DR

MET O 11 TECHNICAL NOTE No 58

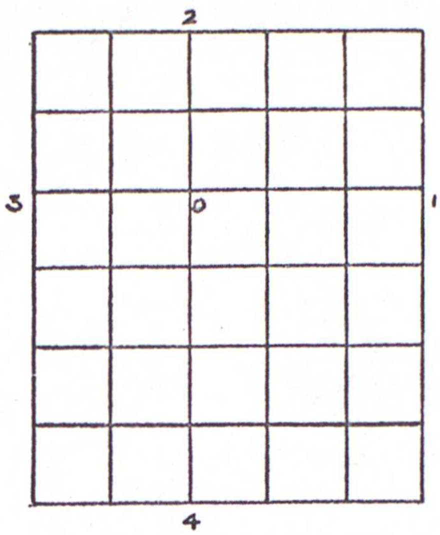
Initial fields for the Dirichlet and Neumann problems

R DIXON

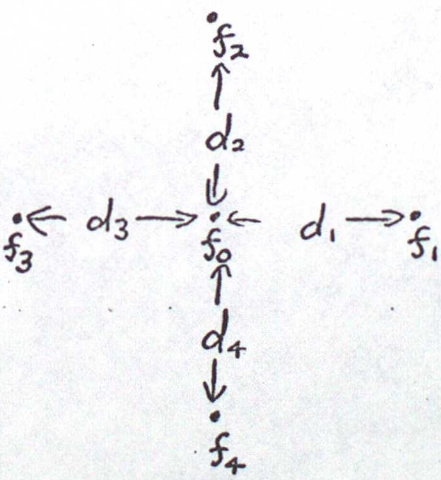
1. Introduction

It sometimes happens that no good first guess field is available for the iterative solution of a Laplace, Poisson, Helmholtz type of equation. This note outlines a simple method for providing a first guess field which may be better than the zero or constant field which is usually assumed in such circumstances.

2. Dirichlet Boundary Conditions



In the grid depicted above take any interior grid point as point 0, the corresponding points on the boundaries in the x direction as points 1 and 3 and the corresponding points on the boundaries in the y direction as points 2 and 4 so that at a typical grid point we have



where f_o is the unknown interior function value and f_1, f_2, f_3, f_4 are the known boundary values. Applying Taylor's series expressed formally as

$$f(\underline{r} + d\underline{r}) = e^{d\underline{r}\cdot\nabla} f(\underline{r}) \qquad (1)$$

where $\underline{r}$ is a 2-d position vector with origin at 0, it is found that cross-product terms disappear and there results.

$$f_1 = \left(1 + d_1\frac{\partial}{\partial x} + \frac{1}{2!}d_1^2\frac{\partial^2}{\partial x^2} + \frac{1}{3!}d_1^3\frac{\partial^3}{\partial x^3} + \text{----}\right) f_0 \qquad (1)$$

$$f_2 = \left(1 + d_2\frac{\partial}{\partial y} + \frac{1}{2!}d_2^2\frac{\partial^2}{\partial y^2} + \frac{1}{3!}d_2^3\frac{\partial^3}{\partial y^3} + \text{---}\right) f_0 \qquad (2)$$

$$f_3 = \left(1 - d_3\frac{\partial}{\partial x} + \frac{1}{2!}d_3^2\frac{\partial^2}{\partial x^2} - \frac{1}{3!}d_3^3\frac{\partial^3}{\partial x^3} + \text{---}\right) f_0 \qquad (3)$$

$$f_4 = \left(1 - d_4\frac{\partial}{\partial y} + \frac{1}{2!}d_4^2\frac{\partial^2}{\partial y^2} - \frac{1}{3!}d_4^3\frac{\partial^3}{\partial y^3} + \text{---}\right) f_0 \qquad (4)$$

Combining these equations in such a way as to eliminate terms in $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ then, discarding terms in d^3 and above

$$\frac{1}{2}\nabla^2 f_0 = \frac{d_3 f_1 + d_1 f_3}{d_1 d_3 (d_1 + d_3)} + \frac{d_4 f_2 + d_2 f_4}{d_2 d_4 (d_2 + d_4)} - \frac{d_1 d_3 + d_2 d_4}{d_1 d_2 d_3 d_4} f_0 \qquad (5)$$

and if we use proportions so that

$$d_1 = p_1 X, \quad d_3 = p_3 X, \quad d_2 = p_2 Y, \quad d_4 = p_4 Y \qquad (6)$$

then bearing in mind that

$$d_1 + d_3 = X \quad \text{and} \quad d_2 + d_4 = Y \qquad (7)$$

it is found that (5) is expressible as

$$\frac{1}{2}\nabla^2 f_0 = \frac{f_1}{p_1 X^2} + \frac{f_2}{p_2 Y^2} + \frac{f_3}{p_3 X^2} + \frac{f_4}{p_4 Y^2} - \left(\frac{1}{p_2 p_4 Y^2} + \frac{1}{p_1 p_3 X^2}\right) f_0 \qquad (8)$$

and of course if X = Y this becomes

$$\frac{1}{2}\nabla^2 f_0 = \frac{1}{X^2}\left[\frac{f_1}{p_1} + \frac{f_2}{p_2} + \frac{f_3}{p_3} + \frac{f_4}{p_4} - \left(\frac{1}{p_1 p_3} + \frac{1}{p_2 p_4}\right) f_0\right] \qquad (9)$$

Taking the X = Y case for simplicity, for the Laplace equation $\nabla^2 f = 0$ we move through the grid point by point applying

$$\left(\frac{1}{p_1 p_3} + \frac{1}{p_2 p_4}\right) f_0 = \frac{f_1}{p_1} + \frac{f_2}{p_2} + \frac{f_3}{p_3} + \frac{f_4}{p_4} \qquad (10)$$

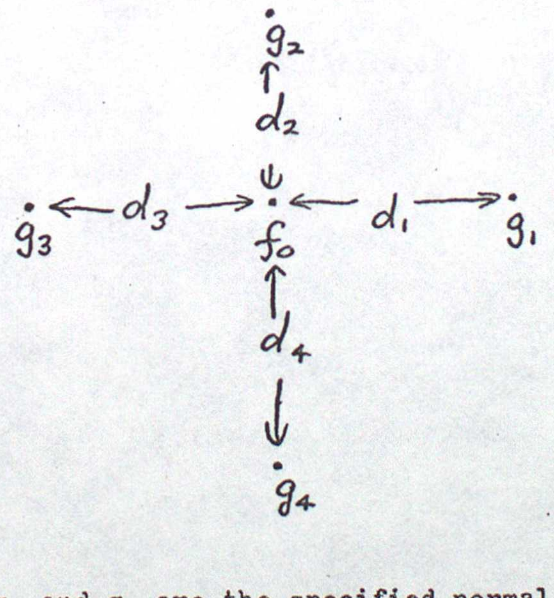
For the Helmholtz eqn. $\nabla^2 f + k^2 f = 0$ we can do the same thing using

$$\left[\frac{1}{2}k^2 - \frac{1}{X^2}\left(\frac{1}{p_1 p_3} + \frac{1}{p_2 p_4}\right)\right] f_0 = \frac{1}{X^2}\left[\frac{f_1}{p_1} + \frac{f_2}{p_2} + \frac{f_3}{p_3} + \frac{f_4}{p_4}\right] \qquad (11)$$

Having thus provided a coarse first guess we can then switch to whatever more sophisticated iterative routine is being used.

3. The Neumann Boundary Conditions

We now have the situation



where g_1, g_2, g_3 and g_4 are the specified normal gradients on the boundaries. The above algebraic process can be repeated but for this case it is scarcely necessary as intuition suggests and inspection confirms the simple result

$$\nabla^2 f_0 = \frac{g_1 - g_3}{X} + \frac{g_2 - g_4}{Y} \qquad (12)$$

and thus as a first guess to the solution of

$$\nabla^2 f + k^2 f = q \tag{13}$$

we have

$$f_0 = \frac{1}{k^2}\left[q - \frac{g_1 - g_3}{X} - \frac{g_2 - g_4}{Y}\right] \tag{14}$$

at each interior grid point. This result looks suspiciously simple but there may be circumstances in which it gives an improvement on q/k^2 as a first guess. It is noted that as $k \rightarrow 0$ the classical singularity of the Laplace and Poisson equations under Neumann conditions asserts itself and the artifice becomes unavailable.

Using Taylor's series, discarding $O(d^3)$ terms, to get these expressions is equivalent to assuming that the field is a second power polynomial. To get a more refined first guess for the Neumann problem it is necessary to ~~ensure~~ assume that the field is a third order polynomial. If this is done it is found that the problem of determining an expression for $\nabla^2 f_0$ is indeterminate. However it can be made determinate by imposing a minimum norm condition on the polynomial. Some tedious algebra then leads to

$$\nabla^2 f_0 = \frac{4}{X}\left[\frac{(g_1 - g_3) + 9 d_1 d_3 (d_3^2 g_1 - d_1^2 g_3)}{9(d_1 - d_3)^2 + 36 d_1^2 d_3^2 + 4}\right] + \frac{4}{Y}\left[\frac{(g_2 - g_4) + 9 d_2 d_4 (d_4^2 g_2 - d_2^2 g_4)}{9(d_2 - d_4)^2 + 36 d_2^2 d_4^2 + 4}\right] \tag{15}$$

and thus as a first guess for the solution to (13) we have in place of (14)

$$f_0 = \frac{1}{k^2}\left[q - \nabla^2 f_0\right] \tag{16}$$

where $\nabla^2 f_0$ is given by (15).

CG

4. Conclusions

The essence of the approach is to calculate a likely value at each interior grid point due to the influence of the known boundary values. This artifice has been used before but formula (16) is new. It has a probabilistic interpretation in so far as it provides the expected value of ∇f at each interior grid point subject to the conditions that f lies on a cubic which satisfies the boundary gradients and that the expected value of ∇f at points very remote from the boundaries is zero. This is an intuitive interpretation. I have not established this rigorously.

R Holden

Met O 11
February 1976