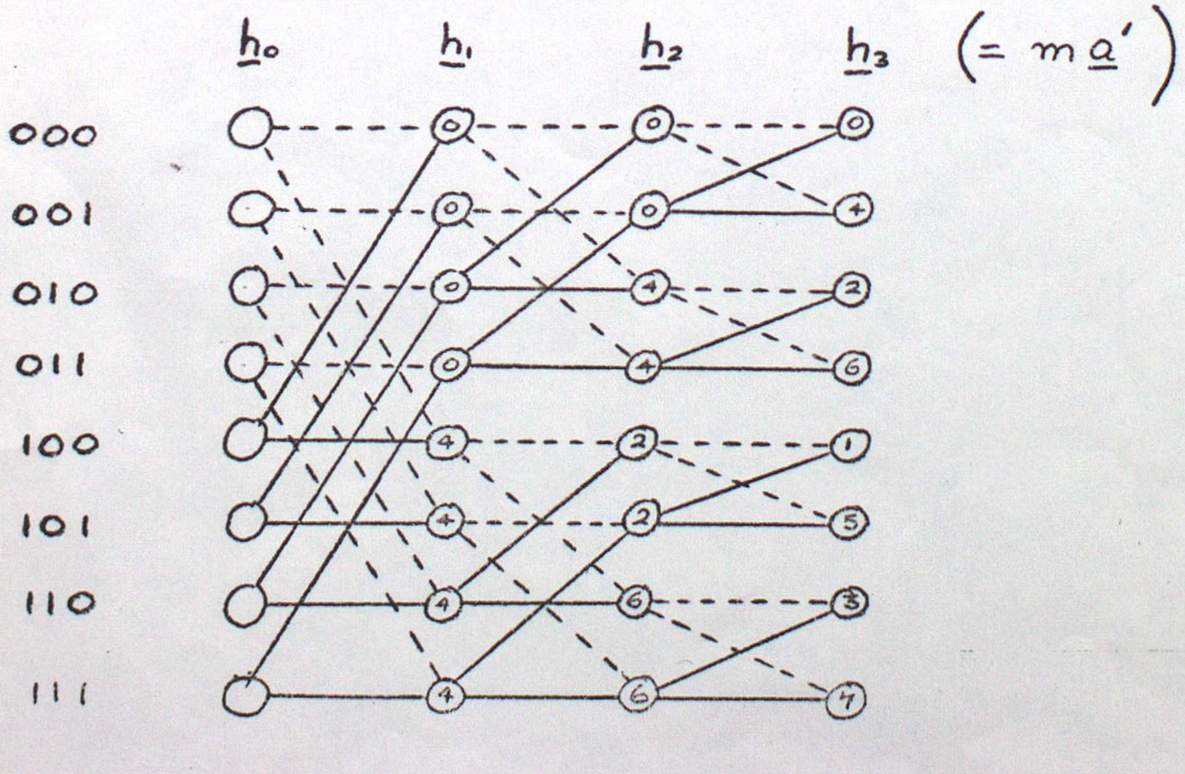
where $\underset{8}{P}$ is the 8 x 8 permutation matrix

$$P_8 = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} \tag{7}$$

The matrix W^1 factorizes so that (5) may be written as

$$\underline{a}^1 = \frac{1}{m}\left[\begin{pmatrix} w^o & w^o & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ w^o & w^4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & w^o & w^o & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & w^o & w^6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & w^o & w^1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & w^o & w^5 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & w^o & w^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & w^o & w^7 \end{pmatrix} \cdot \begin{pmatrix} w^o & \cdot & w^o & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & w^o & \cdot & w^o & \cdot & \cdot & \cdot & \cdot \\ w^o & \cdot & w^4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & w^o & \cdot & w^4 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & w^o & \cdot & w^2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & w^o & \cdot & w^2 \\ \cdot & \cdot & \cdot & \cdot & w^o & \cdot & w^6 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & w^o & \cdot & w^6 \end{pmatrix} \cdot \begin{pmatrix} w^o & \cdot & \cdot & \cdot & w^o & \cdot & \cdot & \cdot \\ \cdot & w^o & \cdot & \cdot & \cdot & w^o & \cdot & \cdot \\ \cdot & \cdot & w^o & \cdot & \cdot & \cdot & w^o & \cdot \\ \cdot & \cdot & \cdot & w^o & \cdot & \cdot & \cdot & w^o \\ w^o & \cdot & \cdot & \cdot & w^4 & \cdot & \cdot & \cdot \\ \cdot & w^o & \cdot & \cdot & \cdot & w^4 & \cdot & \cdot \\ \cdot & \cdot & w^o & \cdot & \cdot & \cdot & w^4 & \cdot \\ \cdot & \cdot & \cdot & w^o & \cdot & \cdot & \cdot & w^4 \end{pmatrix}\right] \cdot \underline{h} \tag{8}$$

an equation which (ignoring the 1/m) has the flow-graph



(9)

In the flow-graph (9) the dotted lines represent additions and the solid lines represent multiplications, the numbers in the nodal circles giving the exponent of the multiplying w. For example, at the first stage of transformation the value of h, (101) is given by

$$h_1\ (101) = h_o\ (001) + w^4 h_1\ (101) \qquad (10)$$

The first stage of the transformation, $\underline{h}_o \rightarrow \underline{h}_1$ in (9) represents the action of the rightmost matrix in (8) on the original data vector $\underline{h}$. The next stage in (9), $\underline{h}_1 \rightarrow \underline{h}_2$ represents the action of the middle matrix of (8) on $\underline{h}_1$. The last stage of (9) $\underline{h}_2 \rightarrow \underline{h}_3$ represents the action of the leftmost matrix in (8). Then finally $\underline{a}^1 = \frac{1}{m}\ \underline{h}_3$.

Thus the FFT tree-graph (9), which is one of the most popular forms of the FFT, is simply a flow-graph of the action of the particular factors of W^1 which appear in (8). A little investigation soon reveals that the factorization in (8) is by no means unique. There are several other ways in which W^1 can be factored. The question then arises as to whether a factorization exists which has some advantage over the customary (8). The next section is concerned with this possibility.

3. An alternative matrix factorization for the FFT

First, it can be verified that W^1 factorizes as

$$W^1_8 = \left(\begin{array}{c|c} W^1_4 & O_4 \\ \hline O_4 & W^1_4 \end{array}\right) \cdot \left(\begin{array}{c|c} I_4 & O_4 \\ \hline O_4 & K_4 \end{array}\right) \cdot \left(\begin{array}{c|c} I_4 & I_4 \\ \hline I_4 & -I_4 \end{array}\right) \qquad (11)$$

In (11) W^1_4 is the 4 x 4 matrix which comprises the top left hand corner of W^1_8 in (4), ie

$$W^1_4 = \begin{pmatrix} w^o & w^o & w^o & w^o \\ w^o & w^4 & w^o & w^4 \\ w^o & w^2 & w^4 & w^6 \\ w^o & w^6 & w^4 & w^2 \end{pmatrix} \qquad (12)$$

I_4 is the 4 x 4 Idemfactor (unit matrix)

O_4 is a 4 x 4 matrix of zeros

K_4 is the 4 x 4 diagonal matrix K = Diag (w^o, w^1, w^2, w^3)

It is evident that the last factor in (11) is the same as the last factor in (8). This follows because $w^o = 1$ and $w^4 = -1$. The other factors in (11) are not the same as the corresponding factors in (8). Equation (11) is a different factorization to (8) and it has a different tree-graph, but its tree-graph is of no special interest in the context of this section.

The introduction of the Kronecker product now facilitates the further development of (11). For any two matrices the Kronecker product is defined as

$$A_{mn} \otimes B_{st} \triangleq a_{ij}\ B \qquad (13)$$

a matrix of dimensions ms x nt. Using (13), equation (11) may be written as

$$W^1_8 = (I_2 \otimes W^1_4) \cdot D_8 \cdot (W^1_2 \otimes I_4) \tag{14}$$

where I_2 is the 2 x 2 Idemfactor

D_8 is Diag (I_4, K_4)

and W^1_2 is the 2 x 2 matrix which comprises the top left hand corner of W^1_4 in (12), ie

$$W^1_2 = \begin{pmatrix} w^o & w^o \\ w^o & w^4 \end{pmatrix} \tag{15}$$

It is then found that W^1_4 can be written as

$$W^1_4 = (I_2 \otimes W^1_2) \cdot D_4 \cdot (W^1_2 \otimes I_2) \tag{16}$$

where $D_4 = \text{Diag}\ (I_2, K_2)$

with $K_2 = \text{Diag}\ (w^o, w^2)$

Since $I_4 = I_2 \otimes I_2$, (16) can now be substituted

into (14) to get

$$W^1_8 = \left[I_2 \otimes \left\{ (I_2 \otimes W^1_2) \cdot D_4 \cdot (W^1_2 \otimes I_2) \right\} \right] \cdot D_8 \cdot \left[W^1_2 \otimes I_2 \otimes I_2 \right] \tag{17}$$

As it stands (17) is somewhat inscrutable but using the matrix identity

$$I \otimes (A \cdot B \cdot C \cdot \text{----}) = (I \otimes A) \cdot (I \otimes B) \cdot (I \otimes C) \cdot \text{-----} \tag{18}$$

it can be brought to the form

$$W^1_8 = (I_2 \otimes I_2 \otimes W^1_2) \cdot (I_2 \otimes D_4) \cdot (I_2 \otimes W^1_2 \otimes I_2) \cdot D_8 \cdot (W^1_2 \otimes I_2 \otimes I_2) \tag{19}$$

This is now a fairly regular looking expression since D_8 and $(I_2 \otimes D_4)$ are simply diagonal matrices and the other factors all involve only I_2 and W^1_2. It is evident that (19) would become even simpler if by some manipulation W^1_2 could be located in the same position in each of the factors in which it occurs. This is possible as follows. Define the permutation matrix Q_8

$$Q_8 = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} \qquad (20)$$

It can now be shown that

$$Q_8 \cdot (I_2 \otimes W^1_2 \otimes I_2) \cdot \tilde{Q}_8 = I_2 \otimes I_2 \otimes W^1_2 \qquad (21)$$

and that

$$Q_8 \cdot (W^1_2 \otimes I_2 \otimes I_2) \cdot \tilde{Q}_8 = I_2 \otimes W^1_2 \otimes I_2 \qquad (22)$$

which means that

$$Q^2_8 \cdot (W^1_2 \otimes I_2 \otimes I_2) \cdot \tilde{Q}^2_8 = I_2 \otimes I_2 \otimes W^1_2 \qquad (23)$$

In other words, each time the operations $Q_8 \cdot$ and $\cdot \tilde{Q}_8$ are simultaneously applied the W^1_2 matrix is shifted one place to the right. It then follows that if we put

$W^1_2 \otimes I_2 \otimes I_2 = C_8$ the use of (21) to (23) brings (19) to the form

$$W^1_8 = Q^2_8 \cdot C_8 \cdot \tilde{Q}^2_8 \cdot (I_2 \otimes D_4) \cdot Q_8 \cdot C_8 \cdot \tilde{Q}_8 \cdot D_8 \cdot C_8 \qquad (24)$$

Now notice that $Q^3 = I$, which means that $Q^2 \cdot Q = I$ and therefore that $Q^2 = Q^{-1}$. But Q is an orthogonal matrix and so $Q^{-1} = \tilde{Q}$, so that in the end

$$Q^2 = \tilde{Q} \qquad (25)$$

Thus if we put

$$\tilde{Q}_8 \cdot (I_2 \otimes D_4) \cdot Q_8 = (I_2 \otimes D_4)^1 \qquad (26)$$

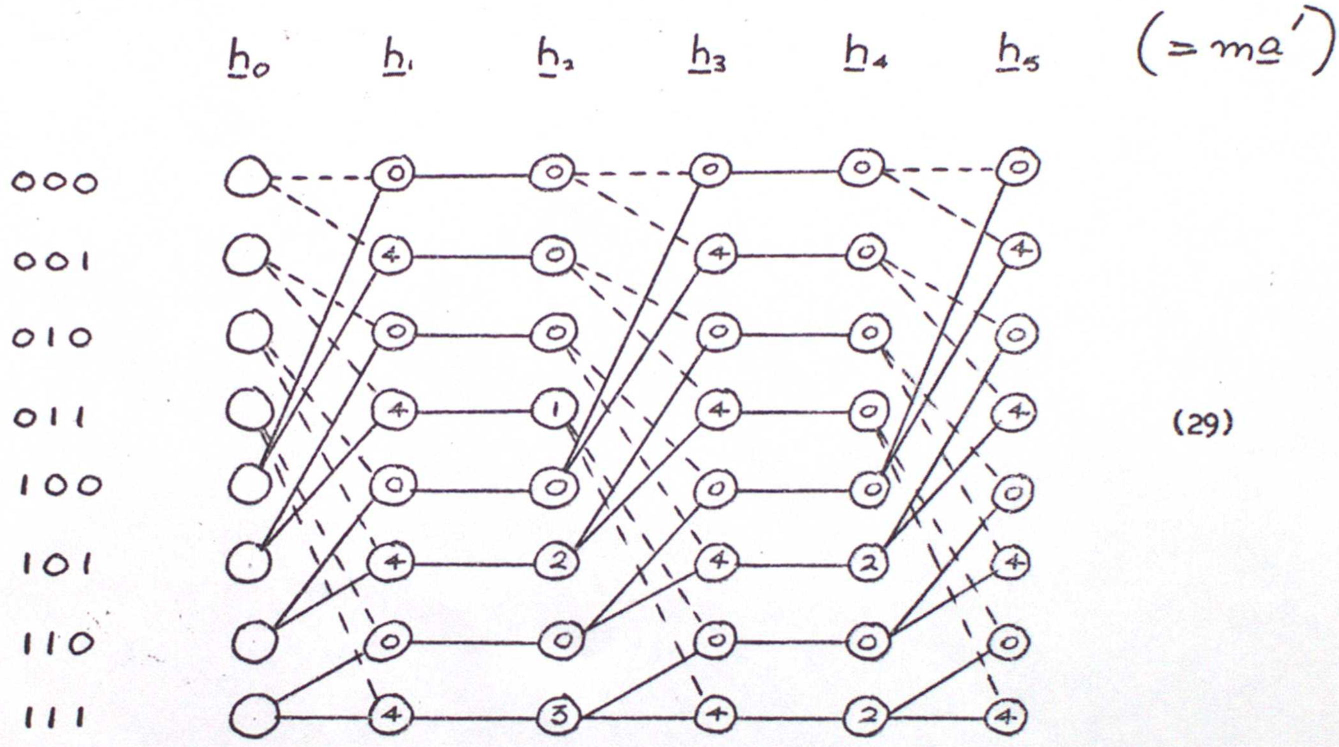
then (24) can be written as

$$W^1_8 = \tilde{Q}_8 \cdot C_8 \cdot \tilde{Q}_8 \cdot (I_2 \otimes D_4)^1 \cdot C_8 \cdot \tilde{Q} \cdot D_8 \cdot C_8 \qquad (27)$$

In (27) $\tilde{Q}_8$ is a permutation matrix, and $(I_2 \otimes D_4)^1$ is, like $(I_2 \otimes D_4)$, diagonal.
The especially simple charteristics of these matrices enables (27) to be expressed in the final very regular form

$$W^1_8 = (\tilde{Q}_8 . C_8). \ \mathrm{Diag}\,(w^0, w^0, w^0, w^0, w^0, w^2, w^0, w^2). (\tilde{Q}_8 . C_8)$$
$$. \ \mathrm{Diag}\,(w^0, w^0, w^0, w^1, w^0, w^2, w^0, w^3) . (\tilde{Q}_8 . C_8) \qquad (28)$$

Since (28) is so regular in form it is inevitable that its tree-graph is also regular from stage to stage, being in fact



(29)

The regularity of the tree-graph (29) stems from the fact that the matrix which determines the "cross-wiring at each stage is always the same matrix $(\tilde{Q}.C)$.
This regularity does not depend on m being 8. For example, for m = 16, omitting the algebraical details the factorization, with C_{16} and Q_{16} suitably defined, is

$$W^1_{16} = (\tilde{Q}_{16} . C_{16}) . \ \mathrm{Diag}\,(w^0, w^0, w^0, w^0, w^0, w^0, w^0, w^0, w^0, w^4, w^0, w^4, w^0, w^4, w^0, w^4). (\tilde{Q}_{16} . C_{16})$$
$$. \ \mathrm{Diag}\,(w^0, w^0, w^0, w^0, w^0, w^2, w^0, w^2, w^0, w^4, w^0, w^4, w^0, w^6, w^0, w^6). (\tilde{Q}_{16} . C_{16})$$
$$. \ \mathrm{Diag}\,(w^0, w^0, w^0, w^1, w^0, w^2, w^0, w^3, w^0, w^4, w^0, w^5, w^0, w^6, w^0, w^7). (\tilde{Q}_{16} . C_{16})$$

4 ~~3~~. <u>Applications</u>

(a) The regular form should enable very compact FFT programs to be written for the IBM 195.

(b) There may be some special advantages in relation to array processors or vector machines.

(c) The regular form raises the possibility that a very simple (and therefore inexpensive) machine could be built, pre-wired to do the FFT for a particular m. The simplicity of the machine would arise from the fact that only one stage would have to be wired up and the machine would be designed to use this stage the necessary number of times.

It will be apparent from a detailed study of the foregoing algebra that the regular form arrived at is not unique. There may well be another regular form having some optimal qualities. It is difficult to know at this stage whether this point is worth pursuing. What seems to be desirable is that scientists in the Office having a problem to which the FFT is or can be made applicable should give some thought to the general possibility outlined in Section 3.

A possible major application might be in the development and implementation of a spectral model using (c) in conjunction with a main computer. It might be possible in this way to achieve a very fast model.

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