Forecasting Research

Met O 11 Technical Note No. 17

Conservative finite difference schemes for a unified forecast/ climate model

by

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July 1988

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CONSERVATIVE
FINITE DIFFERENCE SCHEMES
FOR A
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July 13 1988

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1. INTRODUCTION

This note describes a finite difference formulation which appears to be the most appropriate for the unified forecast/climate model. It seeks to combine the advantages of the present operational scheme in accuracy and efficiency with the conservation properties required for long-term climate integrations. Though the scheme is based on the results of an extended period of research, the effect of changing the integration scheme in a high resolution forecast is very small, and conclusive experiments are rare. The scheme set out represents a modification of the present operational split-explicit scheme on the Arakawa 'B' grid, Bell and Dickinson (1987). Satisfactory results were not obtained with this scheme on a 'C' grid, because of stability problems. Insufficient work has been done in the Office on semi-implicit or spectral methods to justify a change to using them, and results produced by other centres which use such methods do not appear to be better than those of the present operational models when the resolution is similar. Efficient time integration schemes have not yet been applied successfully to a full forecast model on an unstaggered grid, though such grids are more convenient for programming, and for ensuring conservation.

The proposed scheme is to use the Heun time-step for advection, rather than the Lax-Wendroff. This avoids the complexity of time staggering, and is slightly more stable in practice. The scheme is set out in hybrid vertical coordinates, as used at ECMWF, since there is a clear requirement for the numerical models to extend further into the stratosphere. Mass-weighted linear quantities are conserved, and the mass-weighted second moments of advected quantities are also conserved under non-divergent advection.

2. THE FORECAST EQUATIONS

Define a vertical coordinate $\eta = h(p, p_*)$, where $h(0, p_*)=0$ and $h(p_*, p_*)=1$.

$$\frac{\partial u}{\partial t} + \frac{u}{a \cos\varphi}\frac{\partial u}{\partial \lambda} + \frac{v}{a}\frac{\partial u}{\partial \varphi} + \dot{\eta}\frac{\partial u}{\partial \eta} + \frac{1}{a\cos\varphi}\left(\frac{\partial \Phi}{\partial \lambda} + \frac{RT}{p}\frac{\partial p}{\partial \lambda}\right)$$

$$- \left(f + \frac{u \tan\varphi}{a}\right)v = F_u. \qquad (1)$$

$$\frac{\partial v}{\partial t} + \frac{u}{a \cos\varphi}\frac{\partial v}{\partial \lambda} + \frac{v}{a}\frac{\partial v}{\partial \varphi} + \dot{\eta}\frac{\partial v}{\partial \eta} + \frac{1}{a}\left(\frac{\partial \Phi}{\partial \varphi} + \frac{RT}{p}\frac{\partial p}{\partial \varphi}\right)$$

$$+ (f + u \tan\varphi)u = F_v. \qquad (2)$$

$$\frac{\partial \theta}{\partial t} + \frac{u}{a \cos\varphi}\frac{\partial \theta}{\partial \lambda} + \frac{v}{a}\frac{\partial \theta}{\partial \varphi} + \dot{\eta}\frac{\partial \theta}{\partial \eta} = F_\theta. \qquad (3)$$

$$\frac{\partial q}{\partial t} + \frac{u}{a \cos\varphi}\frac{\partial q}{\partial \lambda} + \frac{v}{a}\frac{\partial q}{\partial \varphi} + \dot{\eta}\frac{\partial q}{\partial \eta} = F_q. \qquad (4)$$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial p}{\partial t}\right) + \frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial \lambda}\left(u\frac{\partial p}{\partial \eta}\right) + \frac{\partial}{\partial \varphi}\left(v\cos\varphi\frac{\partial p}{\partial \eta}\right)\right] + \frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right) = 0. \qquad (5)$$

The vertical boundary conditions are:

$$\dot{\eta} = 0 \text{ at } \eta = 0, 1. \qquad (6)$$

Integrating (5) in the vertical from $\eta=0$ to 1 gives:

$$\frac{\partial p_*}{\partial t} = -\int_0^1 \frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial \lambda}\left(u\frac{\partial p}{\partial \eta}\right) + \frac{\partial}{\partial \varphi}\left(v\cos\varphi\frac{\partial p}{\partial \eta}\right)\right]. \qquad (7)$$

Integrating (5) from $\eta=0$ to η gives:

$$\dot{\eta}\frac{\partial p}{\partial \eta} = -\frac{\partial p}{\partial t} - \int_0^\eta \frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial \lambda}\left(u\frac{\partial p}{\partial \eta}\right) + \frac{\partial}{\partial \varphi}\left(v\cos\varphi\frac{\partial p}{\partial \eta}\right)\right] \qquad (8)$$

The hydrostatic relation is given by:

$$\frac{\partial \Phi}{\partial \eta} = -\frac{RT}{p}\frac{\partial p}{\partial \eta} = -C_p\theta\frac{\partial \pi}{\partial \eta}, \text{ where } \pi = (p/1000)^\kappa. \qquad (9)$$

3. THE INTEGRATION SCHEME

The variables are held on the Arakawa 'B' grid as in the present operational model. The variables u, v, θ, q and Φ are held at levels η_k, where k is the vertical grid-length index, while the vertical velocity $\dot{\eta}$ is held at the intermediate levels $\eta_{k+1/2}$. The upper and lower boundaries $\eta=0, 1$ are considered as intermediate levels. The pressure is defined at intermediate levels by

$$p_{k+1/2} = A_{k+1/2} + B_{k+1/2}\,p_*, \qquad (10)$$

where $A_{k+½}$ and $B_{k+½}$ are specified constants. Swinbank (private communication) has proposed a method of choosing these constants for a 20 level model. Thus

$$\left(\frac{\partial p}{\partial p_*}\right)_{k+½} = B_{k+½} \tag{11}$$

and

$$\Delta p_k = (A_{k+½}-A_{k-½}) + (B_{k+½}-B_{k-½})p_*. \tag{12}$$

A split explicit integration scheme is used, similar to that in the operational model. The solution procedure is split into two parts, called the 'adjustment' and 'advection' steps. The adjustment timestep is written as δt, the advection timestep as Δt. In the former, the pressure, temperature, and wind fields are updated using the pressure gradient and Coriolis terms, and the vertical advection of potential temperature. Only the final updated values of surface pressure and horizontal wind are used in the next step. The average horizontal wind from this step is used to define the horizontal advection in the advection step, and, via the continuity equation, the vertical advection. This procedure is needed to ensure conservation. All advection increments are then calculated in the advection step, together with the horizontal diffusion and divergence damping.

3.1 *The adjustment step*

This uses a 'forward-backward' scheme in which a forward step is used for the p_* and θ equations, and the new values of these variables are then used in the u and v equations. The 'forward' part of the integration scheme is:

$$p_*^{n+1} = p_*^n - \frac{\delta t}{a \cos \varphi} \sum_{m=1}^{TOP} D_m \tag{13}$$

$$\theta^{n+1} = \theta^n - \frac{\delta t}{2\Delta p_k} \left[\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+½} (\theta_{k+1}-\theta_k) + \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k-½} (\theta_k-\theta_{k-1})\right] \tag{14}$$

where

$$\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+½} = -\left(\frac{\partial p}{\partial p_*}\right)_{k+½} \sum_{m=1}^{TOP} D_m - \sum_{m=1}^{k} D_m \tag{15}$$

$$D_m = \frac{1}{a\cos\varphi}\left[\delta_\lambda\left(\overline{u_m \Delta\bar{p}_m^{\lambda\varphi}}^{\varphi}\right) + \delta_\varphi\left(\overline{v_m \cos\varphi\, \Delta\bar{p}_m^{\lambda\varphi}}^{\lambda}\right)\right]. \qquad (16)$$

The 'backward' part of the integration scheme is given by

$$u_k^{n+1} = u_k^{n} + \delta t\Big[\tfrac{1}{2}F^{n}\left(v_k^{n} + v_k^{n+1}\right) -$$

$$\frac{1}{a\cos\varphi}\overline{\left\{\delta_\lambda \Phi_k^{n+1} + \frac{C_p \bar{\theta}_k^{\lambda}}{(\kappa+1)}\,\frac{\delta_\lambda[\pi_{k+1/2}p_{k+1/2} - \pi_{k-1/2}p_{k-1/2}]}{\Delta p_k}\right\}}^{\,n+1\,\varphi}, \qquad (17)$$

$$v_k^{n+1} = v_k^{n} - \delta t\Big[\tfrac{1}{2}F^{n}\left(u_k^{n} + u_k^{n+1}\right) +$$

$$\frac{1}{a}\overline{\left\{\delta_\varphi \Phi_k^{n+1} \quad \frac{C_p \bar{\theta}_k^{\varphi}}{(\kappa+1)}\,\frac{\delta_\varphi[\pi_{k+1/2}p_{k+1/2} - \pi_{k-1/2}p_{k-1/2}]}{\Delta p_k}\right\}}^{\,n+1\,\lambda}, \qquad (18)$$

where

$$F^n = (f + u^n \tan\varphi\, /a). \qquad (19)$$

As in the operational scheme, equations (17) and (18) can be arranged to allow explicit integration. The hydrostatic equation is approximated by

$$\Phi_k = \Phi_* + \sum_{m=1}^{k-1} C_p \theta_{vm}(\pi_{m+1/2} - \pi_{m-1/2}) +$$

$$c_p \pi_k \theta_{vk}\left(\pi_{k-1/2} - \frac{(\pi_{k-1/2}p_{k-1/2} - \pi_{k+1/2}p_{k+1/2})}{(\kappa+1)\Delta p_k}\right). \qquad (20)$$

The special form of the last term is chosen to ensure angular momentum conservation.

In order to ensure that θ and q are conserved under advection, it is necessary that all advection is done by a three-dimensional velocity field which satisfies the continuity equation. The average fields of $u_m \Delta p_m$ and $v_m \Delta p_m$ over the adjustment steps must be saved for use in the advection step. The value of θ at the end of the adjustment steps is not used, so that the value at the beginning of the adjustment must be saved. This allows the vertical advection to be repeated in a conservative manner. The error made by not doing this is of the order of the time truncation rather than the space truncation, and therefore may be negligible.

3.2 *The advection step*

The Lax-Wendroff scheme used in the operational model is replaced by the Heun scheme, which avoids the complexity of time staggering. Though the Heun scheme has growing eigensolutions of order $(1+O(\Delta t^4))$, experiments show that it is more stable than the Lax-Wendroff. The scheme has two steps. The advecting velocity for both is the average value saved from the adjustment steps. Mass-weighted increments to θ and q have to be predicted to ensure conservation.

Define

$$(U_k, V_k) = (u_k \overline{\Delta p_k}^{\lambda\varphi},\ v_k \overline{\Delta p_k}^{\lambda\varphi} \cos\varphi), \tag{21}$$

as saved from the adjustment steps. Define

$$\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+½} \equiv E_{k+½}$$

where $E_{k+½}$ is calculated using the finite difference formulae (15) and (16). The second order finite difference equations for the first advection step are then:

$$\Delta p_k^{n+1} \theta_k^{\#} = \Delta p_k^{n+1} \theta_k^{n} - \Delta t\Big[\frac{1}{a\cos\varphi}\{\overline{\overline{U_m}^{\varphi}\delta_\lambda\theta_k}^{\lambda} + \overline{\overline{V_m}^{\lambda}\delta_\varphi\theta_k}^{\varphi}\} +$$

$$½\{E_{k+½}(\theta_{k+1}-\theta_k) + E_{k-½}(\theta_k-\theta_{k-1})\}\Big], \tag{22}$$

$$\Delta p_k^{n+1} q_k^{\#} = \Delta p_k^{n+1} q_k^{n} - \Delta t\Big[\frac{1}{a\cos\varphi}\{\overline{\overline{U_m}^{\varphi}\delta_\lambda q_k}^{\lambda} + \overline{\overline{V_m}^{\lambda}\delta_\varphi q_k}^{\varphi}\} +$$

$$½\{E_{k+½}(q_{k+1}-q_k) + E_{k-½}(q_k-q_{k-1})\}\Big], \tag{23}$$

$$\overline{\Delta p_k}^{\lambda\varphi\,n+1} u_k^{\#} = \overline{\Delta p_k}^{\lambda\varphi\,n+1} u_k^{n} - \Delta t\Big[\frac{1}{a\cos\varphi}\{\overline{\overline{U_m}^{\lambda}\delta_\lambda u_k}^{\lambda} + \overline{\overline{V_m}^{\varphi}\delta_\varphi u_k}^{\varphi}\} +$$

$$½\{\overline{E_{k+½}}^{\lambda\varphi}(u_{k+1}-u_k) + \overline{E_{k-½}}^{\lambda\varphi}(u_k-u_{k-1})\}\Big], \tag{24}$$

$$\overline{\Delta p_k}^{\lambda\varphi\,n+1} v_k^{\#} = \overline{\Delta p_k}^{\lambda\varphi\,n+1} v_k^{n} - \Delta t\Big[\frac{1}{a\cos\varphi}\{\overline{\overline{U_m}^{\lambda}\delta_\lambda v_k}^{\lambda} + \overline{\overline{V_m}^{\varphi}\delta_\varphi v_k}^{\varphi}\} +$$

$$½\{\overline{E_{k+½}}^{\lambda\varphi}(v_{k+1}-v_k) + \overline{E_{k-½}}^{\lambda\varphi}(v_k-v_{k-1})\}\Big]. \tag{25}$$

It is also possible to use fourth order approximations to the horizontal derivatives. In the operational scheme, this is done by only modifying the second advection step. In the Heun scheme, it is necessary to use the same finite difference approximation in both steps, or else there is an $O(\Delta t^2)$ instability. The fourth order scheme for θ is:

$$\frac{1}{a\cos\varphi}\left\{(1+c)\overline{\overline{U}_m^{\varphi}\delta_\lambda\theta_k}^{\lambda} - c\overline{\overline{U}_m^{\varphi}\delta_\lambda\theta_k}^{3\lambda} + (1+c)\overline{\overline{V}_m^{\lambda}\delta_\varphi\theta_k}^{\varphi} - c\overline{\overline{V}_m^{\lambda}\delta_\varphi\theta_k}^{3\varphi}\right\}, \quad (26)$$

instead of the {} bracket in (22), and that for q is similar. The scheme for u is:

$$\frac{1}{a\cos\varphi}\left\{(1+c)\overline{\overline{U}_m^{\lambda}\delta_\lambda u_k}^{\lambda} - c\overline{\overline{U}_m^{\lambda}\delta_\lambda u_k}^{3\lambda} + (1+c)\overline{\overline{V}_m^{\varphi}\delta_\varphi u_k}^{\varphi} - c\overline{\overline{V}_m^{\varphi}\delta_\varphi u_k}^{3\varphi}\right\}, \quad (27)$$

with a similar expression for v. The value $c=1/6$ gives fourth order accuracy, but will increase the amplification rate of the growing solution from $(1+\frac{1}{4}\mu^4)$ to $(1+\frac{1}{4}\nu^4)$ where μ is the Courant number and $\nu=1.48\mu$. This will reduce the maximum timestep that can safely be used. A fixed value must be used for c to allow conservation, but for forecasting use the choice $c=1/6(1-\mu)$, where μ is the Courant number, should avoid the need to reduce the timestep.

The second advection step can be written:

$$\Delta p_k^{n+1}\theta_k^{n+1} = \Delta p_k^{n+1}\theta_k^{n} - \tfrac{1}{2}\Delta t(\underline{U}.\nabla\theta_k^{n} + \underline{U}.\nabla\theta_k^{\#}), \quad (28)$$

with similar equations for q, u and v. This overall scheme is not conservative with respect to time differencing, though this may be a small effect. An alternative is set out in section 4.4.

3.3 *Diffusion and divergence damping*

Experiments have indicated that the grid splitting problem which requires special treatment in the operational model is not significant when the Heun advection scheme is used. However, experience with the operational fine-mesh model and with higher resolution limited area models suggests that divergence damping will be needed in forecast as well as assimilation mode for gridlengths below 100 km. The vertical diffusion will also need to be reassessed, this is not covered in this note.

The operational form of diffusion is not conservative, but is effective in other respects. The most similar conservative scheme for diffusing a variable X can be written:

$$DX = \frac{1}{a^2\cos^2\varphi}\delta_\lambda(|\nabla^2 X|\,\delta_\lambda X) + \frac{1}{a^2\cos\varphi}\delta_\varphi(|\nabla^2 X|\cos\varphi\;\delta_\varphi X), \tag{29}$$

where

$$\nabla^2 X = \frac{1}{a^2\cos^2\varphi}\delta_{\lambda\lambda}X + \frac{1}{a^2\cos\varphi}\delta_\varphi(\cos\varphi\;\delta_\varphi X). \tag{30}$$

The divergence at level k is defined by equation (16). Values D_k are calculated by substituting (U_k, V_k) as defined in equation (21) into equation (16). Increments

$$\frac{K_D}{a\cos\varphi}\delta_\lambda\overline{D}^{\varphi}\,, \quad \text{and} \quad \frac{K_D}{a}\delta_\varphi\overline{D}^{\lambda}$$

are added to equations (24) and (25) respectively in the second advection step.

4. CONSERVATION PROPERTIES

4.1 *Angular momentum conservation*

The requirement is that the pressure gradient term can only change the angular momentum through the surface torque. This means that we must be able to write the approximation to the pressure gradient term in the model which is

$$\sum_{m=1}^{TOP}\left\{\frac{\partial\Phi_m}{\partial\lambda}\Delta p_m + C_p\left(\theta\frac{\partial\pi}{\partial\lambda}\right)_m\Delta p_m\right\}, \tag{31}$$

in the form

$$\frac{\partial}{\partial\lambda}\left(\sum_{m=1}^{TOP}\Phi_m\Delta p_m\right) - \Phi_*\frac{\partial p_*}{\partial\lambda}. \tag{32}$$

The first term in (32) integrates to zero and the second integrates to the surface torque. This requirement determines how the second term in (31) has to be calculated at level m, as in Simmons and Strufing (1981). Cancelling terms gives the requirement

$$\sum_{m=1}^{TOP} C_p\left(\theta_v \frac{\partial\pi}{\partial\lambda}\right)_m \Delta p_m = \sum_{m=1}^{TOP} \Phi_m \frac{\partial}{\partial\lambda}(\Delta p_m) - \Phi_* \frac{\partial p_*}{\partial\lambda}. \qquad (33)$$

We now substitute for Φ in the right hand side of (33) to establish the required form of approximation to the left hand side. Write (20) as

$$\Phi_k = \Phi_* + \sum_{m=1}^{k-1} C_p \theta_{vm} (\Delta\pi_m) + \alpha C_p \pi_k \theta_{vk}. \qquad (34)$$

Then the right hand side of (33) becomes

$$\sum_{m=1}^{TOP} (\alpha_m C_p \pi_m \theta_{vm} + \Phi_{m-1/2}) \frac{\partial \Delta p_m}{\partial\lambda} - \Phi_* \frac{\partial p_*}{\partial\lambda}, \qquad (35)$$

$$\sum_{m=1}^{TOP} \left(\alpha_m C_p \pi_m \theta_{vm} \frac{\partial \Delta p_m}{\partial\lambda}\right) + \sum_{m=1}^{TOP} \sum_{j=1}^{m-1} C_p \theta_{vj} \Delta\pi_j \frac{\partial \Delta p_m}{\partial\lambda}. \qquad (36)$$

The double sum in (36) can be written

$$\sum_{m=1}^{TOP} C_p \theta_{vm} \Delta\pi_m \sum_{j=m+1}^{TOP} \frac{\partial \Delta p_j}{\partial\lambda}, \qquad (37)$$

where we have used the convention that

$$\sum_{j=r}^{r-1} = 0. \qquad (38)$$

The second sum is now just $\partial/\partial\lambda(p_{m+1/2})$. Equation (35) can therefore be written as:

$$\sum_{m=1}^{TOP} \frac{1}{\Delta p_m}\left[C_p \alpha_m \pi_m \theta_{vm} \frac{\partial \Delta p_m}{\partial\lambda} + C_p \theta_{vm} \Delta\pi_m \frac{\partial p_{m+1/2}}{\partial\lambda}\right] \Delta p_m. \qquad (39)$$

Choose α_m such that

$$\alpha_m \pi_m = \pi_{m-1/2} + \frac{\Delta(\pi p)_m}{(\kappa+1)\Delta p}. \qquad (40)$$

Then the summand in (39) becomes

$$\frac{C_p \theta_{vm}}{\Delta p_m}\left[\pi_{m-1/2} \frac{\partial \Delta p_m}{\partial\lambda} + \frac{\Delta(\pi p)_m}{(\kappa+1)\Delta p_m} \frac{\partial \Delta p_m}{\partial\lambda} + \Delta\pi_m \frac{\partial p_{m+1/2}}{\partial\lambda}\right]$$

$$= \frac{C_p \theta_{vm}}{\Delta p_m}\left[\pi_{m+1/2} \frac{\partial p_{m+1/2}}{\partial\lambda} - \pi_{m-1/2} \frac{\partial p_{m-1/2}}{\partial\lambda} + \frac{\Delta(\pi p)_m}{(\kappa+1)\Delta p_m} \frac{\partial \Delta p_m}{\partial\lambda}\right]. \qquad (41)$$

Using the definition of π in terms of p, (41) reduces to

$$\frac{C_p \theta_{vm}}{(\kappa+1)}\left[\frac{\partial}{\partial\lambda}\left(\frac{\Delta(\pi p)_m}{\Delta p_m}\right)\right]. \qquad (42)$$

The expression (42) is then approximated by spatial finite differences and used to approximate the term on the left hand side of (33) in the equation of motion. The above argument can be carried through in finite differences, provided that the approximation used is

$$\frac{C_p \overline{\theta_{\kappa m}}^{\lambda}}{(\kappa+1)} \left[\delta_\lambda \left(\frac{\Delta(\pi p)_m}{\Delta p_m} \right) \right], \tag{43}$$

as used in equations (17) and (18).

4.2 *Conservation of first moments*

The requirement is that the global mass-weighted mean of all advected quantities is conserved. The proof is written out only for meridional advection of θ and u, since this covers all the possible staggerings of variables that occur in the other cases. The continuity equation reduces to:

$$E_{k+½} = -\left(\frac{\partial p}{\partial p_*}\right)_{k+½} \sum_{m=1}^{TOP} D_m - \sum_{m=1}^{k} D_m. \tag{44}$$

where

$$D_m = \frac{1}{a \cos \varphi} \delta_\varphi \overline{V_m}^{\lambda}. \tag{45}$$

A simple forward update of θ by meridional advection, and advection by the vertical motion associated with the meridional motion, is given by

$$\overset{+}{\Delta p_k} \overset{+}{\theta_k} = \overset{+}{\Delta p_k} \theta_k - \Delta t \left[\frac{1}{a \cos \varphi} \overline{\overline{V_k}^{\lambda} \delta_\varphi \theta_k}^{\varphi} + \right.$$

$$\left. ½\{E_{k+½}(\theta_{k+1}-\theta_k) + E_{k-½}(\theta_k-\theta_{k-1})\} \right] . \tag{46}$$

The update of p_* can be written

$$\overset{+}{p_*} = p_* - \frac{\Delta t}{a \cos \varphi} \sum_{m=1}^{TOP} D_m, \tag{47}$$

because of the definition of V_m as the average over the adjustment steps. Then use (12) to give

$$\overset{+}{\Delta p_k} = \Delta A_k + \Delta B_k \overset{+}{p_*}.$$

Equation (11) can be used to rewrite (44) as

$$E_{k+1/2} - E_{k-1/2} = \Delta B_k(\overset{+}{p_*} - p_*)/\Delta t - D_k. \tag{48}$$

Multiplying (48) by θ_k and adding to (46) gives

$$(\Delta A_k + \Delta B_k \overset{+}{p_*})(\overset{+}{\theta_k} - \theta_k) + \theta_k(\overset{+}{p_*} - p_*)\Delta B_k =$$

$$-\Delta t[\frac{1}{a\cos\varphi}\overline{\{\overline{V_k}^{\lambda}\delta_\varphi\theta_k + \theta_k\overline{\delta_\varphi V_k}^{\lambda}\}}^{\varphi} + \tfrac{1}{2}\{E_{k+1/2}(\theta_{k+1} - \theta_k) +$$

$$E_{k-1/2}(\theta_k - \theta_{k-1}) + 2\theta_k(E_{k+1/2} - E_{k-1/2})\}]. \tag{49}$$

This reduces to

$$\overset{+}{\Delta p_k}\overset{+}{\theta_k} - \Delta p_k\theta_k = -\Delta t[\frac{1}{a\cos\varphi}\delta_\varphi\overline{\overline{V_k}^{\lambda}\theta_k}^{\varphi} + \tfrac{1}{2}\{E_{k+1/2}(\theta_{k+1} + \theta_k) -$$

$$E_{k-1/2}(\theta_k + \theta_{k+1})\}], \tag{50}$$

which gives the desired conservation integral when multiplied by $\cos\varphi$ and integrated over φ.

The integral is conserved if the time derivatives are analytic, or if a simple forward step is used. With the scheme used in section 3 there is some loss of conservation due to time truncation, an alternative method which avoids this is descriubed in section 4·4.

The update of u by meridional advection and advection by the associated part of the vertical motion is given by

$$\overset{+}{\Delta p_k}\overset{+}{u_k} = \overset{+}{\Delta p_k}u_k - \Delta t[\frac{1}{a\cos\varphi}\overline{\overline{V_k}^{\varphi}\delta_\varphi u_k}^{\varphi} + \tfrac{1}{2}\{\overline{E_{k+1/2}}^{\lambda\varphi}(u_{k+1} - u_k) +$$

$$\overline{E_{k-1/2}}^{\lambda\varphi}(u_k - u_{k-1})\}]. \tag{51}$$

Multiplying (48) by $\overline{u_k}^{\lambda\varphi}$ and adding gives

$$\overline{\overset{+}{\Delta p_k}\overset{+}{u_k}}^{\lambda\varphi} - \Delta p_k\overline{u_k}^{\lambda\varphi} = -\Delta t[\frac{1}{a\cos\varphi}\overline{\{\overline{V_k}^{\varphi}\delta_\varphi u_k}^{\varphi} + u_k\overline{\overline{\delta_\varphi V_k}^{\lambda}}^{\lambda\varphi}\} +$$

$$\tfrac{1}{2}\{\overline{E_{k+1/2}}^{\lambda\varphi}(u_{k+1} - u_k) + \overline{E_{k-1/2}}^{\lambda\varphi}(u_k - u_{k-1}) + 2u_k(\overline{E_{k+1/2}}^{\lambda\varphi} - \overline{E_{k-1/2}}^{\lambda\varphi})\}]. \tag{52}$$

The horizontal advection terms cannot be rewritten in flux form, they will only take such a form if the term $\overline{V_k}^{\varphi}\delta_\varphi u_k$ is replaced by $\overline{\overline{V_k}^{\lambda}}^{\lambda\varphi}\delta_\varphi u_k$, which may be considerably less accurate. Momentum is, however, conserved in the case $U = V =$ constant, (linear advection), or if V is independent of λ and U is independent of φ. Similar extra averaging is present in the schemes presented by Mesinger (1981) for this grid.

Now consider the fourth order terms in (26) and (27). Conservation cannot be achieved if the constant c is a function of μ, as is necessary to avoid reducing the timestep. Suppose that c is a constant. The terms

$$(1+c)\overline{\overline{U}_k^{\varphi}\delta_\lambda\theta_k}^{\lambda} - c\overline{\overline{U}_k^{\varphi}\delta_\lambda\theta_k}^{3\lambda}$$

can be expanded as

$$(1+c)\overline{U}_k^{\varphi}(\lambda+\tfrac{1}{2}\Delta\lambda)(\theta_k(\lambda+\Delta\lambda)-\theta_k(\lambda)) -$$
$$c\overline{U}_k^{\varphi}(\lambda+3/2\Delta\lambda)(\theta_k(\lambda+2\Delta\lambda)-\theta_k(\lambda+\Delta\lambda)), \qquad (53)$$

with symmetrical terms in $-\Delta\lambda$. When the equations for $\overset{+}{\theta}_k(\lambda)$ and $\overset{+}{\theta}_k(\lambda+\Delta\lambda)$ are added to give the conservation law, the terms multiplied by c in (53) cancel, to give

$$\overline{U}_k^{\varphi}(\lambda+\tfrac{1}{2}\Delta\lambda)(\theta_k(\lambda+\Delta\lambda)-\theta_k(\lambda)).$$

Thus the conservation property follows as in the second order case. A similar argument holds for equation (27) in the case of linear advection, so that the fourth order scheme does not result in any further loss of conservation.

4·3 *Conservation of second moments*

We first demonstrate that the integral of $\Delta p\theta^2$ is conserved using the second order accurate approximation to the advection terms. This will give a reasonable approximation to the true Lagrangian conservation property of θ, which requires that all moments are conserved by advection by an incompressible flow. Multiply (48) by θ_k^2 and add to (46) multiplied by $2\theta_k$:

$$2\Delta p_k^{+}\theta_k(\theta_k^{+} - \theta_k) + \theta_k^2(p_*^{+} - p_*)\Delta B_k =$$

$$-\Delta t[\frac{1}{\cos\varphi}\{2\theta_k\overline{\overline{V_k}^{\lambda}\delta_\varphi\theta_k}^{\varphi} + \theta_k\overline{\delta_\varphi V_k}^{\lambda}\} + \{\theta_k E_{k+1/2}(\theta_{k+1} - \theta_k) +$$

$$\theta_k E_{k-1/2}(\theta_k - \theta_{k-1}) + \theta_k^2(E_{k+1/2} - E_{k-1/2})\}]. \qquad (54)$$

The left hand side is a discrete approximation to

$$\Delta p_k\frac{\partial}{\partial t}(\theta_k^2) + \theta_k^2\frac{\partial}{\partial t}\Delta p_k. \qquad (55)$$

However, it cannot be written as exact conservation of $\Delta p_k\theta_k^2$. The right hand side becomes

$$-\Delta t[\frac{1}{a\cos\varphi}\delta_\varphi\overline{V_k}^{\lambda}(2(\overline{\theta_k}^{\varphi})^2 - \overline{\theta_k^2}^{\varphi}) + \theta_k\theta_{k+1}E_{k+1/2} - \theta_k\theta_{k-1}E_{k-1/2}]. \qquad (56)$$

This is in conservation form. In order to achieve quadratic conservation with the fourth order terms included, the E_k's must be redefined (Fisher, private communication). The resulting scheme is rather less accurate because it uses a broader stencil of gridpoints.

Consider now conservation of $u^2\Delta p$ by meridional advection by a V independent of λ, so that the first moment $u\Delta p$ is conserved. Multiplying (48) by $\overline{u_k}^{\lambda\varphi\,2}$ and adding to (51) multiplied by $2u_k$ gives

$$2\overline{\Delta p_k^{+}}^{\lambda\varphi}\,u_k(u_k^{+} - u_k) + u_k^2\overline{(p_*^{+} - p_*)}^{\lambda\varphi}\Delta B_k =$$

$$-\Delta t[\frac{1}{a\cos\varphi}\{2u_k\overline{\overline{V_k}^{\varphi}\delta_\varphi u_k}^{\varphi} + u_k^2\overline{\overline{\delta_\varphi V_k}^{\lambda}}^{\lambda\varphi}\} + \{u_k\overline{E_{k+1/2}}^{\lambda\varphi}(u_{k+1} - u_k)$$

$$+ u_k\overline{E_{k-1/2}}^{\lambda\varphi}(u_k - u_{k-1}) + u_k^2(\overline{E_{k+1/2}}^{\lambda\varphi} - \overline{E_{k-1/2}}^{\lambda\varphi})\}]. \qquad (57)$$

The left hand side is a finite difference approximation to

$$\Delta p_k\frac{\partial}{\partial t}(u_k^2) + u_k^2\frac{\partial}{\partial t}\Delta p_k.$$

The right hand side, for V independent of λ, becomes

$$-\Delta t[\frac{1}{a\cos\varphi}\delta_\varphi\overline{V_k}^{\varphi}(2(\overline{u_k}^{\varphi})^2 - \overline{u_k^2}^{\varphi}) + u_k u_{k+1}\overline{E_{k+1/2}}^{\lambda\varphi} - u_k u_{k-1}\overline{E_{k-1/2}}^{\lambda\varphi}]. \qquad (58)$$

This is in conservation form. If the averaged form $\overline{V_k}^{\lambda}\,\overline{\delta_\varphi u_k}^{\lambda\varphi}$ is used in (51), the conservation holds for general V. The second moment is not conserved by the fourth order scheme.

4·4 *Time differencing*

The approximation to the time derivatives in the second advection step, equation (28), can be made to conserve the integral of $\Delta p_k \theta_k$ by setting

$$\Delta p_k^{n+1}\,\theta_k^{n+1} = \Delta p_k^{n+1}\left(\theta_k^{n}\frac{\Delta p_k^{n}}{\Delta p_k^{n+1}} + \tfrac{1}{2}\left(\theta_k^{n} - \theta^{\#}\frac{\Delta p_k}{\Delta p_k^{n+1}}\right)\right) - \tfrac{1}{2}\Delta t\left(\underline{U}.\nabla\theta_k^{n} + \underline{U}\cdot\nabla\theta_k^{\#}\right) \qquad (59)$$

There is no need to alter the first step. Using (59) requires extra storage for p_*^n.

5. SUMMARY

This note has outlined a conservative split-explicit finite difference scheme on a B grid. There are some restrictions on what is possible, and choices to be made which require further experiments.

(i) Conservation integrals can only be satisfied with respect to time differencing if (59) is used, and the integral of $\Delta p\theta$ is only conserved if the vertical advection of θ is recalculated. Experiments should be carried out to see if these extra computations are worthwhile in practice. It is likely that they would not be needed for forecasting applications, and the code should be written to allow them to be bypassed.

(ii) The conservation of momentum by advection requires extra averaging which may be particularly damaging at low resolution. If the equations were written in flux form this extra averaging would still occur. This should not be used in forecasting applications and experiments should be conducted to determine the best option for climate integrations.

(iii) The forecast model should use a fourth order scheme with variable c. Experiments should be conducted to see if climate integrations perform better with a second order scheme or the fourth order scheme proposed by Fisher, giving quadratic conservation, or the fourth order scheme given here with constant c, giving more accuracy.

(iv) The scheme has intentionally been written using an approximation to the Lagrangian derivative of momentum in the momentum equation, rather than using the alternative vorticity/ energy form. The latter form can lead to spurious sources or sinks of momentum, though it allows enstrophy to be conserved. If the solutions are not smooth, it is more important to treat the momentum correctly than the vorticity.

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