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Div, Curl, Grad, and the equations of motion in oblique curvilinear coordinates

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Introduction

Whilst it is usually possible to work only in terms of orthogonal coordinate systems it may be necessary for certain special requirements to use an oblique system. This note provides a systematic derivation of the general oblique curvilinear case and highlights the connection with the more familiar orthogonal and rectilinear cases.

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1. Rectangular Cartesian Coordinates

If $\underline{V}$ is a 3-d vector such that

$$\underline{V} = u\underline{i} + v\underline{j} + w\underline{k} \tag{1}$$

then

$$\text{Grad } \underline{V} = \nabla \underline{V} = \underline{i}\frac{\partial \underline{V}}{\partial x} + \underline{j}\frac{\partial \underline{V}}{\partial y} + \underline{k}\frac{\partial \underline{V}}{\partial z} \tag{2}$$

$$\text{Div } \underline{V} = \nabla \cdot \underline{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{3}$$

$$\text{Curl } \underline{V} = \nabla \times \underline{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\underline{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\underline{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\underline{k} \tag{4}$$

and it is worth noting that (4) can be expressed as a determinant

$$\text{Curl } \underline{V} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{5}$$

If $\underline{r}$ is the position vector then

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} \tag{6}$$

and

$$\text{Grad } \underline{r} = \nabla \underline{r} = \underline{i}\frac{\partial \underline{r}}{\partial x} + \underline{j}\frac{\partial \underline{r}}{\partial y} + \underline{k}\frac{\partial \underline{r}}{\partial z} = I \tag{7}$$

Where I is the Idemfactor

$$I = \underline{i}\,\underline{i} + \underline{j}\,\underline{j} + \underline{k}\,\underline{k} \tag{8}$$

This is all standard work such as may be found in, for example, Weatherburn's "Advanced Vector Analysis Vol 2", but there is a relevant point which is not brought out in standard texts -

The Grad operator is

$$\nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z} \tag{9}$$

and therefore, using this on x, y, z in turn we get

$$\nabla x = \underline{i} \quad , \quad \nabla y = \underline{j} \quad , \quad \nabla z = \underline{k} \tag{10}$$

But from (6) by differentiating in turn by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ we get

$$\frac{\partial \underline{r}}{\partial x} = \underline{i} \quad , \quad \frac{\partial \underline{r}}{\partial y} = \underline{j} \quad , \quad \frac{\partial \underline{r}}{\partial z} = \underline{k} \tag{11}$$

and so from (8), (10), and (11) the Idemfactor can be written as

$$I = \nabla x \frac{\partial \underline{r}}{\partial x} + \nabla y \frac{\partial \underline{r}}{\partial y} + \nabla z \frac{\partial \underline{r}}{\partial z} \tag{12}$$

(12) is little more than a curiosity in rectilinear coordinates, but it is a form which matters in curvilinear coordinates.

2. Orthogonal Curvilinear Coordinates

Again this is standard Weatherburn or Rutherford material apart from slight variations in notation. Take θ, ϕ, λ as the independent orthogonal curvilinear coordinates and let $\underline{a}$, $\underline{b}$, $\underline{c}$ be the corresponding orthogonal unit vectors. As in the cartesian case $\underline{a}$, $\underline{b}$, $\underline{c}$ are a right-handed set. Now

$$\text{Grad}\,\underline{V} = \nabla \underline{V} = \frac{\underline{a}}{h_1}\frac{\partial \underline{V}}{\partial \theta} + \frac{\underline{b}}{h_2}\frac{\partial \underline{V}}{\partial \phi} + \frac{\underline{c}}{h_3}\frac{\partial \underline{V}}{\partial \lambda} \tag{13}$$

$$\text{Div}\,\underline{V} = \nabla\cdot\underline{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial\theta}(h_2 h_3 u) + \frac{\partial}{\partial\phi}(h_3 h_1 v) + \frac{\partial}{\partial\lambda}(h_1 h_2 w)\right] \tag{14}$$

$$\begin{aligned}\text{Curl}\,\underline{V} = \nabla\times\underline{V} = {} & \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial\phi}(h_3 w) - \frac{\partial}{\partial\lambda}(h_2 v)\right] \\ & + \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial\lambda}(h_1 u) - \frac{\partial}{\partial\theta}(h_3 w)\right] \\ & + \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial\theta}(h_2 v) - \frac{\partial}{\partial\phi}(h_1 u)\right]\end{aligned} \tag{15}$$

and again the Curl can be expressed as a determinant

$$\text{Curl}\,\underline{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\underline{a} & h_2\underline{b} & h_3\underline{c} \\ \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\lambda} \\ h_1 u & h_2 v & h_3 w \end{vmatrix} \tag{16}$$

In (13) to (16) the h_i are the usual "scale factors" such that if dl_θ is an element of distance along the θ-coordinate, dl_ϕ is an element of distance along the ϕ-coordinate, and dl_λ is an element of distance along the λ- coordinate then

$$dl_\theta = h_1\, d\theta \quad , \quad dl_\phi = h_2\, d\phi \quad , \quad dl_\lambda = h_3\, d\lambda \tag{17}$$

As a cautionary note, whilst the great majority of authors follows this convention there are a few whose h_i are the reciprocals of the above h_i.

The Idemfactor is now

$$I = \underline{a}\underline{a} + \underline{b}\underline{b} + \underline{c}\underline{c} \tag{18}$$

and by manipulations similar to those carried out from (8) to (12) it can be got into the same form as (12). From (13)

$$\nabla = \frac{\underline{a}}{h_1}\frac{\partial}{\partial\theta} + \frac{\underline{b}}{h_2}\frac{\partial}{\partial\phi} + \frac{\underline{c}}{h_3}\frac{\partial}{\partial\lambda} \tag{19}$$

and immediately we have

$$\nabla\theta = \frac{1}{h_1}\underline{a} \quad , \quad \nabla\phi = \frac{1}{h_2}\underline{b} \quad , \quad \nabla\lambda = \frac{1}{h_3}\underline{c} \tag{20}$$

From (6)

$$\frac{\partial \underline{r}}{\partial\theta} = \frac{\partial x}{\partial\theta}\underline{i} + \frac{\partial y}{\partial\theta}\underline{j} + \frac{\partial z}{\partial\theta}\underline{k} \tag{21}$$

But if α, β, γ are the angles made by the θ - direction (ie the $\underline{a}$ direction) with the cartesian axes then

$$\frac{\partial x}{\partial\theta} = h_1\cos\alpha \quad , \quad \frac{\partial y}{\partial\theta} = h_1\cos\beta \quad , \quad \frac{\partial z}{\partial\theta} = h_1\cos\gamma \tag{22}$$

Putting (22) into (21) yields

$$\frac{\partial \underline{r}}{\partial\theta} = h_1(\cos\alpha\,\underline{i} + \cos\beta\,\underline{j} + \cos\gamma\,\underline{k}) .$$

ie

$$\frac{\partial \underline{r}}{\partial\theta} = h_1\underline{a} \tag{23}$$

Clearly differentiating (6) with respect to ϕ and λ will yield corresponding results, so that altogether

$$\frac{\partial \underline{r}}{\partial\theta} = h_1\underline{a} \quad , \quad \frac{\partial \underline{r}}{\partial\phi} = h_2\underline{b} \quad , \quad \frac{\partial \underline{r}}{\partial\lambda} = h_3\underline{c} \tag{24}$$

Therefore, using (20) and (24) in (18) we have

$$I = \nabla\theta\frac{\partial \underline{r}}{\partial\theta} + \nabla\phi\frac{\partial \underline{r}}{\partial\phi} + \nabla\lambda\frac{\partial \underline{r}}{\partial\lambda} \tag{25}$$

The relationships (20), (24) and (25) are the key to the orthogonal curvilinear case. Before moving on to the oblique curvilinear case it is useful to emphasize certain features which link the orthogonal case to the oblique case.

First, if $\underline{l}$, $\underline{m}$, $\underline{n}$ and $\underline{p}$, $\underline{q}$, $\underline{t}$ are <u>any</u> vectors then

$$\underline{l}\,\underline{p} + \underline{m}\,\underline{q} + \underline{n}\,\underline{t} = I \tag{26}$$

is a necessary and sufficient condition for $\underline{l}$, $\underline{m}$, $\underline{n}$ and $\underline{p}$, $\underline{q}$, $\underline{t}$ to be reciprocal sets of vectors. This is true whether the vectors are orthogonal or not. Thus (25) itself establishes that $\nabla\theta, \nabla\phi, \nabla\lambda$ and $\frac{\partial \underline{r}}{\partial \theta}, \frac{\partial \underline{r}}{\partial \phi}, \frac{\partial \underline{r}}{\partial \lambda}$ are reciprocal sets. Of course this can be shown from (20) and (24) but (25) alone is enough.

Second, (20) and (24) show that $\nabla\theta, \nabla\phi, \nabla\lambda$ and $\frac{\partial \underline{r}}{\partial \theta}, \frac{\partial \underline{r}}{\partial \phi}, \frac{\partial \underline{r}}{\partial \lambda}$ are functions of the same set of unit vectors. This is peculiar to the orthogonal case.

Third, a point which is of little importance in the orthogonal case but which becomes valuable in the oblique case is that by back-tracking from (20) to (19) the Grad can be put in the form

$$\nabla = \nabla\theta \frac{\partial}{\partial\theta} + \nabla\phi \frac{\partial}{\partial\phi} + \nabla\lambda \frac{\partial}{\partial\lambda} \tag{27}$$

and the scale factors have been eliminated.

Fourth, although (13) has been presented as an expression for Grad $\underline{V}$, it can just as well be viewed as an expression for the dyadic $\nabla\underline{V}$. In this case it is valid, as a purely formal exercise in Gibbsian vector notation, to obtain expressions for Div $\underline{V}$ and Curl $\underline{V}$ by inserting the $\bullet$ and $\times$ symbols in the dyads on the RHS of (13). Thus for Div $\underline{V}$ we get

$$\nabla\cdot\underline{V} = \underline{a}\cdot\frac{1}{h_1}\frac{\partial\underline{V}}{\partial\theta} + \underline{b}\cdot\frac{1}{h_2}\frac{\partial\underline{V}}{\partial\phi} + \underline{c}\cdot\frac{1}{h_3}\frac{\partial\underline{V}}{\partial\lambda} \tag{28}$$

Different though they look (28) and (14) are exactly equivalent expressions. Similarly we have

$$\nabla\times\underline{V} = \underline{a}\times\frac{1}{h_1}\frac{\partial\underline{V}}{\partial\theta} + \underline{b}\times\frac{1}{h_2}\frac{\partial\underline{V}}{\partial\phi} + \underline{c}\times\frac{1}{h_3}\frac{\partial\underline{V}}{\partial\lambda} \tag{29}$$

and this is equivalent to (15) and (16).

3. Oblique Curvilinear Coordinates

We can use (27) to write Grad $\underline{V}$ (or the dyadic $\nabla\underline{V}$) as

$$\nabla\underline{V} = \nabla\theta\frac{\partial\underline{V}}{\partial\theta} + \nabla\phi\frac{\partial\underline{V}}{\partial\phi} + \nabla\lambda\frac{\partial\underline{V}}{\partial\lambda} \tag{30}$$

This can be done not simply because (27) has an obviously greater generality than the usual forms, but also because (30) can be derived without reference to any coordinate system. For if $\underline{n}$ is a unit vector specifying the direction of greatest increase of any given function and $\frac{\partial}{\partial n}$ represents differentiation along that arc then

$$\nabla \equiv \underline{n}\frac{\partial}{\partial n} \tag{31}$$

If we now take some quite arbitrary direction specified by a unit vector $\underline{e}$ and consider a change $d\underline{V}$ along an arc ds in this direction then

$$\frac{d\underline{V}}{ds} = \underline{e}\cdot\nabla\underline{V} \tag{32}$$

and either (31) or (32) can be taken as a coordinate - free definition for the symbol ∇. Then the chain rule for differentiation can be used to write the LHS of (32) as

$$\frac{d\underline{V}}{ds} = \frac{\partial \underline{V}}{\partial \theta}\frac{d\theta}{ds} + \frac{\partial \underline{V}}{\partial \phi}\frac{d\phi}{ds} + \frac{\partial \underline{V}}{\partial \lambda}\frac{d\lambda}{ds} \tag{33}$$

and the order of the factors may be reversed to give

$$\frac{d\underline{V}}{ds} = \frac{d\theta}{ds}\frac{\partial \underline{V}}{\partial \theta} + \frac{d\phi}{ds}\frac{\partial \underline{V}}{\partial \phi} + \frac{d\lambda}{ds}\frac{\partial \underline{V}}{\partial \lambda} \tag{34}$$

ie

$$\frac{d\underline{V}}{ds} = \underline{e}\cdot\nabla\theta\frac{\partial \underline{V}}{\partial \theta} + \underline{e}\cdot\nabla\phi\frac{\partial \underline{V}}{\partial \phi} + \underline{e}\cdot\nabla\lambda\frac{\partial \underline{V}}{\partial \lambda}$$

$$\frac{d\underline{V}}{ds} = \underline{e}\cdot\left[\nabla\theta\frac{\partial \underline{V}}{\partial \theta} + \nabla\phi\frac{\partial \underline{V}}{\partial \phi} + \nabla\lambda\frac{\partial \underline{V}}{\partial \lambda}\right]$$

so that from (32)

$$\underline{e}\cdot\nabla\underline{V} = \underline{e}\cdot\left[\nabla\theta\frac{\partial \underline{V}}{\partial \theta} + \nabla\phi\frac{\partial \underline{V}}{\partial \phi} + \nabla\lambda\frac{\partial \underline{V}}{\partial \lambda}\right]$$

but, as $\underline{e}$ is purely arbitrary, this implies that

$$\nabla\underline{V} = \nabla\theta\frac{\partial \underline{V}}{\partial \theta} + \nabla\phi\frac{\partial \underline{V}}{\partial \phi} + \nabla\lambda\frac{\partial \underline{V}}{\partial \lambda}$$

which is (30). Note that the derivation depends entirely on the coordinate-free (32). No use have been made of θ, ϕ, λ as coordinates. Thus (27) is the general form for ∇ and the more familiar orthogonal and rectilinear expressions are simply the special cases which arise from (27) as a result of acknowledging that θ, ϕ, λ are, in those cases, coordinates having special properties.

Similarly (27) can be applied to the position vector $\underline{r}$ to get

$$\nabla\underline{r} = \nabla\theta\frac{\partial \underline{r}}{\partial \theta} + \nabla\phi\frac{\partial \underline{r}}{\partial \phi} + \nabla\lambda\frac{\partial \underline{r}}{\partial \lambda} \tag{35}$$

Since ∇ does not depend on the coordinate system and since it has already been shown that $\nabla\underline{r} = \underline{I}$ in rectilinear and orthogonal curvilinear coordinates it follows that

$$\nabla\theta\frac{\partial \underline{r}}{\partial \theta} + \nabla\phi\frac{\partial \underline{r}}{\partial \phi} + \nabla\lambda\frac{\partial \underline{r}}{\partial \lambda} = \underline{I} \tag{36}$$

whatever the coordinates are. As before, this means that the antecedents and consequents of (36) are reciprocal sets. We therefore have the relationships

$$J \nabla\theta = \frac{\partial \underline{r}}{\partial \phi} \times \frac{\partial \underline{r}}{\partial \lambda} \quad , \quad J \nabla\phi = \frac{\partial \underline{r}}{\partial \lambda} \times \frac{\partial \underline{r}}{\partial \theta} \quad , \quad J \nabla\lambda = \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \tag{37}$$

$$\frac{\partial \underline{r}}{\partial \theta} = J \nabla\phi \times \nabla\lambda \quad , \quad \frac{\partial \underline{r}}{\partial \phi} = J \nabla\lambda \times \nabla\theta \quad , \quad \frac{\partial \underline{r}}{\partial \lambda} = J \nabla\theta \times \nabla\phi \tag{38}$$

where J is the Jacobian

$$J = \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \lambda} \tag{39}$$

$$\frac{1}{J} = \nabla\theta \times \nabla\phi \cdot \nabla\lambda \tag{40}$$

From (39) and (40) we have the obvious

$$\left[\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \lambda}\right]\left[\nabla\theta \times \nabla\phi \cdot \nabla\lambda\right] = 1 \tag{41}$$

Furthermore if the position vector $\underline{r}$, which marks a particular point in the coordinate system, is allowed small elemental increments $d\underline{r}_\theta$, $d\underline{r}_\phi$, $d\underline{r}_\lambda$ in the coordinate directions then

$$d\underline{r}_\theta = \frac{\partial \underline{r}}{\partial \theta} d\theta \quad , \quad d\underline{r}_\phi = \frac{\partial \underline{r}}{\partial \phi} d\phi \quad , \quad d\underline{r}_\lambda = \frac{\partial \underline{r}}{\partial \lambda} d\lambda \tag{42}$$

and so the volume $d\tau$ of the elemental parallelopiped defined by $d\underline{r}_\theta$, $d\underline{r}_\phi$, $d\underline{r}_\lambda$ is

$$d\tau = d\underline{r}_\theta \times d\underline{r}_\phi \cdot d\underline{r}_\lambda = \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \lambda} d\theta\, d\phi\, d\lambda = J\, d\theta\, d\phi\, d\lambda \tag{43}$$

Of course $d\tau = J\,d\theta\,d\phi\,d\lambda$ is a matter of elementary calculus, but the $d\tau = \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \lambda} d\theta\,d\phi\,d\lambda$ is important as it connects $d\tau$ with the vector quantities we are dealing with. The relationships (37) to (43) are vital for handling the oblique curvilinear case. Note that nothing in (37) to (43) implies the simple relationships (20) and (24) which hold in the orthogonal case. In order to get (20) and (24) the additional conditions, required for orthogonality

$$\frac{\partial \underline{r}}{\partial \theta} \cdot \frac{\partial \underline{r}}{\partial \phi} = \frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \lambda} = \frac{\partial \underline{r}}{\partial \lambda} \cdot \frac{\partial \underline{r}}{\partial \theta} = 0 \tag{44}$$

have to be used. It is not immediately obvious that the use of (44) reduces (37) and (38) to (20) and (24) and it is worth digressing a bit to see precisely why this happens. Taking the variable as typical we have from (37) and (38)

$$\nabla\theta = \frac{\dfrac{\partial \underline{r}}{\partial \phi} \times \dfrac{\partial \underline{r}}{\partial \lambda}}{J} = \left|\frac{\dfrac{\partial \underline{r}}{\partial \phi} \times \dfrac{\partial \underline{r}}{\partial \lambda}}{J}\right| \underline{a}' = |\nabla\theta|\,\underline{a}' = h_1'\,\underline{a}' \tag{45}$$

and

$$\frac{\partial \underline{r}}{\partial \theta} = \left|\frac{\partial \underline{r}}{\partial \theta}\right| \underline{a} = h_1 \underline{a} \tag{46}$$

and it is required to show that if (44) holds then $\underline{a}' = \underline{a}$ and $h_1' = \frac{1}{h_1}$ From (45) and (39)

$$(\nabla\theta)^2 = \frac{\left(\frac{\partial \underline{r}}{\partial \phi} \times \frac{\partial \underline{r}}{\partial \lambda}\right)^2}{\left(\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \lambda}\right)^2} \tag{47}$$

If $\underline{l}$, $\underline{m}$, and $\underline{p}$ are any vectors then we have the two vector identities

$$(\underline{l} \times \underline{m})^2 = l^2 m^2 - (\underline{l} \cdot \underline{m})^2 \tag{48}$$

and

$$\begin{aligned}(\underline{l} \times \underline{m} \cdot \underline{p})^2 = {} & l^2 m^2 p^2 - (\underline{p} \cdot \underline{l})^2 m^2 - (\underline{l} \cdot \underline{m})^2 p^2 - (\underline{m} \cdot \underline{p})^2 l^2 \\ & + 2(\underline{l} \cdot \underline{m})(\underline{m} \cdot \underline{p})(\underline{p} \cdot \underline{l})\end{aligned} \tag{49}$$

and thus using (48) on the numerator and (49) on the denominator of (47), we have on taking account of (44)

$$(\nabla\theta)^2 = \frac{\left(\frac{\partial \underline{r}}{\partial \phi}\right)^2 \left(\frac{\partial \underline{r}}{\partial \lambda}\right)^2}{\left(\frac{\partial \underline{r}}{\partial \theta}\right)^2 \left(\frac{\partial \underline{r}}{\partial \phi}\right)^2 \left(\frac{\partial \underline{r}}{\partial \lambda}\right)^2} = \frac{1}{\left(\frac{\partial \underline{r}}{\partial \theta}\right)^2} \tag{50}$$

which shows that $h_1' = \frac{1}{h_1}$

To show that $\underline{a}' = \underline{a}$ if (44) holds then all that is necessary is to show that $\nabla\theta \times \frac{\partial \underline{r}}{\partial \theta} = 0$. From (45) we have

$$\nabla\theta \times \frac{\partial \underline{r}}{\partial \theta} = \frac{1}{J}\left(\frac{\partial \underline{r}}{\partial \phi} \times \frac{\partial \underline{r}}{\partial \lambda}\right) \times \frac{\partial \underline{r}}{\partial \theta} \tag{51}$$

but by the standard formula for a triple vector product this is

$$\nabla\theta \times \frac{\partial \underline{r}}{\partial \theta} = \frac{1}{J}\left[\left(\frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \theta}\right)\frac{\partial \underline{r}}{\partial \lambda} - \left(\frac{\partial \underline{r}}{\partial \lambda} \cdot \frac{\partial \underline{r}}{\partial \theta}\right)\frac{\partial \underline{r}}{\partial \phi}\right] \tag{52}$$

and clearly this is zero if (44) holds. Although as far as the orthogonal case is concerned this analysis does no more than verify something which is geometrically obvious by visualization, yet the formalism serves to pinpoint the difference between the orthogonal and oblique cases.

However, for Grad, Div and Curl in the oblique curvilinear case we can simply take the expression (30) for $\nabla\underline{V}$ and insert $\bullet$ and $\times$ in the dyads getting

$$\nabla \underline{V} = \nabla\theta \frac{\partial \underline{V}}{\partial \theta} + \nabla\phi \frac{\partial \underline{V}}{\partial \phi} + \nabla\lambda \frac{\partial \underline{V}}{\partial \lambda} \tag{53}$$

$$\nabla \cdot \underline{V} = \nabla\theta \cdot \frac{\partial \underline{V}}{\partial \theta} + \nabla\phi \cdot \frac{\partial \underline{V}}{\partial \phi} + \nabla\lambda \cdot \frac{\partial \underline{V}}{\partial \lambda} \tag{54}$$

$$\nabla \times \underline{V} = \nabla\theta \times \frac{\partial \underline{V}}{\partial \theta} + \nabla\phi \times \frac{\partial \underline{V}}{\partial \phi} + \nabla\lambda \times \frac{\partial \underline{V}}{\partial \lambda} \tag{55}$$

Alternatively by using (37) they may be expressed as

$$\nabla \underline{V} = \frac{1}{J}\left[\left(\frac{\partial \underline{t}}{\partial \phi} \times \frac{\partial \underline{t}}{\partial \lambda}\right)\frac{\partial \underline{V}}{\partial \theta} + \left(\frac{\partial \underline{t}}{\partial \lambda} \times \frac{\partial \underline{t}}{\partial \theta}\right)\frac{\partial \underline{V}}{\partial \phi} + \left(\frac{\partial \underline{t}}{\partial \theta} \times \frac{\partial \underline{t}}{\partial \phi}\right)\frac{\partial \underline{V}}{\partial \lambda}\right] \tag{56}$$

$$\nabla \cdot \underline{V} = \frac{1}{J}\left[\left(\frac{\partial \underline{t}}{\partial \phi} \times \frac{\partial \underline{t}}{\partial \lambda}\right)\cdot\frac{\partial \underline{V}}{\partial \theta} + \left(\frac{\partial \underline{t}}{\partial \lambda} \times \frac{\partial \underline{t}}{\partial \theta}\right)\cdot\frac{\partial \underline{V}}{\partial \phi} + \left(\frac{\partial \underline{t}}{\partial \theta} \times \frac{\partial \underline{t}}{\partial \phi}\right)\cdot\frac{\partial \underline{V}}{\partial \lambda}\right] \tag{57}$$

$$\nabla \times \underline{V} = \frac{1}{J}\left[\left(\frac{\partial \underline{t}}{\partial \phi} \times \frac{\partial \underline{t}}{\partial \lambda}\right)\times\frac{\partial \underline{V}}{\partial \theta} + \left(\frac{\partial \underline{t}}{\partial \lambda} \times \frac{\partial \underline{t}}{\partial \theta}\right)\times\frac{\partial \underline{V}}{\partial \phi} + \left(\frac{\partial \underline{t}}{\partial \theta} \times \frac{\partial \underline{t}}{\partial \phi}\right)\times\frac{\partial \underline{V}}{\partial \lambda}\right] \tag{58}$$

The vector formalism of (53) to (58) opens up various manipulative possibilities. For example, by using the triple vector product identity on each term in (58) it can be put in the form

$$\begin{aligned}\nabla \times \underline{V} = \frac{1}{J}\Bigg\{&\left(\frac{\partial \underline{t}}{\partial \phi}\cdot\frac{\partial \underline{V}}{\partial \theta}\right)\frac{\partial \underline{t}}{\partial \lambda} - \left(\frac{\partial \underline{t}}{\partial \lambda}\cdot\frac{\partial \underline{V}}{\partial \theta}\right)\frac{\partial \underline{t}}{\partial \phi} \\ &+ \left(\frac{\partial \underline{t}}{\partial \lambda}\cdot\frac{\partial \underline{V}}{\partial \phi}\right)\frac{\partial \underline{t}}{\partial \theta} - \left(\frac{\partial \underline{t}}{\partial \theta}\cdot\frac{\partial \underline{V}}{\partial \phi}\right)\frac{\partial \underline{t}}{\partial \lambda} \\ &+ \left(\frac{\partial \underline{t}}{\partial \theta}\cdot\frac{\partial \underline{V}}{\partial \lambda}\right)\frac{\partial \underline{t}}{\partial \phi} - \left(\frac{\partial \underline{t}}{\partial \phi}\cdot\frac{\partial \underline{V}}{\partial \lambda}\right)\frac{\partial \underline{t}}{\partial \theta}\Bigg\}\end{aligned} \tag{59}$$

and this can be expressed as the determinant

$$\text{Curl } \underline{V} = \frac{1}{J}\begin{vmatrix} \frac{\partial \underline{t}}{\partial \theta} & \frac{\partial \underline{t}}{\partial \phi} & \frac{\partial \underline{t}}{\partial \lambda} \\ \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \lambda} \\ \frac{\partial \underline{t}}{\partial \theta}\cdot\underline{V} & \frac{\partial \underline{t}}{\partial \phi}\cdot\underline{V} & \frac{\partial \underline{t}}{\partial \lambda}\cdot\underline{V} \end{vmatrix} \tag{60}$$

which clearly reduces to (16) and (5) in the appropriate circumstances. However in (60)

$$\frac{\partial \underline{r}}{\partial \theta} \cdot \underline{V} = \left| \frac{\partial \underline{r}}{\partial \theta} \right| \left[u + (\underline{a} \cdot \underline{b}) v + (\underline{a} \cdot \underline{c}) w \right]$$

and similarly for the $\frac{\partial \underline{r}}{\partial \phi} \cdot \underline{V}$ and $\frac{\partial \underline{r}}{\partial \lambda} \cdot \underline{V}$ terms, and so the oblique case is quite complicated.

4. A mixed Oblique and Orthogonal System

The system considered here is one where θ and ϕ are curvilinear coordinates in a plane and z is a rectilinear coordinate perpendicular to the plane. Without repeating the argument the essential algebra can be set down as

$$\nabla = \nabla\theta \frac{\partial}{\partial \theta} + \nabla\phi \frac{\partial}{\partial \phi} + \underline{k} \frac{\partial}{\partial z} \tag{61}$$

$$\nabla\theta \frac{\partial \underline{r}}{\partial \theta} + \nabla\phi \frac{\partial \underline{r}}{\partial \phi} + \underline{k}\,\underline{k} = \mathrm{I} \tag{62}$$

$$J \nabla\theta = \frac{\partial \underline{r}}{\partial \phi} \times \underline{k} \quad , \quad J \nabla\phi = \underline{k} \times \frac{\partial \underline{r}}{\partial \theta} \quad , \quad J \underline{k} = \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \tag{63}$$

$$\frac{\partial \underline{r}}{\partial \theta} = J \nabla\phi \times \underline{k} \quad , \quad \frac{\partial \underline{r}}{\partial \phi} = J \underline{k} \times \nabla\theta \quad , \quad \underline{k} = J \nabla\theta \times \nabla\phi \tag{64}$$

$$J = \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \cdot \underline{k} \tag{65}$$

$$\frac{1}{J} = \nabla\theta \times \nabla\phi \cdot \underline{k} \tag{66}$$

$$d\tau = \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \cdot \underline{k} \, d\theta \, d\phi \, dz \tag{67}$$

$$\frac{\partial \underline{r}}{\partial \theta} \cdot \frac{\partial \underline{r}}{\partial \phi} \neq 0 \quad , \quad \frac{\partial \underline{r}}{\partial \phi} \cdot \underline{k} = \underline{k} \cdot \frac{\partial \underline{r}}{\partial \theta} = 0 \tag{68}$$

$$(\nabla\theta)^2 = \frac{\left(\frac{\partial \underline{r}}{\partial \phi}\right)^2}{\left(\frac{\partial \underline{r}}{\partial \theta}\right)^2 \left(\frac{\partial \underline{r}}{\partial \phi}\right)^2 - \left(\frac{\partial \underline{r}}{\partial \theta} \cdot \frac{\partial \underline{r}}{\partial \phi}\right)^2} \tag{69}$$

$$\nabla\theta \times \frac{\partial \underline{r}}{\partial \theta} = \frac{1}{J} \left[\left(\frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \theta} \right) \underline{k} \right] \tag{70}$$

$$(\nabla\phi)^2 = \frac{\left(\frac{\partial \underline{t}}{\partial\theta}\right)^2}{\left(\frac{\partial \underline{t}}{\partial\theta}\right)^2\left(\frac{\partial \underline{t}}{\partial\phi}\right)^2 - \left(\frac{\partial \underline{t}}{\partial\theta}\cdot\frac{\partial \underline{t}}{\partial\phi}\right)^2} \tag{71}$$

(69), (70) and (71) show that although this mixed system is simpler than the general oblique case it is nothing like as simple as the completely orthogonal case.

$$\nabla\underline{V} = \nabla\theta\frac{\partial\underline{V}}{\partial\theta} + \nabla\phi\frac{\partial\underline{V}}{\partial\phi} + \underline{k}\frac{\partial\underline{V}}{\partial z} \tag{72}$$

$$\nabla\cdot\underline{V} = \nabla\theta\cdot\frac{\partial\underline{V}}{\partial\theta} + \nabla\phi\cdot\frac{\partial\underline{V}}{\partial\phi} + \underline{k}\cdot\frac{\partial\underline{V}}{\partial z} \tag{73}$$

$$\nabla\times\underline{V} = \nabla\theta\times\frac{\partial\underline{V}}{\partial\theta} + \nabla\phi\times\frac{\partial\underline{V}}{\partial\phi} + \underline{k}\times\frac{\partial\underline{V}}{\partial z} \tag{74}$$

and, alternatively

$$\nabla\underline{V} = \frac{1}{J}\left[\left(\frac{\partial \underline{t}}{\partial\phi}\times\underline{k}\right)\frac{\partial\underline{V}}{\partial\theta} + \left(\underline{k}\times\frac{\partial \underline{t}}{\partial\theta}\right)\frac{\partial\underline{V}}{\partial\phi} + \left(\frac{\partial \underline{t}}{\partial\theta}\times\frac{\partial \underline{t}}{\partial\phi}\right)\frac{\partial\underline{V}}{\partial z}\right] \tag{75}$$

$$\nabla\cdot\underline{V} = \frac{1}{J}\left[\left(\frac{\partial \underline{t}}{\partial\phi}\times\underline{k}\right)\cdot\frac{\partial\underline{V}}{\partial\theta} + \left(\underline{k}\times\frac{\partial \underline{t}}{\partial\theta}\right)\cdot\frac{\partial\underline{V}}{\partial\phi} + \left(\frac{\partial \underline{t}}{\partial\theta}\times\frac{\partial \underline{t}}{\partial\phi}\right)\cdot\frac{\partial\underline{V}}{\partial z}\right] \tag{76}$$

$$\nabla\times\underline{V} = \frac{1}{J}\left[\left(\frac{\partial \underline{t}}{\partial\phi}\times\underline{k}\right)\times\frac{\partial\underline{V}}{\partial\theta} + \left(\underline{k}\times\frac{\partial \underline{t}}{\partial\theta}\right)\times\frac{\partial\underline{V}}{\partial\phi} + \left(\frac{\partial \underline{t}}{\partial\theta}\times\frac{\partial \underline{t}}{\partial\phi}\right)\times\frac{\partial\underline{V}}{\partial z}\right] \tag{77}$$

and (77) may be expressed as

$$\begin{aligned}\nabla\times\underline{V} = \frac{1}{J}\Bigg[&\left(\frac{\partial \underline{t}}{\partial\phi}\cdot\frac{\partial\underline{V}}{\partial\theta}\right)\underline{k} - \left(\underline{k}\cdot\frac{\partial\underline{V}}{\partial\theta}\right)\frac{\partial \underline{t}}{\partial\phi}\\ &+\left(\underline{k}\cdot\frac{\partial\underline{V}}{\partial\phi}\right)\frac{\partial \underline{t}}{\partial\theta} - \left(\frac{\partial \underline{t}}{\partial\theta}\cdot\frac{\partial \underline{t}}{\partial\phi}\right)\underline{k}\\ &+\left(\frac{\partial \underline{t}}{\partial\theta}\cdot\frac{\partial\underline{V}}{\partial z}\right)\frac{\partial \underline{t}}{\partial\phi} - \left(\frac{\partial \underline{t}}{\partial\phi}\cdot\frac{\partial\underline{V}}{\partial z}\right)\frac{\partial \underline{t}}{\partial\theta}\Bigg]\end{aligned} \tag{78}$$

ie

$$\nabla\times\underline{V}=\frac{1}{J}\begin{vmatrix}\dfrac{\partial\underline{r}}{\partial\theta} & \dfrac{\partial\underline{r}}{\partial\phi} & \underline{k}\\[2ex] \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z}\\[2ex] \dfrac{\partial\underline{r}}{\partial\theta}\cdot\underline{V} & \dfrac{\partial\underline{r}}{\partial\phi}\cdot\underline{V} & \underline{k}\cdot\underline{V}\end{vmatrix} \qquad (79)$$

One use for these curvilinear expressions for the invariants is that they enable the equations of motion to be transcribed into some unusual coordinate system. The 3-d equation of motion is

$$\frac{d\underline{V}}{dt}+2\underline{\Omega}\times\underline{V}=-\alpha\nabla p-g\underline{k} \qquad (80)$$

and this can be expressed as

$$\frac{\partial\underline{V}}{\partial t}+\underline{V}\cdot\nabla\underline{V}+2\underline{\Omega}\times\underline{V}=-\alpha\nabla p-g\underline{k} \qquad (81)$$

in which case (56) can be used, or as

$$\frac{\partial\underline{V}}{\partial t}+\frac{1}{2}\nabla V^2-\underline{V}\times\mathrm{Curl}\,\underline{V}+2\underline{\Omega}\times\underline{V}=-\alpha\nabla p-g\underline{k} \qquad (82)$$

and now (56) and (60) can be used. If, say, (60) is used in (82) it may be desirable to use (24) to put the Grad and Curl in scale factor form again, eg

$$\mathrm{Curl}\,\underline{V}=\frac{1}{h_1 h_2 h_3\,\underline{a}\times\underline{b}\cdot\underline{c}}\begin{vmatrix}h_1\underline{a} & h_2\underline{b} & h_3\underline{c}\\[2ex] \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial\lambda}\\[2ex] h_1\underline{a}\cdot\underline{V} & h_2\underline{b}\cdot\underline{V} & h_3\underline{c}\cdot\underline{V}\end{vmatrix} \qquad (83)$$

This is quite legitmate, but is must be remembered that the unit vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ are not orthogonal and that therefore $\underline{a}\times\underline{b}\cdot\underline{c}\neq 1$, $\underline{a}\cdot\underline{V}\neq u$, $\underline{b}\cdot\underline{V}\neq v$ and $\underline{c}\cdot\underline{V}\neq w$.
Similarly,

$$\nabla=\frac{1}{h_1h_2h_3\,\underline{a}\times\underline{b}\cdot\underline{c}}\left[h_2h_3(\underline{b}\times\underline{c})\frac{\partial}{\partial\theta}+h_3h_1(\underline{c}\times\underline{a})\frac{\partial}{\partial\phi}+h_1h_2(\underline{a}\times\underline{b})\frac{\partial}{\partial\lambda}\right] \qquad (84)$$

observing the caution that $\underline{b}\times\underline{c}\neq\underline{a}$ etc.

Up to the present meteorologists have not been greatly concerned with the 3-d equations and a problem more likely to arise is that of transcribing into oblique curvilinear coordinates a set of equations such as

$$\frac{\partial \underline{V}_H}{\partial t} + \frac{1}{2}\nabla_H V_H^2 + \zeta \underline{k} \times \underline{V}_H + \omega \frac{\partial \underline{V}_H}{\partial p} + f \underline{k} \times \underline{V}_H = -g \nabla_H h \tag{85}$$

$$\nabla_H \cdot \underline{V}_H + \frac{\partial \omega}{\partial p} = 0 \tag{86}$$

$$\frac{\partial h'}{\partial t} + \underline{V}_H \cdot \nabla_H h' + \omega \beta = 0 \tag{87}$$

in customary notation. We now use

$$\nabla_H V_H^2 = \frac{1}{J}\left[\left(\frac{\partial \underline{t}}{\partial \phi} \times \underline{k}\right)\frac{\partial V_H^2}{\partial \theta} + \left(\underline{k} \times \frac{\partial \underline{t}}{\partial \theta}\right)\frac{\partial V_H^2}{\partial \phi}\right] \tag{88}$$

$$\nabla_H \cdot \underline{V}_H = \frac{1}{J}\left[\left(\frac{\partial \underline{t}}{\partial \phi} \times \underline{k}\right)\cdot\frac{\partial \underline{V}_H}{\partial \theta} + \left(\underline{k} \times \frac{\partial \underline{t}}{\partial \theta}\right)\cdot\frac{\partial \underline{V}_H}{\partial \phi}\right] \tag{89}$$

$$\zeta = \underline{k} \cdot \nabla_H \times \underline{V}_H = \frac{1}{J}\left[\left(\frac{\partial \underline{t}}{\partial \phi}\cdot\frac{\partial \underline{V}_H}{\partial \theta}\right) - \left(\frac{\partial \underline{t}}{\partial \theta}\cdot\frac{\partial \underline{V}_H}{\partial \phi}\right)\right] \tag{90}$$

from (75), (76) and (78). J is given by (65). In terms of unit vectors and scale factors (88) to (90) become

$$\nabla_H V_H^2 = \frac{1}{h_1 h_2 \sin\alpha}\left[h_2(\underline{b} \times \underline{k})\frac{\partial V_H^2}{\partial \theta} + h_1(\underline{k} \times \underline{a})\frac{\partial V_H^2}{\partial \phi}\right] \tag{91}$$

$$\nabla_H \cdot \underline{V}_H = \frac{1}{h_1 h_2 \sin\alpha}\left[h_2(\underline{b} \times \underline{k})\cdot\frac{\partial \underline{V}_H}{\partial \theta} + h_1(\underline{k} \times \underline{a})\cdot\frac{\partial \underline{V}_H}{\partial \phi}\right] \tag{92}$$

$$\zeta = \frac{1}{h_1 h_2 \sin\alpha}\left[h_2\, \underline{b}\cdot\frac{\partial \underline{V}_H}{\partial \theta} - h_1\, \underline{a}\cdot\frac{\partial \underline{V}_H}{\partial \phi}\right] \tag{93}$$

where α is the angle between $\underline{a}$ and $\underline{b}$.

In this system

$$\underline{V}_H = u\underline{a} + v\underline{b} \tag{94}$$

and therefore

$$V_H^2 = \underline{V}_H \cdot \underline{V}_H = (u\underline{a} + v\underline{b})\cdot(u\underline{a} + v\underline{b}) = u^2 + v^2 + 2uv\,\underline{a}\cdot\underline{b}$$

ie

$$V_H^2 = u^2 + v^2 + 2uv\cos\alpha \tag{95}$$

Also

$$\frac{\partial \underline{V}_H}{\partial \theta} = \frac{\partial}{\partial \theta}(u\underline{a} + v\underline{b}) = \frac{\partial u}{\partial \theta}\underline{a} + u\frac{\partial \underline{a}}{\partial \theta} + \frac{\partial v}{\partial \theta}\underline{b} + v\frac{\partial \underline{b}}{\partial \theta} \tag{96}$$

and likewise

$$\frac{\partial \underline{V}_H}{\partial \phi} = \frac{\partial u}{\partial \phi}\underline{a} + u\frac{\partial \underline{a}}{\partial \phi} + \frac{\partial v}{\partial \phi}\underline{b} + v\frac{\partial \underline{b}}{\partial \phi} \tag{97}$$

Now if the θ and ϕ lines are curved as well as being oblique then the unit vectors $\underline{a}$ and $\underline{b}$ change direction from point to point along the coordinate lines and so $\frac{\partial \underline{a}}{\partial \theta}, \frac{\partial \underline{a}}{\partial \phi}, \frac{\partial \underline{b}}{\partial \theta}, \frac{\partial \underline{b}}{\partial \phi}$ are not zero. Thus (91) to (93) are still quite complicated. However, in the case where the coordinate lines are oblique but straight then $\frac{\partial \underline{a}}{\partial \theta}, \frac{\partial \underline{a}}{\partial \phi}, \frac{\partial \underline{b}}{\partial \theta}, \frac{\partial \underline{b}}{\partial \phi}$ are zero and we have

$$\frac{\partial \underline{V}_H}{\partial \theta} = \frac{\partial u}{\partial \theta}\underline{a} + \frac{\partial v}{\partial \theta}\underline{b} \tag{98}$$

$$\frac{\partial \underline{V}_H}{\partial \phi} = \frac{\partial u}{\partial \phi}\underline{a} + \frac{\partial v}{\partial \phi}\underline{b} \tag{99}$$

and then taking into account that

$$\underline{b} = \cos\alpha\,\underline{a} + \sin\alpha(\underline{k} \times \underline{a}) \tag{100}$$

$$\underline{a} = \cos\alpha\,\underline{b} + \sin\alpha(\underline{b} \times \underline{k}) \tag{101}$$

it is found that (91), (92) and (93) can be written as

$$\nabla_H V_H^2 = \frac{1}{\sin^2\alpha}\left[(\underline{a} - \cos\alpha\,\underline{b})\frac{1}{h_1}\frac{\partial V_H^2}{\partial \theta} + (\underline{b} - \cos\alpha\,\underline{a})\frac{1}{h_2}\frac{\partial V_H^2}{\partial \phi}\right] \tag{102}$$

$$\nabla_H \cdot \underline{V}_H = \frac{1}{h_1}\frac{\partial u}{\partial \theta} + \frac{1}{h_2}\frac{\partial v}{\partial \phi} \tag{103}$$

$$\zeta = \frac{1}{\sin\alpha}\left[\left(\frac{1}{h_1}\frac{\partial v}{\partial \theta} - \frac{1}{h_2}\frac{\partial u}{\partial \phi}\right) + \cos\alpha\left(\frac{1}{h_1}\frac{\partial u}{\partial \theta} - \frac{1}{h_2}\frac{\partial v}{\partial \phi}\right)\right] \tag{104}$$

Equations (95), (102, (103), (104) provide all the information needed to transform (85), (86) and (87).

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