M.O.11 Technical Note (B) No. 7 (May 1958).

Finite difference approximations in numerical prediction

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Introduction

The differential equations which occur in numerical prediction of the pressure field cannot be solved by means of analytic solutions except in a few trivial cases: they are usually solved by replacing the differential equations by their finite difference analogues and then carrying out the solution numerically. The link between the differential and difference equations is provided by the Taylor series e.g.

$$X(x+h) - X(x-h) = 2h\left\{\frac{dX}{dx} + \frac{h^2}{3!}\frac{d^3X}{dx^3} + \ldots\ldots\right\} \tag{1}$$

For practical use such series as that on the right must be truncated after a few terms, usually after a single term. The solution of the difference equations is then only an approximation to the solution of the differential equations and the errors arising from such approximations to the derivative are known as truncation errors.

In forming equation (1) the values of the function X have been considered at points distant h apart. The finite difference analogues of the differential equations will usually refer to values of the co-ordinates $x_1, x_2, x_3 \ldots.$ at distances apart of $h_1, h_2, h_3 \ldots\ldots$ and the solution of the finite difference equations will be functions of $h_1, h_2, h_3 \ldots\ldots$ as well as of $x_1, x_2, x_3 \ldots\ldots$ It is sometimes possible to choose $h_1, h_2, h_3 \ldots\ldots$ so that as $h_1, h_2, h_3 \ldots \to 0$ the solution of the finite difference equations approaches that of the differential equations and this is the solution that we require. It is also possible for the choice to be made in such a way that the solution of the difference equations does not approach that of the differential equations, and indeed so that small errors are magnified indefinitely and ultimately the solution appears as rapidly oscillating with increasing amplitudes; the process of solution is then said to be computationally unstable.

The object of the present work was to examine a number of finite difference approximations, which could be of practical value, in order to determine the truncation errors introduced and the criteria for stability in computation.

A simple conservation equation

The typical equation arising in numerical prediction is

$$\frac{\partial X}{\partial t} = -V.\nabla X + F \tag{2}$$

where the fields of X and F are known and V is a function of X. Whatever limitations there are in solving this equation when V is a constant must apply also to the case in which V depends on X and we will investigate the much simpler equation

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} = 0 \tag{3}$$

where U is constant. Later we will test the conclusions drawn by computations using equations similar to (2) rather than (3).

The finite difference equation usually associated with (3) is

$$\frac{\phi(x,t+\tau) - \phi(x,t-\tau)}{2\tau} + U\frac{\phi(x+h,t) - \phi(x-h,t)}{2h} = 0 \tag{4}$$

where the time and space derivatives have been replaced by centred differences. We assume that the solution is of the form

$$\phi = c(t)\cos kx + s(t)\sin kx \tag{5}$$

where the right hand side may be generalised by summation. Since the equation is linear there is no interaction between terms of different wave number. It is convenient to write $t = n\tau$ and $c(n\tau) + is(n\tau) = A(n\tau) = A_n$.
Then

$$\phi = \text{real part}\left[A_n e^{-ikx}\right] \tag{6}$$

and (4) gives the recurrence relationship

$$A_{n+1} - A_{n-1} - 2i\mu A_n = 0 \tag{7}$$

where

$$\mu = \frac{U\tau \sin kh}{h} \tag{8}$$

The solution of (7) is

$$A_n = A\left[i\mu + \sqrt{1-\mu^2}\right]^n + B\left[i\mu - \sqrt{1-\mu^2}\right]^n \tag{9}$$

where A and B are arbitrary constants, independent of n.

If μ exceeds unity we may put $\mu = \cosh\psi$ where ψ is real and (9) becomes

$$A_n = i^n\left[Ae^{n\psi} + Be^{-n\psi}\right] \tag{10}$$

Hence

$$|A_n|^2 = \{c(t)\}^2 + \{s(t)\}^2 \longrightarrow \infty \quad \text{with } n.$$

It can also be shown that if μ is unity, $|A_n|^2 \rightarrow \infty$ with n^2 and in neither of these cases will the finite difference computation converge. Clearly it is necessary that h and τ be chosen so that the inequality $\frac{U\tau \sin kh}{h} < 1$ is satisfied. In the more general case obtained by summation of the right hand side of (5) this inequality must hold for all values of k and hence the inequality $\frac{U\tau}{h} < 1$ must be satisfied. This condition is well-known for hyperbolic equations of the second order, such as the wave equation. If $|\mu| < 1$, we may write $\mu = \sin\psi$ and

$$A_n = Ae^{in\psi} + B(-1)^n \times e^{-in\psi} \tag{11}$$

$|A_n|^2$ lies between the values $\{|A|+|B|\}^2$ and $\{|A|-|B|\}^2$ so that the amplitude of the solution is bounded; the phase speeds may, of course, be in error.

The differential equation (3) corresponds to a single wave of constant shape moving along the x-axis with constant velocity U. The solution of the finite difference equation given by (11) corresponds to two waves moving along the x-axis. The first of these is represented by $A \exp i(n\psi - kx)$ and has a velocity $\psi/k\tau$ where $\sin\psi = \frac{U\tau}{h}\sin kh$. As h and τ approach zero this velocity approaches U. The second is represented by $B \exp -i(n\psi \pm n\pi + kx)$ and has a velocity $-(\psi \pm \pi)/k\tau$ which may be regarded as moving in either direction. This second wave has no counterpart in the solution of the differential equation and can only vanish under very special conditions.

Simple forward time step.

It is possible to set up a finite difference analogue of (3) using a forward-time step instead of the centred-time difference of (4); it is necessary that some such analogue exist for the difference formula (4) cannot be applied until ϕ is known at two times, say $t = 0$ and $t = \tau$, and the value of ϕ at $t = \tau$ is often computed by using a forward-time step. We have

$$\frac{\phi(x,t+\tau) - \phi(x,t)}{\tau} + U\,\frac{\phi(x+h,t) - \phi(x-h,t)}{2h} = 0 \tag{12}$$

and after inserting (5) we find

$$A_n = (1+\mu)^n A_0 \tag{13}$$

so that computation is unstable since $|A_n| \rightarrow \infty$ with n. Since μ increases with k, the amplitudes of the short waves are increased more quickly than those of the long waves: this gives rise to typical computational instability shown by a pattern of short waves of increasing amplitude swamping the long waves.

Nevertheless, 24 hour forecasts have been successfully carried out by use of formulae similar to (12) and the instability has not been apparent; this must have been because μ was generally quite small, owing to the small velocities and long wave lengths.

As a simple example illustrating the use of a forward-time step followed by the use of a centred-time step consider the solution of (3) which has the value $\cos kx$ at $t = 0$; it is $\cos k(x - Ut)$.

Using (13) we have

$$\begin{aligned} \phi(0) &= \cos kx \\ \phi(\tau) &= \cos kx + \mu \sin kx \end{aligned} \tag{14}$$

and these equations are sufficient to solve for the arbitrary constants in (11) giving

$$\begin{aligned} \phi(2n\tau) &= \cos 2n\psi \cos kx + \sin 2n\psi \sec\psi \sin kx \\ \phi(\overline{2n+1}\,\tau) &= \sec\psi \cos(2n+1)\psi \cos kx + \sin(2n+1)\psi \sin kx \end{aligned} \tag{15}$$

The amplitude varies between unity and $\sec^2\psi$ or $1/(1-\mu^2)$ and since μ increases with k the amplitudes of the short waves have the greatest variation. If we write (15) as

$$\phi(n\tau) = \text{real part}\ \frac{1}{2}\left\{(1+\sec\psi)e^{i(n\psi-kx)} + (1-\sec\psi)(-)^n e^{-i(n\psi+kx)}\right\} \tag{16}$$

the first term appears as a wave with velocity $\psi/k\tau$. The additional term, which it does not seem possible to avoid, has its amplitude reduced as k is reduced, so that it adds less truncation error for the long waves.

<u>An implicit finite difference scheme.</u>

A centred-time step with the space derivatives computed at the central time will always produce two waves insead of the single wave which is required. A forward-time step will produce a single wave, whether the space derivatives are computed at the initial or final time. As we have seen above, if the space derivatives are computed at the initial time the computation is unstable· similarly, if the space derivatives are computed at the final time the amplitude of the wave decreases to zero with time. If we take the formula

$$\frac{\phi(x,t+\tau)-\phi(x,t)}{\tau}+\frac{U}{4h}\left\{\phi(x+h,t+\tau)-\phi(x-h,t+\tau)+\phi(x+h,t)-\phi(x-h,t)\right\}=0 \tag{17}$$

writing G_n as the amplitude factor after a time $n\tau$, then

$$G_n = G_0\left(\frac{1+\frac{i\mu}{2}}{1-\frac{i\mu}{2}}\right) = G_0 e^{i\delta n} \tag{18}$$

where

$$\tan \delta/2 = \mu/2 \tag{19}$$

Hence $|G_n| = |G_0|$ whatever the value of μ. Here μ corresponds to the tangent of an angle and hence may run from zero to infinity. In the previous criterion μ corresponded to the sine of an angle and hence could lie only between zero and unity. The phase speed from (18) is $\delta/k\tau$, which is less than that found for the wave using the centred-time step, in itself an underestimate. Some actual values are given in Table I and there is a serious underestimation of the phase speed at high velocities. At velocities of 50 Kt. or less there is little to choose between this implicit method and that of the centred-time step. The gain in using the implicit method is the absence of a second wave solution, amplitude preservation and ability to move forward in comparatively large time steps; the loss is in the time needed to perform the computations.

One-sided space derivatives

When carrying out computations on the model including tropopause effects (**Knighting (1)**) noted that short wave features of large amplitude appeared where gradients were large. In regions of rapid space change the two-sided space difference which we have used hitherto is not a good approximation to the derivative. Figure 1 shows the effect of advecting a field defined at the grid points for three time steps with a speed equal to 1/3 grid length per unit time. The field should then move on one grid length, but large discrepancies are apparent and at the grid point marked A the pressure has risen to 300 mb. instead of falling slightly. One might expect a better result by approximating to the space derivative by a finite difference using grid points upwind of the point in question rather than a finite difference which uses symmetrical points. There are two simple approximations. The first has a centred-time step and is

$$\frac{\phi(x,t+\tau)-\phi(x,t-\tau)}{2\tau} + U\,\frac{\phi(x,t)-\phi(x-h,t)}{h} = 0 \tag{20}$$

The amplitude increases exponentially and the computation is unstable. The second has a non-centred-time step and is

$$\frac{\phi(x,t+\tau)-\phi(x,t)}{\tau} + U\,\frac{\phi(x,t)-\phi(x-h,t)}{h} = 0 \tag{21}$$

The amplitude factor is given by

$$H_n = \left\{1-\frac{U\tau}{h}\left(1-e^{ikh}\right)\right\}^n \tag{22}$$

and $|H_n| \to \infty$ with n if $\frac{U\tau}{h} > 1$. If $\frac{U\tau}{h} < 1$ $|H_n| \to 0$ as $n \to \infty$ so that the wave is damped with time. Writing $\frac{U\tau}{h} = \alpha$ we find

$$|H_n| = \left\{1-2\alpha(1-\alpha)(1-\cos kh)\right\}^{n/2} |H_0| \tag{23}$$

so that the amplitudes of the long waves are less reduced than those of the short waves and short wave features due to errors (perhaps round-off errors) are removed. The phase speed is given by $\tan^{-1}\left\{\alpha \sin kh / 1-\alpha(1-\cos kh)\right\}/k\tau$ which is correct for $\alpha = \frac{1}{2}, 1$. As $k \to 0$ the phase speed approaches the correct value. Phase speeds for a particular wave length are given in Table I. It is possible to use an implicit time derivative together with the one-sided space derivatice but the results are similar: the phase speeds are given in Table I.

Quadratic interpolation

It is possible to approximate more closely to the time derivative with little extra trouble. Taking the first two terms of the Taylor expansion

$$\begin{aligned} \phi(t+\tau)-\phi(t) &= \tau\frac{\partial\phi}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2\phi}{\partial t^2} \\ &= \tau\left(-\frac{\partial\phi}{\partial x} + \tau\frac{U^2}{2}\frac{\partial^2\phi}{\partial x^2}\right) \end{aligned} \tag{24}$$

by use of (3). Introducing finite difference approximations for the space derivatives

$$\phi(t+\tau)-\phi(t) = -\frac{U\tau}{2h}\left\{\phi(x+h)-\phi(x-h)\right\} + \frac{U^2\tau^2}{2h^2}\left\{\phi(x+h)+\phi(x-h)-2\phi(x)\right\} \tag{25}$$

a formula which uses only three points in the space derivatices. The amplitude factor is given by

$$|J_n| = \left\{1-\alpha^2(1-\alpha^2)(1-\cos kh)^2\right\}^{n/2} |J_0|$$

showing that the amplitude decreases with time, provided that $\alpha = \frac{U\tau}{h} < 1$, but less rapidly than in the forward-time backward-space formula. The phase speed is $\tan^{-1}\left\{\alpha \sin kh / 1-\alpha^2(1-\cos kh)\right\}/k\tau$ and some values are given in Table I.

Some general considerations

Table I gives the phase speeds computed for various actual speeds taking $kh = \pi/3$, h = 200 n.mi. corresponding to a wavelength of 1200 n.mi., and τ = 1 hr. These numbers have been chosen as representative of large scale atmospheric motions and of the grid size which we employ. The conclusion to be drawn is that as far as phase-speed is concerned the forward-time backward-space finite difference scheme is the best, the quadratic-time interpolation the next best, followed by the usual centred-time, centred-space formula. All formulae give much the same results for winds of 20 kt., and 50 kt. Table II(a) shows how the phase speed varied with the time interval employed, keeping the velocity and space interval constant. The upper lines in this table give the phase speeds corresponding to an actual speed of 50 kt. and the lower lines give those corresponding to an actual phase speed of 100 kt. Table II(b) gives similar data for an actual phase speed of 50 kt. and a shorter wavelength of 800 n.mi. corresponding to $kh = \pi/2$. It appears from both these tables that if the centred difference formula is used then the phase speed is more nearly correct as the time interval approaches the maximum allowed. Use of the forward-time step leads to exactly the opposite result, that the time step used should be as short as convenient. If the implicit time derivative is used the time step should be as short as possible; quadratic-time interpolation formulae behave similarly to the centred-time difference formulae while the backward-space forward-time formulae show that the phase speed may be exceeded in certain cases. Use of the forward-time and implicit-time derivatives leads to phase speeds which are less than $U \sin kh / kh$, whereas use of the other formulae leads to speeds greater than $U \sin kh / kh$, and generally less than $\underline{U}$. For forecasting the position of troughs and ridges the latter are preferable.

Table III(a) shows the amplitude factors, i.e. the change of amplitude which occurs in one time step, for various finite difference schemes. The amplitude factors given for the centred-time difference do not refer to the change in amplitude in one time step but to the upper limit which the amplitude may reach; the lower limit is unity. The advantage of using as short a time step as is convenient is shown up in this table; for example, using $\frac{1}{4}$ hr. time steps, one may advance 5 steps or $1\frac{1}{4}$ hr. making the same error as a single $\frac{1}{2}$ hr. time step. The exception appears to be with the forward-time, backward-space where little advantage is gained in using very short time steps and where it may be a positive advantage to use a long time step. If the amplitude of the phenomenon in question is of importance the centred-time derivative and the implicit-time derivative are superior to the others.

So far we have been considering the effect of replacing the differential equation by the difference equation on a single wave. The displacements that are considered in numerical weather prediction will be made up of the sum of such waves, and each wavelength will have a corresponding phase speed and amplitude factor, so that distortion of the original pattern cannot be avoided. We have also been considering a wave of infinite extent whereas in numerical weather prediction we usually have boundaries where the values are fixed without reference to the advective motion. Waves are then reflected in both directions and may obscure the differences arising between the various finite difference analogues of the differential equations.

Some experimental results

Explicit results could be obtained for a series of waves in place of a single wave but the expressions become very complicated even for two or three waves. It is easier to avoid the analysis and compute directly from the difference equations themselves. Computations have been made in this way using actual data representative of the large scale atmospheric motions. A steady west to east velocity of one third of a grid length per hour has been assumed, so that the effect should be to translate each value one grid length to the east after each 3 time steps. The assumption corresponds to a grid length of 180 n.mi. velocity of 60 kt. and a time step of 1 hr., except in the computations with the implicit-time derivative. The data were nine east-west cross sections of a field of tropopause pressure which had shown instability when used in a time integration. Most of the cross sections had sharp gradients and changes of gradient and a typical one (row 1, which will be used as an example throughout the paper) is shown in Fig.2. For computational purposes the value at the left hand end point was assumed to remain constant, and where the finite difference equations required the value at the right hand end point to be specified that value was also assumed to remain constant.

Striking differences between the different solutions have not had time to arise in 6 hr., but there is already a grouping of the curves according to the space difference used. Theory predicts that the centred-time, non-centred-space finite difference scheme will be unstable and there is a hint of this instability in one of the rows. The amplitude of the one-sided space and time derivatives appears to have been reduced as predicted but there is no sign of instability in the curve for the centred-space, forward-time scheme.

The results of computing with the various finite difference schemes over a 12 hr. interval (see Fig.3) show that the centred-time, non-centred-space scheme is clearly unstable and of no further interest. The other curves are well in phase, the differences in position of maxima and minima being only one grid length, except near the end-points. The non-centred-time, non-centred-space scheme shows a decrease in amplitude as expected, but is well in phase with the actual curve. The curves corresponding to the centred-space differences are nearly always in phase and agree remarkably well except that the amplitude of the curve corresponding to non-centred-time differences is greater than that corresponding to centred-time differences, as predicted.

It is possible to apply the implicit-time derivative scheme in a single step of 12 hr. or two steps of 6 hr; each computation was carried out. The under-estimation of the phase speed is apparent in Fig. 3. In each case the forecast made by using two 6 hr./steps was better than that made in one 12 hr. step as regards phase while the amplitudes were well-forecast and of similar values. The finite difference scheme using quadratic-time interpolation gives good fore-casts both as regards phase speed and amplitude, better than are given by using an implicit-time derivative.

The twelve-hour changes are shown in Fig. 4, the clearly unstable scheme having been omitted. The agreement in phase is well-marked, except for the two curves corresponding to implicit derivatives, while the decrease in amplitude of the backward-space forward-time difference scheme and the increase in amplitude of the centred-space, forward-time scheme are apparent.

The correlation between the actual and computed 12 hr. changes are given in Table V; the rows have been grouped for the computation. The 12 hr. statistics show the instability of the non-centred-space, centred-time difference scheme. The non-centred space and time scheme has a high correlation coefficient at 12 hr. as has the quadratic-time interpolation scheme and the usual centred-space and centred-time scheme. The position of the waves is well-forecast by these schemes. The comparatively large regression coefficients for the non-centred space and time scheme compared with the other two indicate that the amplitude of the disturbance is underestimated. On the other hand the centred-space non-centred time finite difference scheme leads to an overestimation of the amplitude.

Although theory predicts that the centred-space forward-time scheme should be unstable with increasing amplitude, some early time integrations using the Sawyer-Bushby model showed little difference in the forecasts made by this scheme and that of the centred-time, centred-space scheme. It was decided to test the two non-centred-space finite difference schemes by carrying out a time integration using the barotropic model with the data of 1500Z January 31st, 1953. The non-centred-space, centred-time scheme failed after 6 hr., owing to non-convergence of the solution of the Poisson equation. The non-centred space and time scheme led to a good forecast of the position of the trough over the North Sea but the amplitude was grossly underestimated. Fig. 5(a) shows this forecast, the actual contour chart to which it corresponds and the forecast made using the usual centred-space centred-time scheme. Clearly the latter forecast is much superior as regard the amplitude of the trough; the error in the former exceeds 1000 ft.

Summary: Of the finite difference schemes examined those using centred-space and centred-time or quadratic time interpolation appear to give the best all round results, and the former is preferable on account of its simplicity.

Table I

Computed and actual phase speeds (kt.$^{-1}$)

			Actual speed			
Time step used	τ	200	150	100	50	20
			Computed speed			
Centred	1 hr.	200	135	86	42	17
Forward	1 hr.	136	110	78	41	17
Implicit forward	1 hr.	156	120	81	41	17
	3 hr.	116	98	73	40	17
Implicit-time, backward-space	1 hr.	164	124	83	41	17
Quadratic	1 hr.	200	140	88	42	17
Forward-time backward-space	1 hr.	200	154	100	47	18

h = 200 n.mi. $kh = \pi/3$ Wavelength = 1200 n.mi.

Table II (a)

The variation of phase speed with time interval τ (hr.)

Time step used		$3\frac{1}{2}$	3	$2\frac{1}{2}$	2	$1\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$
		Speed (kt.)							
Centred difference	A	47	45	44	43	42	42	41	41
	B	--	--	--	100	90	86	83	83
Forward difference	A	36	37	38	39	40	41	41	41
	B	54	58	63	68	73	78	81	83
Implicit time derivative	A	40	42	40	42	41	41	41	41
	B	71	73	76	78	80	81	83	83
Backward-space, forward-time	A	51	51	51	50	48	46	44	43
	B	81	88	95	100	102	100	92	88
Quadratic-time interpolation	A	48	47	45	44	43	42	41	41
	B	--	--	--	100	94	88	86	83

h = 200 n.mi. $kh = \pi/3$ wavelength = 1200 n.mi.

A actual phase speed 50 kt.

B actual phase speed 100 kt.

Table II. (b)

The variation of phase speed with time interval τ (hr.)

Time step used	$3\frac{1}{2}$	3	$2\frac{1}{2}$	2	$1\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$
	Speed (Kt)							
Centred difference	39	36	34	33	33	32	32	32
Forward difference	26	27	28	30	31	31	32	32
Implicit time derivative	30	31	31	31	31	32	32	32
Backward-space, forward-time	52	53	54	50	46	41	36	34
Quadratic time-interpolation	48	44	41	37	35	33	32	32

h = 200 n.mi. $kh = \pi/2$ wavelength = 800 n.mi.

Actual phase speed = 50 kt.

Table III (a)

Amplitude factors

Time step used	$3\frac{1}{2}$	3	$2\frac{1}{2}$	2	τ(hr.) $1\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$
Centred difference *	2·35	1·729	1·414	1·232	1·118	1·049	1·012	1·003
Forward time	1·255	1·192	1·137	1·089	1·051	1·023	1·005	1·001
Implicit time derivative	1	1	1	1	1	1	1	1
Forward-time, backward-space	·9447	·901	·876	·866	·875	·901	·944	·970
Quadratic time interpolation	·977	·969	·970	·976	·985	·994	·998	1·000

h = 200 n.mi. $kh = \pi/3$ U = 50 kt. Wavelength = 1200 n.mi.

Table III (b)

Amplitude factors

Time step used	τ(hr.) 2	1	$\frac{1}{2}$	$\frac{1}{4}$
* Centred difference	1·073	1·019	1·005	1·001
Forward time	1·181	1·040	1·010	1·003
Forward time, backward-space	·957	·957	·973	·985
Quadratic interpolation	·999	1·000	1·000	1·000

h = 180 n.mi. $kh = \pi/5$ U = 60 kt. Wavelength = 1800 n.mi.

Table IV

Phase speeds corresponding to Table III (b)

Time step used	τ(hr.) 2	1	$\frac{1}{2}$	0
Centred difference	57	57	56	56
Forward difference	54	56	56	56
Implicit time derivative	57	56	56	56
Backward-space, forward-time	61	59	58	56
Quadratic time interpolation	58	57	56	56

h = 180 n.mi. $kh = \pi/5$ U = 60 kt. Wavelength = 1800 n.mi.

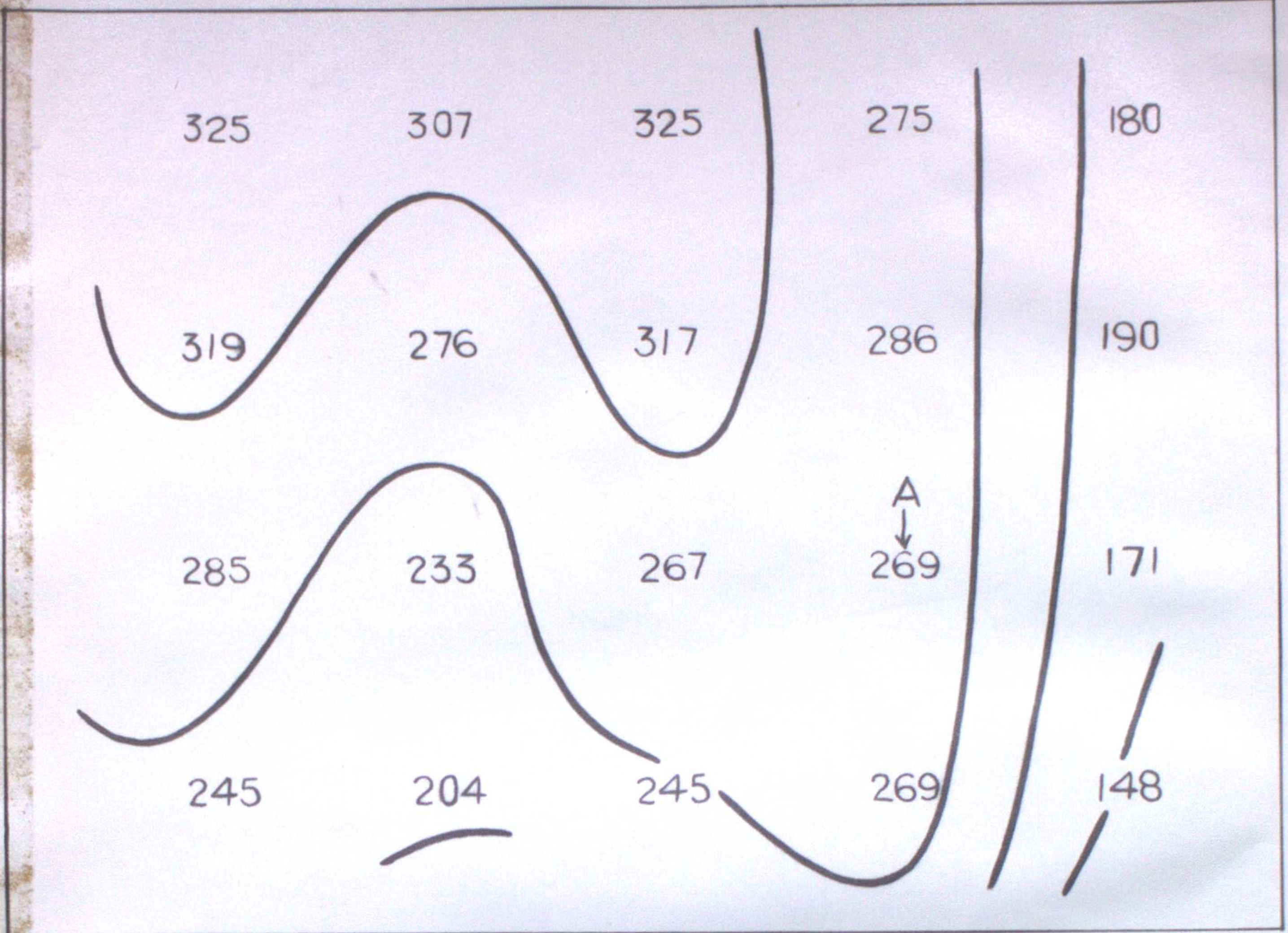
* The amplitude varies between unity and the tabular value.

Table V

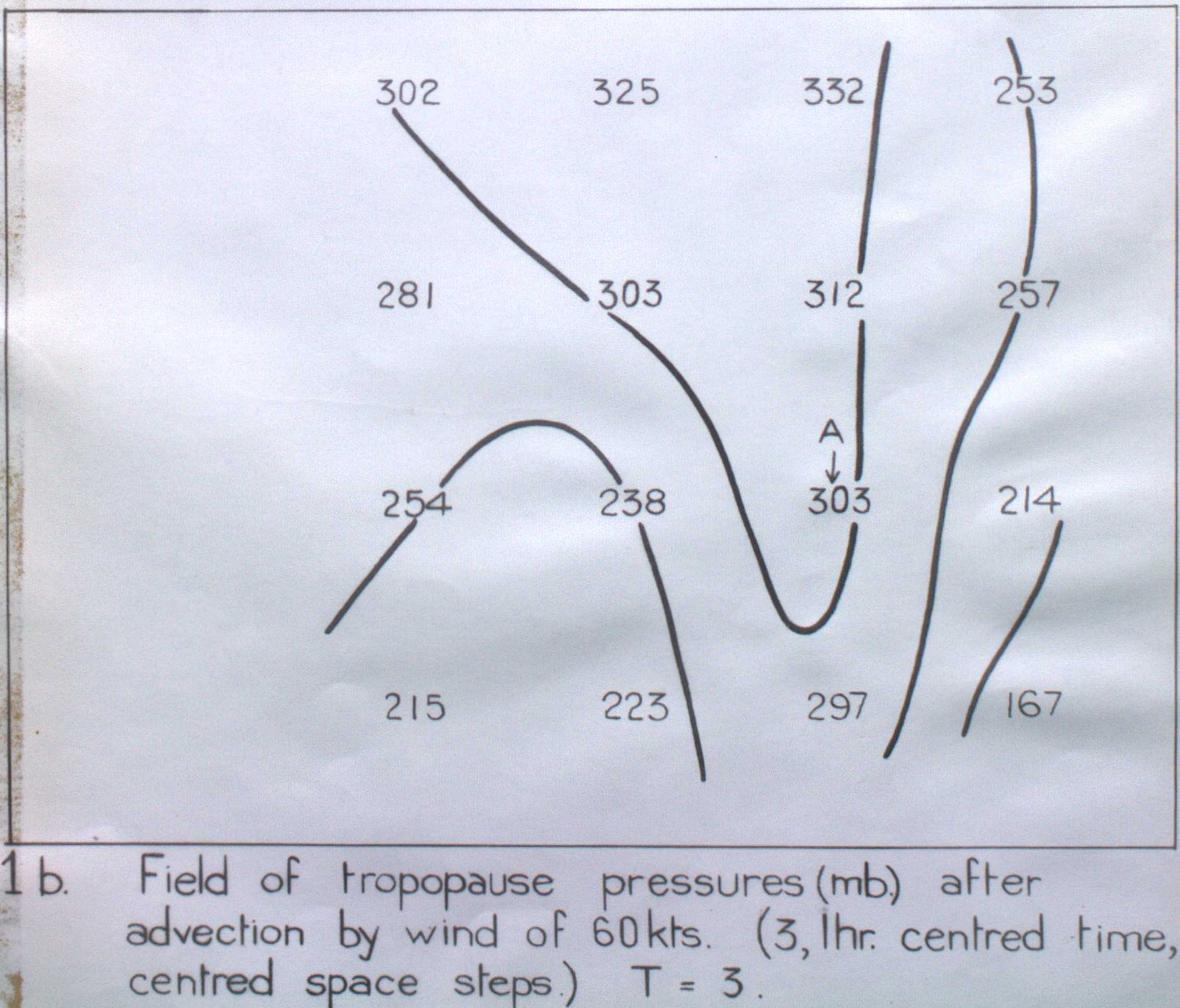
	Comp. Scheme		Corr. Coeff.			Regression coefficients Actual = A (computed) + B 3 hr.		6 hr.		12 hr.	
	Space	Time	3hr.	6hr.	12hr.	A	B	A	B	A	B
Rows 1,4,7	centred	centred	·964	·980	·989	1·11	0·20	1·01	-1·44	1·00	-0·83
	non-centred	centred	·960	·966	·222	1·12	1·36	1·10	0·85	0·10	10·11
	non-centred	non-centred	·966	·981	·998	1·07	1·07	1·09	0·88	1·21	6·53
	centred	non-centred	·950	·962	·973	1·07	-0·12	0·97	-2·27	0·90	3·22
	Quadratic	t	·980	·990	·994	1·12	0·27	1·05	-0·93	1·04	-0·38
	Implicit time 6 hr.			·966	·925			1·11	1·54	1·02	-3·16
	Implicit time 12 hr.				·897					1·07	-3·75
Rows 2,5,8	centred	centred	·970	·983	·990	1·10	0·60	1·01	-1·15	1·02	2·28
	non-centred	centred	·962	·965	·185	1·11	1·55	1·09	1·15	0·05	0·12
	non-centred	non-centred	·969	·982	·996	1·07	1·19	1·09	0·71	1·23	0·51
	centred	non-centred	·961	·968	·977	1·07	0·41	0·97	2·13	0·93	-6·18
	Quadratic	t	·982	·990	·994	1·10	0·66	1·06	-0·38	1·04	-0·91
	Implicit time 6 hr.			·969	·963			1·10	1·29	1·16	5·24
	Implicit time 12 hr.				·914					1·03	-5·07
Rows 3,6,9,	centred	centred	·968	·987	·989	1·10	0·32	1·03	0·71	0·99	-2·70
	non-centred	centred	·958	·973	·272	1·07	0·06	0·99	1·78	0·11	-9·57
	non-centred	non-centred	·968	·984	·997	1·07	0·90	1·10	1·76	1·22	7·03
	centred	non-centred	·963	·968	·982	1·12	1·34	1·12	2·56	0·92	-5·37
	Quadratic	t	·972	·994	·996	1·09	0·15	1·05	-0·47	1·03	-0·48
	Implicit time 6 hr.			·975	·932			1·13	0·49	1·06	12·20
	Implicit time 12 hr.				·925					1·15	7·89

References

(1) E. Knighting. An atmospheric model including the tropopause effects. - Sonderdruck aus: Bericht des Deutschen Wetterdienstes Nr. 38 Symposium uber Numerische Wettervorhersage in Frankfurt a.M. . 1956.



1a. Initial field of tropopause pressures (mb.)



1 b. Field of tropopause pressures (mb.) after advection by wind of 60 kts. (3, 1hr. centred time, centred space steps.) T = 3.

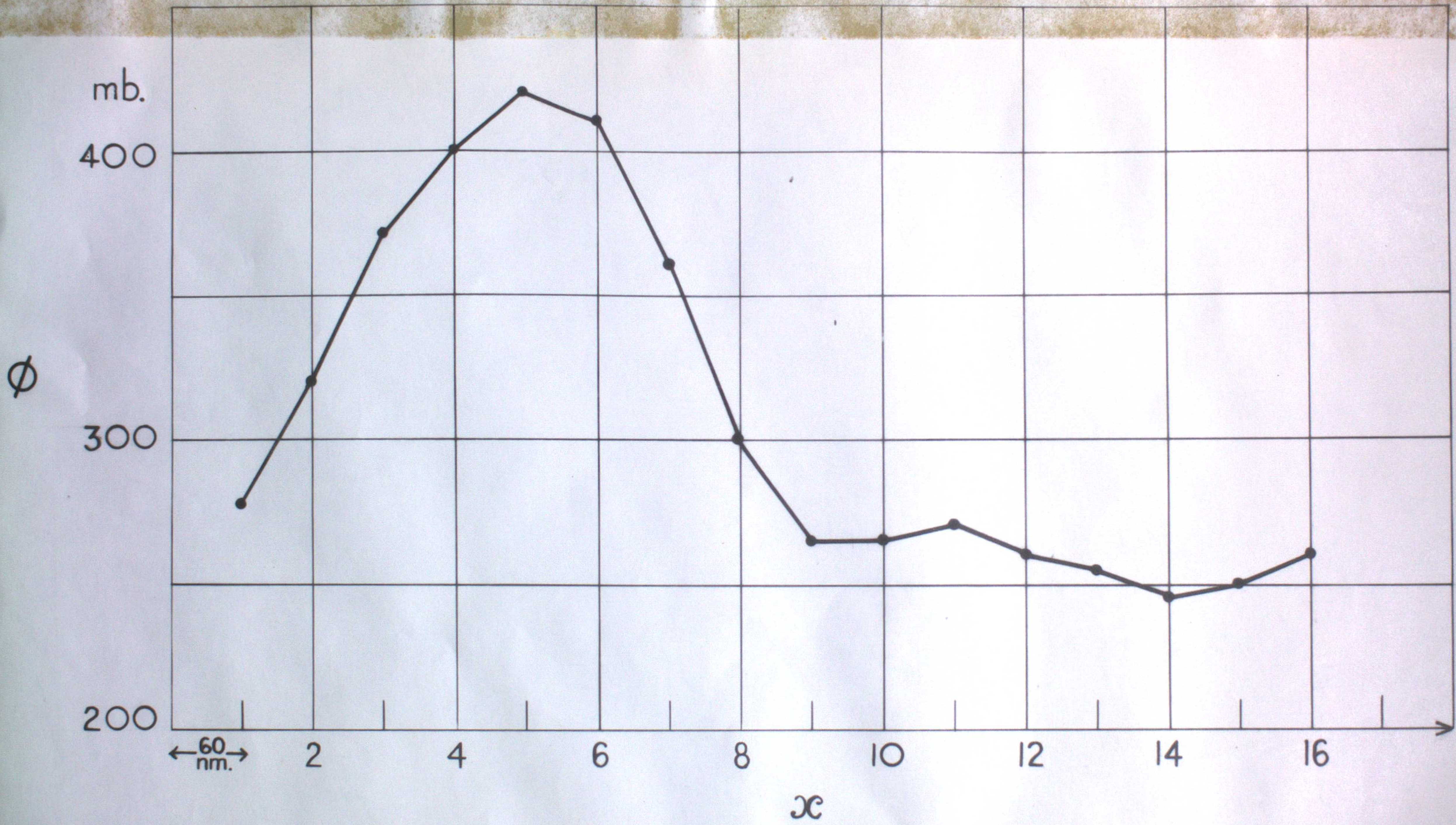


Fig. 2 Initial values of φ (row 1)

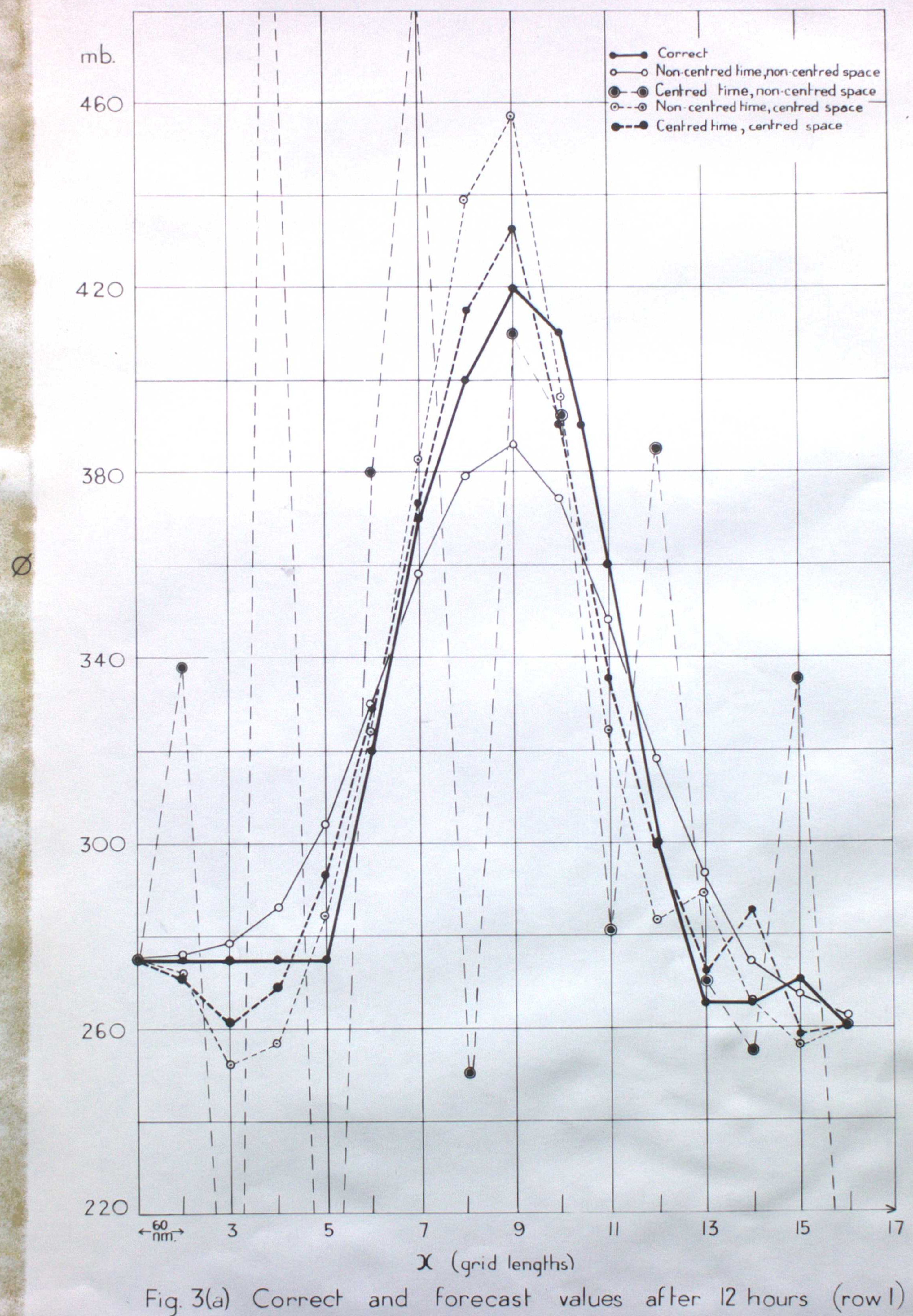


Fig. 3(a) Correct and forecast values after 12 hours (row 1)

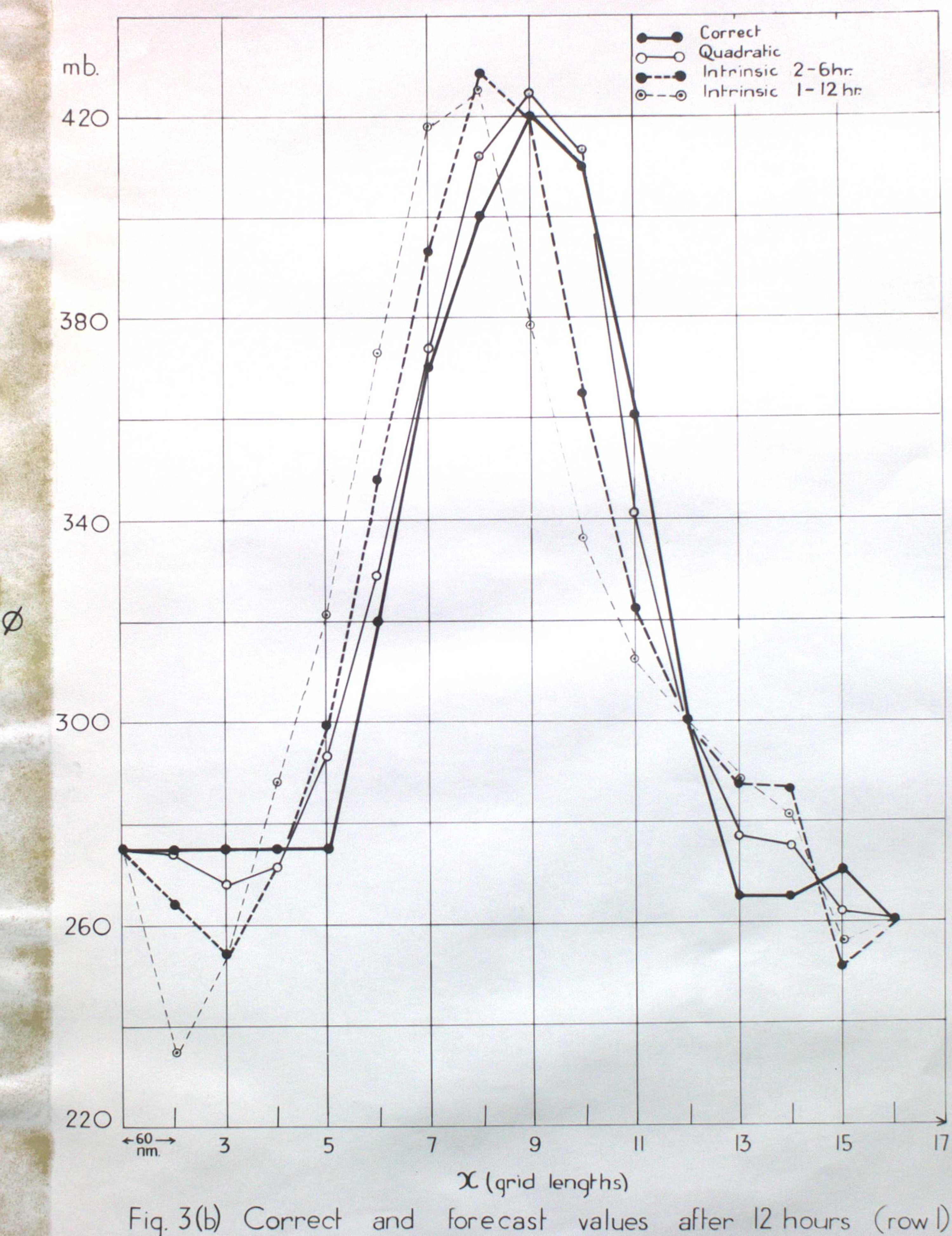


Fig. 3(b) Correct and forecast values after 12 hours (row I)

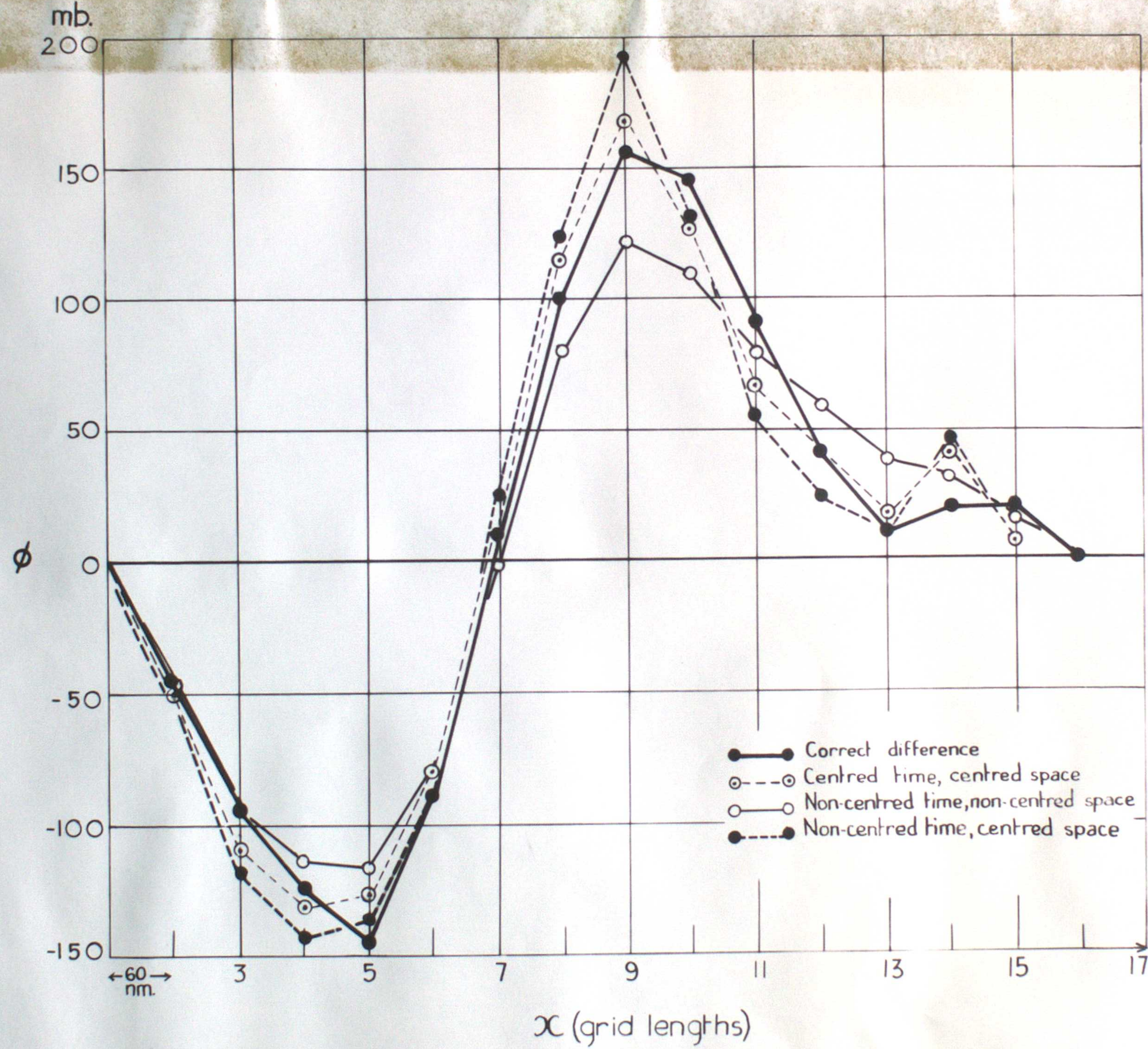


Fig. 4(a) Correct and forecast 12 hr. changes (row 1)

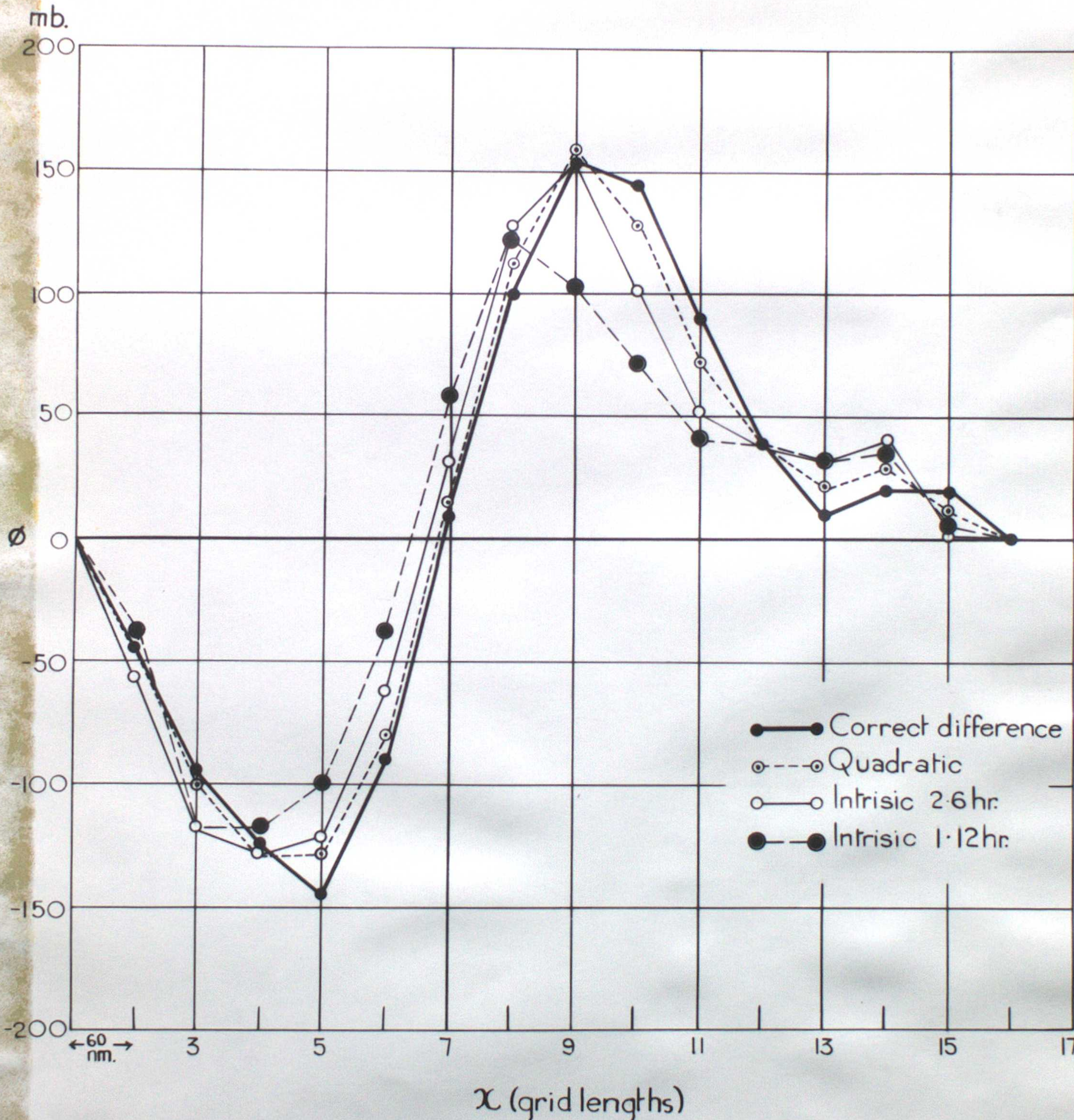


Fig. 4(b) Correct and forecast 12 hr. changes (row 1)

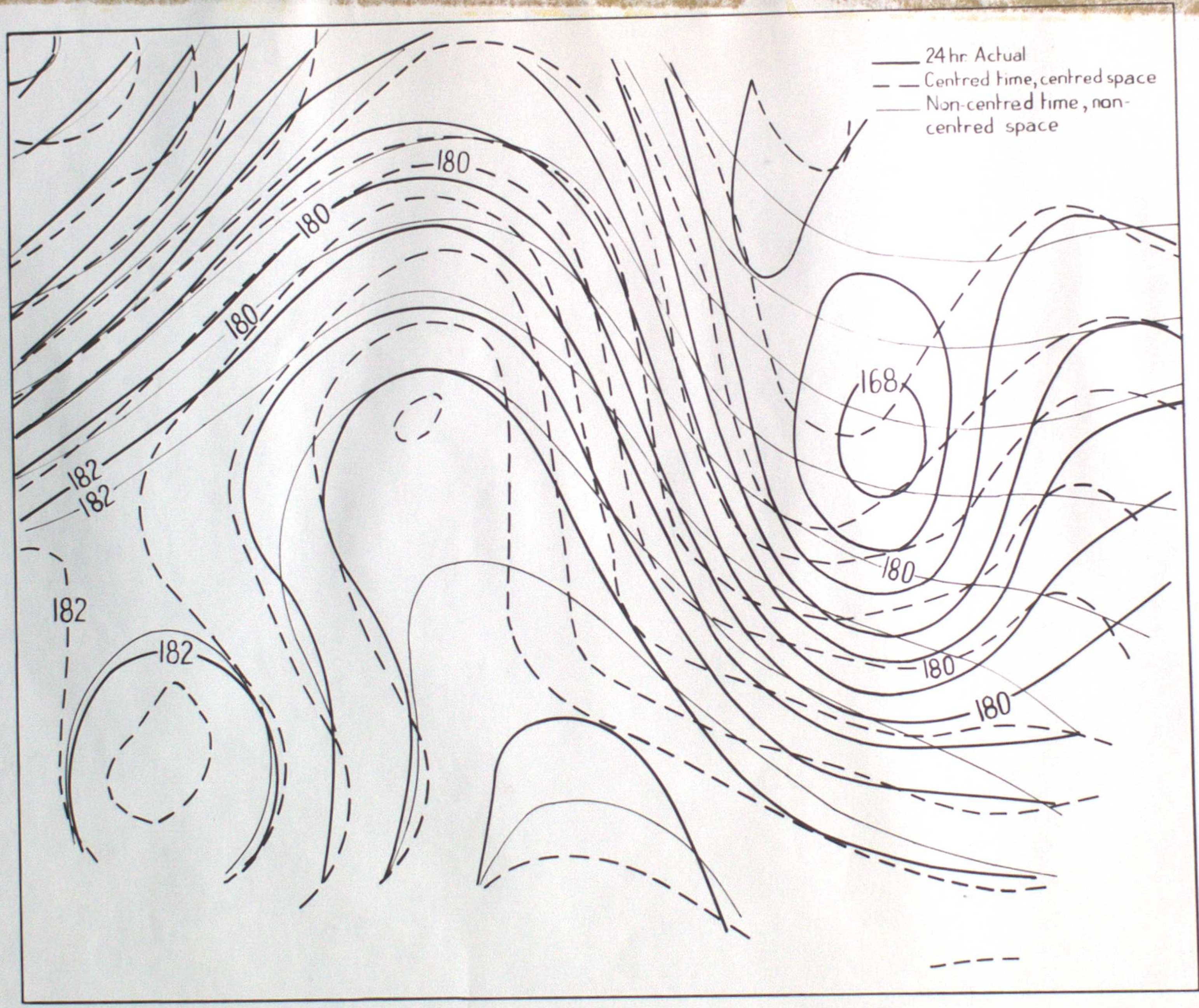


Fig.5 Comparison of 24 hr. forecasts of 500 mb. surface made, using centred time, centred space, and forward time, backward space finite difference schemes with actual field 1500 G.M.T. 31:1:53 (100's of feet)