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CAT AND THE RICHARDSON NUMBER

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1. Introduction

Sections 5, 6, 7, and 8 of a recent voluminous report by W R Sparks, S G Crawford and J K Gibson entitled "A report of the 1972 Clear Air Turbulence Campaign" comprise a clear and readable account of the state of the art in a difficult problem. The theoretical attack has largely consisted of attempts to link CAT with indices concocted from Ri and/or its rate of decrease. Intuitive physical reasoning provides a very strong motivation for this line and the authors comment that the relative failure of such indices is probably due more to resolution difficulties than any genuine lack of validity in the approach.

The Richardson number Ri is easily applied to such matters as scale analysis a la Haltiner, chapter three, but it is an ungainly creature to handle in more involved derivations. Such derivations can rapidly become quite complicated and difficult to interpret. This leads in turn to doubts and uncertainties when experimental evidence gives contrary indications.

The purpose of this note is to place on record that proceeding via the vertical vector wind-shear equation and introducing a shear stability ratio vector affords a somewhat smoother theoretical approach to the problem.

Note: This paper has not been published. Permission to quote from it must be obtained from the Assistant Director of the above Meteorological Office branch.

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2. Notation

f, g, h, p, ω, θ, T, ζ, ρ, and R have their usual meanings.
$\underline{k}$ is the unit vertical vector
$\underline{V}$ is the horizontal wind vector $\underline{V} = u\underline{i} + v\underline{j}$
∇ is the 2-d grad operator
I is the 2-d Idemfactor $\underline{ii} + \underline{jj}$

$\underline{V}$ is the dyadic $$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}$$

F is the dyadic $$\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

where A and B are the familiar "components" of deformation
x indicates the vector product
• indicates the scalar product
Ri is the Richardson number Ri = $-\frac{1}{\rho\theta}\frac{\partial\theta}{\partial p}\Big/\left(\frac{\partial \underline{V}}{\partial p}\right)^2$
J is J(u,v) the customary Jacobian.

3. The vertical vector wind shear equation

In isobaric co-ordinates the equation of motion is

$$\frac{d\underline{V}}{dt} + f\underline{k}\times\underline{V} = -g\nabla h \qquad (1)$$

Differentiating (1) with respect to p gives

$$\frac{\partial}{\partial p}\frac{d\underline{V}}{dt} + f\underline{k}\times\frac{\partial \underline{V}}{\partial p} = \frac{R}{p}\nabla T \qquad (2)$$

Reversing the order of $\frac{\partial}{\partial p}\frac{d\underline{V}}{dt}$ gives

$$\frac{d}{dt}\frac{\partial \underline{V}}{\partial p} + \frac{\partial \underline{V}}{\partial p}\cdot\nabla\underline{V} + \frac{\partial\omega}{\partial p}\frac{\partial \underline{V}}{\partial p} + f\underline{k}\times\frac{\partial \underline{V}}{\partial p} = \frac{R}{p}\nabla T \qquad (3)$$

Using the vector-dyadic identity

$$\underline{a}\cdot\nabla\underline{b} \equiv \nabla\underline{b}\cdot\underline{a} - \underline{a}\times \mathrm{Curl}\,\underline{b} \qquad (4)$$

and the continuity equation

$$D = \mathrm{div}_p\,\underline{V} = -\frac{\partial\omega}{\partial p} \qquad (5)$$

transforms (3) to

$$\frac{d}{dt}\frac{\partial \underline{V}}{\partial p}\cancel{\frac{d\underline{V}}{dt}} + \nabla\underline{V}\cdot\frac{\partial \underline{V}}{\partial p} - D\frac{\partial \underline{V}}{\partial p} + (\zeta+f)\underline{k}\times\frac{\partial \underline{V}}{\partial p} = \frac{R}{p}\nabla T \qquad (6)$$

Using the identity

$$(\zeta+f)\underline{k}\times\frac{\partial \underline{V}}{\partial p} \equiv (\zeta+f)(I\times\underline{k})\cdot\frac{\partial \underline{V}}{\partial p} \qquad (7)$$

we can now isolate $\frac{\partial \underline{V}}{\partial p}$ by writing (6) as

$$\frac{d}{dt}\frac{\partial \underline{V}}{\partial p}\cancel{\frac{d\underline{V}}{dt}} + \left[\nabla\underline{V} - DI + (\zeta+f)(I\times\underline{k})\right]\cdot\frac{\partial \underline{V}}{\partial p} = \frac{R}{p}\nabla T \qquad (8)$$

Then, using the dyadic identity

$$\nabla \underline{V} = \frac{1}{2}\left(F + D I - \zeta I \times \underline{k}\right) \tag{9}$$

equation (8) is brought to the form

$$\boxed{\frac{d}{dt}\frac{\partial V}{\partial p} + \frac{1}{2} C \cdot \frac{\partial V}{\partial p} = \frac{R}{p}\nabla T} \tag{10}$$

where C is the dyadic

$$C = \begin{bmatrix} (A - D) & B - (\zeta + 2f) \\ B + (\zeta + 2f) & -(A + D) \end{bmatrix} \tag{11}$$

Equation (10) is the dynamical vertical wind-shear vector equation. It is as exact as equation (1). No assumptions or approximations other than those implicit in (1) have been introduced.

4. <u>The inverse Richardson vector equation</u>

Define a stability index

$$\sigma = -\frac{1}{\rho\theta}\frac{\partial \theta}{\partial p} \tag{12}$$

and a vector $\underline{s}$ which we will call the inverse Richardson vector

$$\underline{s} = \frac{1}{\sqrt{\sigma}}\frac{\partial V}{\partial p} \tag{13}$$

the point being that the magnitude of $\underline{s}$ is the negative of the inverse of the Richardson number Ri, i.e.

$$\frac{1}{Ri} = +s^2 = +\underline{s} \cdot \underline{s} \tag{14}$$

Multiply through (10) by $\frac{1}{\sqrt{\sigma}}$, use the formula for the differentiation of a product and obtain

$$\frac{ds}{dt} - \frac{\partial V}{\partial p}\frac{d\sigma^{-\frac{1}{2}}}{dt} + \frac{1}{2} C \cdot \underline{s} = \frac{R}{p\sqrt{\sigma}}\nabla T$$

i.e.

$$\frac{d\underline{s}}{dt} - \frac{\partial V}{\partial p}\left[-\frac{1}{2}\sigma^{-\frac{3}{2}}\frac{d\sigma}{dt}\right] + \frac{1}{2} C \cdot \underline{s} = \frac{R}{p\sqrt{\sigma}}\nabla T$$

i.e.

$$\frac{d\underline{s}}{dt} + \frac{1}{2}\underline{s}\frac{1}{\sigma}\frac{d\sigma}{dt} + \frac{1}{2} C \cdot \underline{s} = \frac{R}{p\sqrt{\sigma}}\nabla T$$

and finally

$$\boxed{\frac{d\underline{s}}{dt} + \frac{1}{2}\left(\frac{d}{dt}\ln\sigma\right)\underline{s} + \frac{1}{2} C \cdot \underline{s} = \frac{R}{p\sqrt{\sigma}}\nabla T} \tag{15}$$

and this is the sought for dynamical equation. Like (10) it is exact. The equation for the rate of change of the inverse of the Richardson number itself is obtained simply by taking the scalar product of $\underline{s}$ through (15) to get

$$\underline{s} \cdot \frac{d\underline{s}}{dt} + \frac{1}{2}\left(\frac{d}{dt}\ln\sigma\right)\underline{s} \cdot \underline{s} + \frac{1}{2}\underline{s} \cdot C \cdot \underline{s} = \frac{R}{p\sqrt{\sigma}}\underline{s} \cdot \nabla T$$

i.e.

$$\boxed{\frac{ds^2}{dt} + s^2\frac{d\ln\sigma}{dt} + \underline{s} \cdot C \cdot \underline{s} = \frac{2R}{p\sqrt{\sigma}}\underline{s} \cdot \nabla T} \tag{16}$$

Apart from a factor the vector $\underline{s}$ is the same as introduced by Spilhaus many years ago.

5. Comments

Notice that equation (1) itself can be written so that it is formally very similar to (10). We have from (1)

$$\cancel{\frac{dV}{dt} + G\cdot V = -g\nabla h} \quad \frac{d\underline{V}}{dt} + f(I\times \underline{k})\cdot\underline{V} = -g\nabla h \qquad (17)$$

and by defining the dyadic G as

$$G = f(I\times\underline{k}) = \begin{pmatrix} 0 & -f \\ f & 0 \end{pmatrix} \qquad (18)$$

(17) becomes

$$\frac{d\underline{V}}{dt} + G\cdot\underline{V} = -g\nabla h \qquad (19)$$

and the formal similarity between (19) and (10) is evident. This similarity is not surprising since all we have done to (1) to get (10) is to differentiate it. Now regard (19) as an equation for $\underline{V}$ and write it as

$$\underline{V} = G^{-1}\cdot\left[-g\nabla h - \frac{d\underline{V}}{dt}\right] \qquad (20)$$

Likewise, regard (10) as an equation for $\frac{\partial \underline{V}}{\partial p}$ and write it as

$$\frac{\partial V}{\partial p} = C^{-1}\cdot\left[\frac{2R}{p}\nabla T - 2\frac{d}{dt}\frac{\partial \underline{V}}{\partial p}\right] \qquad (21)$$

We have

$$G^{-1} = \frac{\mathrm{Adj}\, G}{\mathrm{Det}\, G} \qquad (22)$$

$$C^{-1} = \frac{\mathrm{Adj}\, C}{\mathrm{Det}\, C} \qquad (23)$$

and now a crucial difference between the two equations emerges for

$$\mathrm{Det}\, G = f^2 \qquad (24)$$

and the inverse G^{-1} thus always exists away from the equator, but

$$\mathrm{Det}\, C = \left[D^2 + (\zeta + 2f)^2 - (A^2 + B^2)\right] \qquad (25)$$

and clearly this may be zero. The inverse C^{-1} does not then exist, the flow has reached a singular condition and the situation is resolved by a turbulent exchange of momentum between layers.

In practice it will not be a matter of Det C being zero but of it falling to less than some small value ε, which in most circumstances will result, from (23) and (21), in an unduly large value for $\frac{\partial V}{\partial p}$. In a strong flow zone it is likely that $D^2 + (\zeta + 2f)^2$ will dominate $A^2 + B^2$ in (25) unless ζ goes significantly negative. This supports the findings of Sparks and Co. It is probable that (25) is the true theoretical form of the index $(0.3\,\zeta_A^2 + A^2 + B^2)$ which has turned up in the literature. Again, if the term $D^2 + (\zeta + 2f)^2$ usually dominates, the value of $A^2 + B^2$ will be in the main irrelevant, as the report finds. Furthermore it is apparent from (25), (23) and (21) that if Det C is used as an index then in most circumstances Det C and $\frac{\partial V}{\partial p}$ are virtually equivalent pieces of information. If Det C is very small then it is highly probable that $\frac{\partial V}{\partial p}$ is very large and vice versa. And since $\frac{\partial V}{\partial p}$ will tend to have large values in strong wind zones it is probably this that accounts for the observation that $\underline{V}$ and $\frac{\partial V}{\partial p}$ seem to perform as well as other indices. For completeness it may be worth mentioning that by using the 2-d Hamel identity

$$D^2 + \zeta^2 - 4J - A^2 - B^2 \equiv 0 \qquad (26)$$

(25) may be put in the form

$$\mathrm{Det}\, C = 4(\zeta f + f^2 + J) \qquad (27)$$

The matter can obviously be pursued further, but it is probably best for the foregoing theory to be considered at this stage by those who are in touch with the physical problem and evidence.

6. Conclusions

If we may take it as a matter of faith that the resolution of detail in our observed and forecast fields will steadily improve then theoretically derived indices for CAT may in the end prove more valuable than empirical relationships. This note suggests that equations (10), (15) and (16) are an appropriate starting point for the necessary theoretical and practical investigations.

It seems likely to me that equations (10) and (15) are but 2-d sections of a more general 3-d relationship which should be sought. This 3-d relationship would account for turbulence arising from horizontal as well as vertical shear and it is probably a purely dyadic equation.