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# THE PLANETARY BOUNDARY LAYER MODEL - BASIC FORMULATION

by

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## 1. INTRODUCTION

This paper describes the basic formulation of a meso-scale numerical model of the planetary boundary layer, outlines the progress that has been achieved so far and discusses proposals for the future development and use of the model.

After several years' experience with the Bushby-Timpson fine mesh 10-level model (Benwell et al, 1971) it has become clear that, although such finite difference models are able to predict the broad structure of certain sub-synoptic scale phenomena, a grid length of 100 km is too coarse to describe the detailed structure and local modification of meso-scale weather systems. For this reason it has been decided to develop an experimental limited area model with a grid length considerably smaller than has been used so far. The model is designed to forecast meso-scale flow in the lower troposphere and its interaction with the planetary boundary layer. Synoptic scale motion will be introduced by specifying time dependent lateral and upper boundary conditions provided by the 10-level model, but it is intended that the meso-scale model itself should contain a comprehensive description of the physics and dynamics of the boundary layer.

Because the eventual requirements of the model are not yet known, the computer programs have been written in a general way so that the configuration of the model may be altered easily without the need for major re-programming. At present the model has a 20km horizontal grid length and ten unequally spaced levels confined to the lowest 2 km of the atmosphere. However the spacing of the levels is arbitrary and may be altered at will, the number of levels can be changed up to a maximum of 20 and the grid length may be reduced or increased as desired. The number of grid points in the horizontal may be also changed subject to certain computer hardware limitations. With a 10 km grid an array of about 60x100 grid points is sufficient to cover the UK, although this would have to be increased to 100x100 to include the whole of Ireland. Most of the experiments so far have used a 61x61 array.

On the synoptic scale it is sufficient to regard the atmosphere as being in hydrostatic balance, but this assumption must be re-examined when considering flow with horizontal length scales of only a few kilometres. Examination of the linear theory of gravity waves suggests that non-hydrostatic effects are significant when the horizontal length scale is less than the vertical length scale. Although the model proposed (with a depth of 2 km and a horizontal grid length of 20 km) does not permit such waves, it may be argued that the small errors due to the hydrostatic approximation are such as to make the flow more sensitive to vertical accelerations than would otherwise be the case. Furthermore effects which depend on the dispersive qualities of gravity waves (such as the trapping of gravity waves by stable layers) are represented incorrectly in a hydrostatic model. It has therefore been decided to retain the full non-hydrostatic vertical equation of motion. This has the added advantage that future development of the model to shorter grid lengths and greater depths (where the hydrostatic approximation would be seriously in error) is possible.

The principal difficulty in using the non-hydrostatic equations is that they permit the vertical propagation of sound waves. The combination of a rapid phase speed (of order 300 m $sec^{-1}$) and a small vertical separation between the levels imposes severe stability restrictions on the time step for conventional explicit finite difference schemes. In order to avoid the excessive amount of computation that such a procedure would require, a a semi-implicit finite difference scheme has been developed which treats the main terms responsible for sound wave propagation implicitly and the remaining terms explicitly. The stability criterion is then governed by the wind speed and the speed of internal gravity waves, both of which are considerably less than the speed of sound. Gravity waves for example (with vertical length scales of less than 2 km) have horizontal propagation speeds of little more than 10 m $sec^{-1}$, even under conditions of extreme stability. Their vertical propagation speeds, however, may be of the order of 1 m $sec^{-1}$ and may impose limitations on the time step if the level separation is small. With the model described in this paper a time step of 100 secs is sufficient to ensure stability.

At the present time initial data for testing the model are being obtained by interpolation from the 10-level fine mesh model. Steps are taken to ensure that the wind and temperature fields satisfy some sort of approximate balance conditions in order to prevent the spurious development of gravity or sound waves in the early part of the forecasts. This is described in section 4.

Considerable difficulty was experienced, during the early attempts at integrating the model, in the specification of suitable boundary conditions. The set described in section 4 is not regarded necessarily as the best boundary conditions to use, but they do appear to give sufficiently stable results to enable integrations to be performed to at least 6 hours. They have been designed in such a way that changes on the synoptic scale can be fed into the model from the 10-level model, although in the forecasts performed so far all the external tendencies have been assumed to be zero.

At the time of writing a fully interactive parameterisation of the surface fluxes has not been included in the model. Section 5 contains a discussion of the proposed scheme. Tests conducted on observed data suggest that the method - a combination of Penman's evaporation equation and Clarke's method I (Penman, 1948, Clarke, 1971) - gives reasonable results.

## 2. BASIC EQUATIONS

The basic dynamical equations for the model are the three equations of motion, the continuity equation and the thermodynamic equation. Additional equations describing the transfer of water vapour etc. will be excluded from the current discussion. The main respect in which this set differs from those used in synoptic scale modelling lies in the inclusion of the non-hydrostatic terms in the vertical equation. The model also uses the height z as the vertical coordinate rather than pressure.

With the usual notation the three equations of motion are

$$\frac{Du}{Dt} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = K\nabla_H^2 u + \frac{\partial}{\partial z}\left(K_M \frac{\partial u}{\partial z}\right) \qquad (1)$$

$$\frac{Dv}{Dt} + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = K\nabla_H^2 v + \frac{\partial}{\partial z}\left(K_M \frac{\partial v}{\partial z}\right) \qquad (2)$$

$$\frac{Dw}{Dt} + g + \frac{1}{\rho}\frac{\partial p}{\partial z} = K\nabla_H^2 w \qquad (3)$$

where the Lagrangian derivative D/Dt is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}.\nabla$$

and the velocity $\underline{v}$ is given in component form by $\underline{v}$ = (u, v, w). $\nabla_H^2$ is the horizontal Laplacian operator. The continuity equation is

$$\frac{D\rho}{Dt} + \rho\,\nabla.\underline{v} = 0 \qquad (4)$$

and the thermodynamic equation is

$$\frac{D\theta}{Dt} = K\nabla_H^2\theta + \frac{\partial}{\partial z}\left(K_H \frac{\partial \theta}{\partial z}\right) \qquad (5)$$

The only source terms considered at present are those due to horizontal and vertical diffusion, although additional terms representing the effects of radiation and cloud physics will be included at a later date. The equations are somewhat simplified if a new pressure variable P is introduced where

$$P = \left(\frac{p}{p_s}\right)^{\kappa}$$

and $\kappa = R/c_p$, $R$ is the gas constant, $c_p$ is the specific heat at constant pressure and $p_s$ is a constant reference value of the pressure. Since the potential temperature $\theta$ is given by

$$\Theta = \frac{1}{\rho} \frac{p_s}{R} P^{(\kappa^{-1}-1)}$$

the pressure force term $\rho^{-1}\nabla p$ may be written as $c_p \Theta \nabla P$, and $\rho$ may be eliminated from the continuity equation (4) to give an equation for P of the form

$$c_p \frac{DP}{Dt} + \gamma R P \nabla . \underline{v} = \frac{\gamma R P}{\Theta}\left[ K \nabla_H^2 \Theta + \frac{\partial}{\partial z}\left( K_H \frac{\partial \Theta}{\partial z}\right)\right]$$

It is convenient at this stage to write the variables $P$ and $\Theta$ in terms of deviations $P_1$ and $\Theta_1$ from a steady neutrally stable state of the atmosphere $P_o(z)$ and $\Theta_o$ = constant, i.e.

$$P = P_o(z) + P_1(x, y, z, t)$$

$$\Theta = \Theta_o + \Theta_1(x, y, z, t)$$

We deduce from the hydrostatic equation that $P_o$ is given by

$$P_o = 1 - z/H$$

where we have chosen $P_o = 1$ at $z = 0$, and $H = c_p\Theta_o/g$ is a scale height for the basic atmosphere.

With this transformation, the equations (1) - (5) become

$$\frac{Du}{Dt} = fv - c_p(\Theta_o + \Theta_1)\frac{\partial P_1}{\partial x} + K\nabla_H^2 u + \frac{\partial}{\partial z}\left(K_H \frac{\partial u}{\partial z}\right) \tag{6}$$

$$\frac{Dv}{Dt} = -fu - c_p(\Theta_o + \Theta_1)\frac{\partial P_1}{\partial y} + K\nabla_H^2 v + \frac{\partial}{\partial z}\left(K_H \frac{\partial v}{\partial z}\right) \tag{7}$$

$$\frac{Dw}{Dt} = g\frac{\Theta_1}{\Theta_o} - c_p(\Theta_o + \Theta_1)\frac{\partial P_1}{\partial z} + K\nabla_H^2 w \tag{8}$$

$$\frac{D\Theta_1}{Dt} = K\nabla_H^2 \Theta + \frac{\partial}{\partial z}\left(K_H \frac{\partial \Theta}{\partial z}\right) \tag{9}$$

$$c_p\frac{DP_1}{Dt} = c_p\frac{w}{H} - \gamma R(P_o + P_1)\nabla . \underline{v} + \gamma R\frac{(P_o+P_1)}{(\Theta_o+\Theta_1)}\left[K\nabla_H^2\Theta_1 + \frac{\partial}{\partial z}\left(K_H\frac{\partial \Theta_1}{\partial z}\right)\right] \tag{10}$$

These equations describe the rates of change of the three wind components u, v, w and the perturbation potential temperature and pressure variables $\Theta_1$ and $P_1$. The absolute temperature

T may be calculated at any stage from

$$T = (\Theta_0 + \Theta_1)(P_0 + P_1)$$

The interesting features of this system are a) the inclusion of the non-hydrostatic terms in the vertical momentum equation and b) the fact that the pressure is predicted from the continuity equation rather than from a combination of the hydrostatic and thermodynamic equations as in most synoptic type models.

## 3. THE FINITE DIFFERENCE SCHEME

In the current form the model has ten levels set at unequal intervals in the vertical between 30 m and 2000 m above ground level, as indicated in Fig 1. The lowest levels are closer together in order to increase the vertical resolution there. The horizontal grid structure is given in Fig 2. Pressure and potential temperature are held at the main levels for each grid point and the horizontal wind components are kept midway between adjacent grid points as indicated. The vertical velocities are stored at intermediate levels midway between the main levels.

Since the fastest waves described by the system of equations (6) - (10) are sound waves (propagating both horizontally and vertically), the method of computation must be chosen to take this into account. If c denotes the speed of sound and $\delta z$ the minimum vertical grid separation (i.e. 50 m) then there is a stability restriction on the time step for a conventional explicit integration scheme of the form

$$\delta t < \delta z / c \approx 0.17 \text{ secs}$$

This is unreasonably restrictive and recourse must be made to semi-implicit methods in order to increase the integration time step. The procedure adopted here is to use an implicit scheme on the terms which involve the propagation of sound waves, while the remaining terms are treated explicitly. Provided a reasonably representative value of $\Theta_0$ is chosen, the speed of sound waves is determined to a high degree of accuracy by the temperature distribution of the basic atmosphere itself, since the departures amount to only a few per cent. Consequently we adopt a method similar to that described by Kwizak and Robert (1971) for the semi-implicit treatment of gravity waves; that is we treat implicitly only those terms which involve sound wave propagation in the basic adiabatic atmosphere, while all perturbation quantities are represented explicitly.

The finite difference approximations are centred in both space and time. The implicit terms are represented as an average between time levels (n+1) and (n-1), i.e. the pressure gradient term $c_p \Theta_0 \nabla P$ is replaced by
$\frac{1}{2} c_p \Theta_0 \nabla \left[ P^{(n+1)} + P^{(n-1)} \right]$ and the terms $c_p w/H - \gamma R P_0 \nabla . \underline{v}$ in the continuity equation appear as

$$\frac{1}{2} \left\{ g \left[ w^{(n+1)} + w^{(n-1)} \right] / H - \gamma R P_0 \nabla . \left[ \underline{v}^{(n+1)} + \underline{v}^{(n-1)} \right] \right\}$$

The explicit terms are evaluated either at time level (n) or (n-1) depending on stability requirements. After some rearrangement the finite difference forms of equations (6) - (10) may be written

$$u^{(n+1)} = u^{(n-1)} + X - \Theta_0 \delta t \frac{\partial \Pi}{\partial x} \qquad (11)$$

$$v^{(n+1)} = v^{(n-1)} + Y - \Theta_0 \delta t \frac{\partial \Pi}{\partial y} \qquad (12)$$

$$w^{(n+1)} = w^{(n-1)} + Z_1 - \Theta_0 \delta t \frac{\partial \Pi}{\partial z} \qquad (13)$$

$$\Theta_1^{(n+1)} = \Theta_1^{(n-1)} + \textcircled{H} \qquad (14)$$

$$c_p P_1^{(n+1)} = c_p P^{(n-1)} + \Phi + \frac{1}{\Theta_0} \delta t \left\{ g \, \delta w - \Gamma_0 \nabla . \delta \underline{v} \right\} \qquad (15)$$

where

$$\delta \underline{v} = \underline{v}^{(n+1)} - \underline{v}^{(n-1)}$$

$$\Pi = c_p \left[ P_1^{(n+1)} - 2 P_1^{(n)} + P_1^{(n-1)} \right]$$

and $\Gamma_0(z)$ is the square of the speed of sound in the adiabatic atmosphere defined by

$$\Gamma_0 \equiv c_0^2 = \gamma R \Theta_0 P_0(z)$$

The terms $X$ , $Y$ , $Z$ , $\textcircled{H}$ and $\Phi$ are given in Appendix 2.

If we eliminate $\delta w$ and $\delta \underline{v}$ between equations (11) - (15) we arrive at the three dimensional partial differential equation

$$\nabla_H^2 \Pi + \left\{ \frac{\partial^2}{\partial z^2} - \frac{g}{\Gamma_0} \frac{\partial}{\partial z} - \frac{1}{\Gamma_0 \delta t^2} \right\} \Pi = F \qquad (16)$$

where F is a function of the variables at time levels (n) and (n-1) only. The algebraic form of $F$ is given in Appendix 1. With the values of the variables at time levels (n) and (n-1) known, the new pressure field at time level (n+1) can be found by solving (16), and the values of the wind components $u^{(n+1)}$, $v^{(n+1)}$ and $w^{(n+1)}$ determined from (11) - (13). Since the thermodynamic equation (14) does not contain terms involved in sound wave propagation it can be handled in a direct explicit fashion.

The pressures at levels 1 - 9 are obtained from equation (16) while the pressure at the top level is obtained from the boundary conditions. If finite differences are substituted for the bracketed term in (16) we obtain the vector equation

$$\nabla_H^2 \underline{\Pi} + \underline{\underline{A}}\, \underline{\Pi} = \underline{F} \qquad (17)$$

$\underline{\Pi}$ is a column vector representing the unknown pressures at each of the levels and $\underline{\underline{A}}$ is a 9x9 matrix which is the finite difference equivalent of the vertical differential operator. The form of the matrix $\underline{\underline{A}}$, which is given in Appendix 1, is the same at each grid point in the horizontal.

Equation (17) represents a set of 9 coupled Helmholtz equations to solve for $\underline{\Pi}$. In order to reduce the problem to one of solving a set of two-dimensional equations we calculate the row eigenvectors $\underline{\eta}_i$ and eigenvalues $\lambda_i$ of $\underline{\underline{A}}$ where

$$\underline{\eta}_i \underline{\underline{A}} = \lambda_i \underline{\eta}_i$$

If $\underline{\underline{H}}$ is the matrix of eigenvectors

$$\underline{\underline{H}} = \begin{pmatrix} \underline{\eta}_0 \\ \vdots \\ \underline{\eta}_8 \end{pmatrix}$$

then

$$\underline{\underline{H}}\,\underline{\underline{A}}\,\underline{\underline{H}}^{-1} = \underline{\underline{\mathrm{diag}}}(\lambda_i)$$

By pre-multiplying (17) by $\underline{\underline{H}}$ and writing $\underline{\zeta} = \underline{\underline{H}}\,\underline{\Pi}$ and $\underline{\beta} = \underline{\underline{H}}\,\underline{F}$ we have the uncoupled system

$$\nabla_H^2 \zeta_i + \lambda_i \zeta_i = \beta_i \qquad i = 0, 1, \cdots, 8 \qquad (18)$$

This set of 9 two-dimensional Helmholtz equations can be solved by ADI techniques. Each equation represents a different sound wave mode which, because of the linearity of the system, is separable from the other modes.

The computation may however be shortened by noticing that for the current model configuration all the eigenvalues except one $\lambda_0$ (corresponding to the horizontally propagating mode) are much larger than the eigenvalues of the $\nabla_H^2$ operator. This suggests that only for this root need the Helmholtz equation be solved. For the other (vertically propagating) modes we write

$$\lambda_i \zeta_i = \beta_i \qquad i = 1, 2, \cdots, 8 \qquad (19)$$

Although this appears at first sight to be an approximation, we note that since the variable $\Pi$ is a second order difference in time of the pressure $P$, the use of (19) instead of (18) for the internal sound wave modes simply results in different expressions for the higher order terms in the finite difference expansion. Indeed it may be shown that it is equivalent to treating the uncoupled horizontal pressure gradients explicitly and the vertical gradients implicitly (this is discussed further in Appendix 2). The theoretical examination of the system suggests that it is stable provided that the overall depth of the model is sufficiently small compared with the horizontal grid length. For a model with a larger depth than the present model only those waves with a sufficiently small vertical wavelength can be treated by this simplified procedure. This method of using different finite difference schemes for the uncoupled modes has many similarities with the selective semi-implicit treatment of gravity waves described by Burridge. It reduces the computational and core storage requirements considerably since only one Helmholtz equation need be solved.

The final pressure at time level (n+1) may be recovered by recoupling the modes using

$$\underline{\Pi} = \underline{\underline{H}}^{-1} \underline{\xi}$$

## 4. INITIAL DATA AND BOUNDARY CONDITIONS

a) Initial data.

Since the model is in its early stages of development no effort has yet been devoted to the analysis of observed data. At present, the basis of the initial data is an interpolated pressure field extracted from the 10-level fine mesh model output. The various steps used in producing consistent fields of wind, pressure and potential temperature are as follows:

i) Extraction of geopotentials at the 1000 mb, 900 mb, 800 mb, and 700 mb levels of the 10-level model.

ii) Representation in the horizontal of the height fields by a least squares fit in terms of Chebyshev polynomials (see Appendix 3).

iii) Evaluation of the Chebyshev representation at the grid points of the planetary boundary layer model.

iv) Interpolation of the pressure vertically (using a cubic fit) to model levels.

v) Evaluation of the potential temperature field from the hydrostatic equation.

vi) Solution of the Ekman equations (which express a balance between the Coriolis, friction and pressure gradient terms) to obtain the horizontal winds.

vii) Evaluation of the vertical velocity from a balance equation (see Appendix 4) obtained by eliminating time derivaties between the thermodynamic and continuity equations and the equation

$$\frac{D}{Dt}\left(\rho g + \frac{\partial p}{\partial z}\right) = 0$$

(v) and (vii) ensure that vertically propagating sound waves are filtered out during the first time step, while (vi) gives an approximate balance condition between the horizontal wind and the pressure fields.

b) Boundary conditions.

Although it is eventually intended to feed in tendencies at the lateral and upper boundaries of the model using results obtained from the 10-level model, the test integrations performed so far have used zero tendencies. At present the normal wind component is specified at all points round the lateral boundary and the tangential wind component, the pressure and the potential temperature are specified at inflow points. At outflow points the potential temperature changes are calculated by **upstream** advection, while the tendencies of the tangential wind and the pressure on the boundary are set equal to the values of the tendencies one grid point inside. The horizontal diffusion coefficient $K$ is also increased in a zone 3 grid lengths wide round the boundary. At the bottom of the model the vertical velocity is set equal to zero (in the absence of topography) and at the upper level all variables are held constant.

## 5. PARAMETERISATION OF THE SURFACE FLUXES

Near the earth's surface vertical gradients tend to be large and the structure of the atmosphere is dominated by turbulent diffusion. In this section the proposed representation of the surface layer is discussed. The lowest 30 m are parameterised as a 'constant flux layer' which is assumed to adjust instantaneously to changes in external parameters, and whose bulk tendencies are ignored in the energy and momentum budgets. The method discussed below for determining the vertical fluxes of heat, moisture and momentum in the surface layer is a combination of Clarke's method I (Clarke, 1970) and Penman's evapo-transpiration techniques (Penman, 1948, Monteith, 1964). The procedures over land differ from those over the sea and will be dealt with separately.

a) Surface fluxes over land.

There are five equations used to describe the transfer processes in the surface layer

(i) the surface evapo-transpiration equation

$$E_0 = \rho \frac{\hat{q}_0 - q_0}{r_s} \tag{20}$$

(ii) the momentum flux equation

$$\tau_0 = \rho C_D u_1^2 \tag{21}$$

(iii) the heat flux equation

$$H_0 = -\rho c_p C_H u_1 (\theta_1 - \theta_0) \tag{22}$$

(iv) the water vapour flux equation

$$E_0 = -\rho C_H u_1 (q_1 - q_0) \tag{23}$$

(v) the radiation balance equation

$$R_N - G = H_0 + L E_0 \tag{24}$$

where $R_N$ is the net radiation, $G$ is the heat flux into the ground and $L$ is the latent heat. The subscript 0 refers to ground level and the subscript 1 refers to the first level of the main planetary boundary layer model (i.e. 30 m). $\hat{q}_0$ is the saturated specific humidity at the surface. It is related to the saturated value at level 1, $\hat{q}_1$, by the approximate form of the Clausius-Clapeyron equation

$$\hat{q}_0 = \hat{q}_1 [1 - \Delta (T_1 - T_0)]$$

where

$$\Delta = \frac{0.622\, L}{R T_1^2}$$

Assuming for the present that the drag coefficient $C_D$ and the heat (and moisture) transfer coefficient $C_H$ are known, then the five equations are sufficient to determine the three surface fluxes $\tau_0$, $H_0$ and $E_0$ together with the surface temperature $T_0$ and humidity $q_0$, in terms of $(R_N - G)$ and values of parameters given by the finite difference model at level 1.

After some algebraic re-arrangement we obtain

$$T_1 - T_0 = \frac{\hat{q}_1 - q_1 - [1 + r_s C_H u_1]\{(R_N - G)/\rho C_H u_1 + g(z_1 - z_0)\}/L}{\hat{q}_1 \Delta + [1 + r_s C_H u_1] c_p / L} \tag{25}$$

$$\theta_1 - \theta_0 = T_1 - T_0 + g(z_1 - z_0)/c_p \qquad (26)$$

$$q_1 - q_0 = -c_p(\theta_1 - \theta_0)/L - (R_N - G)/\rho C_H u_1 L \qquad (27)$$

The parameter $r_s$ appearing in equation (20) is known as the surface resistance (measured in units of secs $cm^{-1}$). It is essentially a function of the leaf structure of the plants and has a large diurnal variation, with a minimum during the day and a maximum at night when transpiration virtually ceases.

In order to assess values of $r_s$ for each grid point, the type of surface has been extracted every $3\,{}^1/3$ km over the UK from $\frac{1}{4}$ inch Ordinance Survey maps according to a four fold categorisation: 1) towns, 2) woods, 3) open countryside 4) water. The results, which are shown in Fig 3, can be used for estimating the fraction of a grid square covered by a particular type of surface. Measurements (see Monteith, 1964) suggest a minimum day-time value of the surface resistance for most crops and grasses of about 0.3 secs $cm^{-1}$ and a value of 1.0 secs $cm^{-1}$ for pine forests. Little is known about the influence of towns and one must guess a value from an assumed proportion of vegetation within the urban area, say 10 secs $cm^{-1}$.

The method of using this information in the model has not yet been finally agreed but it is suggested that for the land categories 1) - 3) we form a composite surface resistance $R_s$

$$\frac{1}{R_s} = \sum_{i=1}^{3} \frac{a_i}{r_{s_i}}$$

where $a_i$ and $r_{s_i}$ are the fraction of a grid square and the value of the surface resistance appropriate to category i. Where part of the grid square is covered by water we may perform the calculation as before for the land part of the grid square, but then augment the evaporation by writing

$$E_{total} = E_{land}(1 - a_4) - \rho C_H u_1 (q_1 - \hat{q}_0) a_4$$

where $a_4$ is the fraction of the grid square occupied by water. When the entire grid square is covered by water it is assumed that the fetch is sufficient to allow a different turbulent regime to be established and this is discussed in the next subsection.

These calculations lead to a value for the potential evaporation - that is the evaporation which would occur if an adequate supply of soil moisture were available to the plants. In general the evaporation depends on the soil moisture deficit, and this can be taken into account, as suggested by Priestley and Taylor (1972), by multiplying the potential evaporation by an empirical function of the water content of the ground.

b) Surface fluxes over the sea.

Over the sea the air is assumed to have the same temperature as the sea surface and be saturated at that temperature. Four equations are used to describe the surface fluxes

(i) Charnock's equation

$$z_0 = \alpha \tau_0 / \rho g$$

(ii) the momentum flux equation

$$\tau_0 = \rho C_D u_1^2$$

(iii) the heat flux equation

$$H_0 = -\rho C_H u_1 (\theta_1 - \theta_0)$$

(iv) the water vapour flux equation

$$E_0 = -\rho C_H u_1 (q_1 - \hat{q}_0)$$

Charnock's equation, which relates the roughness length $z_0$ to the surface stress, is intended to describe the changes in the nature of the sea surface under different atmospheric conditions. However it is essentially a hypothetical relationship based on dimensional arguments and there is considerable disagreement about the value of the empirical constant $\alpha$.

c) The determination of the drag and heat transfer coefficients.

The assumptions of similarity theory for the constant flux layer lead to relationships between vertical gradients and vertical fluxes within the layer. Integrated throughout the depth of the surface layer they become

$$C_D^{-1/2} = \int_{z_0}^{z_1} \frac{1}{kz} \Phi_M\left(\frac{z}{L}\right) dz \qquad (28)$$

$$C_H^{-1} C_D^{1/2} = \int_{z_0}^{z_1} \frac{1}{kz} \Phi_H\left(\frac{z}{L}\right) dz \qquad (29)$$

where $L = C_D^{3/2} C_H^{-1} / ks$ , $k$ is the Von Karman constant and $s$ is the bulk Richardson number defined by

$$s = g z_1 \left[ (\theta_1 - \theta_0)/\theta_1 + .61 (q_1 - q_0) \right] / u_1^2$$

Using empirically derived functions $\Phi_M$ and $\Phi_H$ we may solve the simultaneous equations (28) and (29) numerically to obtain

$$C_D = C_D(r, s)$$

$$C_H = C_H(r, s)$$

where $r = \log(z_1/z_0)$. Values of $C_D$ and $C_H$ have been tabulated in the computer as a look-up table, where the function $\Phi_M$ and $\Phi_H$ have been taken as the Webb-Dyer-Clarke combination used by Clarke (1970). The variation of the roughness length over the UK can be assessed from the estimated values of the neutral geostrophic drag coefficient obtained by Carson and Smith (1974) for 10 km squares on the national grid.

Since both $C_D$ and $C_H$ depend on $(\theta_1 - \theta_0)$ and $(q_1 - q_0)$, an iterative procedure is required to solve equations (25) - (27) for the surface temperature and humidity. A similar iterative procedure is required over the sea since $C_D$ and $C_H$ are also functions of $z_0$. However it is probably sufficiently accurate to use the values of $C_D$ and $C_H$ for the previous time step in equations (25) - (27) and then to calculate new values of $C_D$ and $C_H$ consistent with the derived surface temperatures and humidities for use at the next time step.

## 6. DISCUSSION

All the integrations performed so far have been tests to prove the viability of the dynamical system and the numerical scheme, rather than experiments to examine meteorological phenomena themselves. Fig 4 shows the variation of $u$, $v$, $p$ and $\theta$ at level 1 and $w$ at level 5 for an arbitrary grid point during the first 2½ hours of an integration using a grid length of 20 km and a time step of 100 secs. Constant values of $K_M$ = 2.5 m$^2$ sec$^{-1}$ and $K$ = 5x10$^2$ m$^2$ sec$^{-1}$ have been used (except near the boundary where $K$ is increased to 5x10$^4$). $K_H$ has been taken to be zero.

The initial adjustment, which is most noticeable in $u$ and $v$, appears to be quite rapid, suggesting that the initialisation procedure described in section 4 is adequate. Otherwise the variations are quite smooth although there is an indication of a 30 min oscillation in the wind field. A detailed examination of the vertical velocities however reveals a small amplitude oscillation (about 0.05 cm sec$^{-1}$) with a period of $4\delta t$. The analysis of appendix 2 suggests that this consists of truncated sound waves whose frequencies are too large to be represented accurately in a model with a time step of 100 secs. The variation of the wind hodograph at an arbitrary grid point is illustrated in Fig 5. The Ekman spiral shape is maintained and there is no evidence of roughness developing in the vertical except near the top where the maintenance of constant boundary conditions has led to the development of large shears.

The next stage in the development of the model will be to include a fairly comprehensive parameterisation of physical processes. These include (i) the variation of $K_M$ and $K_H$ with height and the matching of the fluxes in the main model to those in the constant flux layer, (ii) the physics of cloud and rain, (iii) a radiation scheme, and (iv) topographic effects.

Some progress has been made on the design of a scheme for including topography by using the height above ground level (z - H) as the vertical coordinate instead of z. The topographic height over the UK has been extracted from Ordinance Survey ¼ in maps every 3 1/3 km and this has been used to provide mean values for each grid square (Fig 6). Difficulties are likely to arise due to the steepness of the slope in some parts of the country.

Methods of parameterising radiation, cloud physics and vertical fluxes have not yet been decided on, though several methods have been suggested in the scientific literature, for example the parameterisation of cloud physics used by Miller and Pearce (1974) in a dynamical model of a cumulo-nimbus cloud.

It is intended to use the model in the first instance as a research tool to examine particular cases which have been the subject of intensive field study - for example sea breezes, mesoscale structure of fronts, lee waves. In addition the model may be used to investigate simple artificial flows in order to tie up linear theory with finite amplitude development, leading to a better understanding of the type of meso-scale phenomena which may arise. This is a necessary preliminary step before meso-scale effects may be parameterised in synoptic scale and general circulation models. The ultimate use of the model as a forecasting tool is a long way off and it would be wise to assess its performance before commenting on its operational possibilities.

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## APPENDIX 1

The terms $X$ , $Y$ , $Z$ , $\Theta$ , and $\Phi$ which appear in the finite difference scheme given in section 3 are

$$X = 2\delta t\Big[f v^{(n)} - \underline{v}^{(n)}\cdot\nabla u^{(n)} - (\theta_0 + \theta_1^{(n)})\, c_p \frac{\partial p_1^{(n)}}{\partial x} + K\nabla_H^2 u^{(n-1)} + \frac{\partial}{\partial z}\Big(K_M \frac{\partial u^{(n-1)}}{\partial z}\Big)\Big]$$

$$Y = 2\delta t\Big[-f u^{(n)} - \underline{v}^{(n)}\cdot\nabla v^{(n)} - (\theta_0 + \theta_1^{(n)})\, c_p \frac{\partial p_1^{(n)}}{\partial y} + K\nabla_H^2 v^{(n-1)} + \frac{\partial}{\partial z}\Big(K_M \frac{\partial u^{(n-1)}}{\partial z}\Big)\Big]$$

$$Z = 2\delta t\Big[g\theta_1^{(n)}/\theta_0 - \underline{v}^{(n)}\cdot\nabla w^{(n)} - (\theta_0 + \theta_1^{(n)})\, c_p \frac{\partial p_1^{(n)}}{\partial z} + K\nabla_H^2 w^{(n-1)}\Big]$$

$$\Theta = 2\delta t\Big[-\underline{v}^{(n)}\cdot\nabla\theta_1^{(n)} + K\nabla_H^2\theta_1^{(n-1)} + \frac{\partial}{\partial z}\Big(K_H \frac{\partial\theta_1^{(n-1)}}{\partial z}\Big)\Big]$$

$$\Phi = 2\delta t\Big[-c_p\underline{v}^{(n)}\cdot\nabla p_1^{(n)} - \gamma R p_1^{(n)}\nabla\cdot\underline{v}^{(n)} + c_p w^{(n-1)}/H - \gamma R p_0 \nabla\cdot\underline{v}^{(n-1)} + \frac{\gamma R(p_0 + p_1^{(n)})}{\theta_0 + \theta_1^{(n)}}\Big\{K\nabla_H^2\theta_1^{(n-1)} + \frac{\partial}{\partial z}\Big(K_H\frac{\partial\theta_1^{(n-1)}}{\partial z}\Big)\Big\}\Big]$$

In the above expressions the diffusion terms have been evaluated at time level (n-1) to ensure computational stability in the numerical scheme.

The function $F$ appearing in the Helmholtz equation (16) is

$$F = \frac{1}{\Gamma_0\delta t}\Big[\frac{2c_p}{\delta t}\big(p_1^{(n)} - p_1^{(n-1)}\big) - \frac{1}{\delta t}\Phi - c_p Z/H + \gamma R p_0\Big(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\Big)\Big]$$

The matrix $\underset{\approx}{A}$ , introduced in section 3, can be written in the form

$$A_{k,j} = E_{k,j} - \frac{1}{\delta t^2 c_0^2(k)}\,\delta_{k,j}$$

where $A_{k,j}$ is the component in row $k$ and column $j$ , $c_0(k)$ is the speed of sound at the k'th level and

$$\delta_{k,j} = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases}$$

The components of the matrix $\underline{\underline{E}}$ may be written as

$$E_{k,k+1} = B_4(k) - \frac{g}{c_0^2(k)} B_1(k)$$

$$E_{k,k} = B_5(k) - \frac{g}{c_0^2(k)} B_2(k)$$

$$E_{k,k-1} = B_6(k) - \frac{g}{c_0^2(k)} B_3(k)$$

for k = 2, ......... 8. For k = 1 $E_{1,1}$ and $E_{1,2}$ are given by

$$E_{1,1} = -2 / b^2(1)$$

$$E_{1,2} = - E_{1,1}$$

and for k = 9

$$E_{9,8} = B_6(9) - \frac{g}{c_0^2(9)} B_3(9)$$

$$E_{9,9} = - E_{9,8}$$

all other elements of $\underline{\underline{E}}$ being zero. $b(1)$ denotes the separation between levels 1 and 2 of the model. $B_1$ , $B_2$ , $B_3$ , $B_4$ , $B_5$, and $B_6$ are the coefficients appearing in the finite difference expressions for the first and second order vertical derivatives. They are obtained by means of a Taylor series expansion about level k to give

$$\frac{\partial \phi}{\partial z}(k) = B_1(k)\,\phi(k+1) + B_2(k)\,\phi(k) + B_3(k)\,\phi(k-1) + \cdots\cdots$$

$$\frac{\partial^2 \phi}{\partial z^2}(k) = B_4(k)\,\phi(k+1) + B_5(k)\,\phi(k) + B_6(k)\,\phi(k-1) + \cdots\cdots$$

## APPENDIX 2

## THE TREATMENT OF SOUND WAVES BY THE FINITE DIFFERENCE SCHEME

a) The continuous solution

We consider equations (6) - (10) and apply them to the study of sound wave propagation in the basic adiabatic atmosphere. The linearised form of the equations is

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + c_p \Theta_0 \frac{\partial P_1}{\partial x} &= 0 \\ \frac{\partial w}{\partial t} + c_p \Theta_0 \frac{\partial P_1}{\partial z} &= 0 \\ c_p \Theta_0 \frac{\partial P_1}{\partial t} - g w + c_0^2\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) &= 0 \end{aligned}\right\} \qquad (2.1)$$

where $c_0$ is the speed of sound in the unperturbed adiabatic atmosphere. The motion has been confined to the x-z plane for simplicity and the depth of the model is assumed to be h.

If we eliminate u and w from equations (2.1) we obtain an equation for $P_1$ of the form

$$\frac{\partial^2 P_1}{\partial t^2} + M P_1 - c_0^2 \frac{\partial^2 P_1}{\partial x^2} = 0 \qquad (2.2)$$

where the operator $M = -c_0^2 \frac{\partial^2}{\partial z^2} + g\frac{\partial}{\partial z}$ . The eigen values $\mu_r$ of $M$ are given by

$$M \xi_r = \mu_r \xi_r$$

where $\xi_r$ is the r'th eigen function of $M$ .

For purely vertical oscillations of the form $\xi_r = \phi(z)\, e^{i\sigma_r t}$ we find from equation (2.2) $\mu_r = \sigma_r^2$ , where $\sigma_r$ is the frequency of the r'th harmonic of the acoustic oscillations of a column of air of depth h. It can be shown (c.f. Lamb, 1932) that the frequencies are given approximately by

$$\sigma_r \approx \frac{r \pi c_s}{h} \qquad r = 0, 1, 2, \ldots\ldots$$

where $c_s$ is a mean value of the speed of sound.

In the general case where there is a horizontal dependence of the form $e^{ikx}$, the frequency $\sigma$ satisfies the equation

$$\sigma^2 \approx \sigma_r^2 + c_s^2 k^2$$

Since the maximum value of k is $\pi/\delta x$, and $\delta x = 20\,km$ and $h = 2\,km$, we have

$$\frac{\sigma_r^2}{c_s^2 k^2} \geqslant \frac{\delta x^2}{h^2} \gg 1 \qquad \text{for } r \neq 0$$

b) The finite difference solution.

It is convenient for the examination of the finite difference solution to make the transformation

$$w = \frac{\partial \omega}{\partial z}$$

Equations (2.1) then take the form (after replacing vertical derivatives by finite differences)

$$\left.\begin{aligned} &\frac{\partial \underline{u}}{\partial t} + c_p \Theta_0 \frac{\partial \underline{P_1}}{\partial x} = 0 \\ &\frac{\partial \underline{\omega}}{\partial t} + c_p \Theta_0 \underline{P_1} = 0 \\ &c_p \Theta_0 \frac{\partial \underline{P_1}}{\partial t} - \underline{\underline{M}}\, \underline{\omega} + c_s^2 \frac{\partial \underline{u}}{\partial x} = 0 \end{aligned}\right\} \qquad (2.3)$$

where $\underline{\underline{M}}$ is the matrix representing the finite difference form of the operator $M$. $\underline{u}$, $\underline{\omega}$ and $\underline{P_1}$ are the vectors whose components are the values of u, w and $P_1$ at each of the levels of the model. Equations (2.3) may be de-coupled by multiplying by $\underline{\underline{X}}$ (the matrix of eigenvectors of $\underline{\underline{M}}$), where

$$\underline{\underline{X}}\,\underline{\underline{M}}\,\underline{\underline{X}}^{-1} = \underline{\underline{\mathrm{diag}}}\,(\mu_r)$$

$$\underline{\alpha} = \underline{\underline{X}}\,\underline{u}$$

$$\underline{\beta} = \underline{\underline{X}}\,\underline{\omega}$$

$$\underline{\gamma} = \underline{\underline{X}}\,\underline{P_1}$$

to give

$$\frac{\partial \alpha_r}{\partial t} + c_p \Theta_0 \frac{\partial \gamma_r}{\partial x} = 0$$

$$\frac{\partial \beta_r}{\partial t} + c_p \Theta_0 \gamma_r = 0 \qquad (2.4)$$

$$c_p \Theta_0 \frac{\partial \gamma_r}{\partial t} - \mu_r \beta_r + c_s^2 \frac{\partial \alpha_r}{\partial x} = 0$$

We now consider the following two finite difference schemes for equations (2.4)

a) fully implicit

$$\frac{1}{2\delta t}\left[\alpha_r^{(n+1)} - \alpha_r^{(n-1)}\right] + c_p \Theta_0 \frac{1}{2}\left[\frac{\partial \gamma_r^{(n+1)}}{\partial x} + \frac{\partial \gamma_r^{(n-1)}}{\partial x}\right] = 0$$

$$\frac{1}{2\delta t}\left[\beta_r^{(n+1)} - \beta_r^{(n-1)}\right] + c_p \Theta_0 \frac{1}{2}\left[\gamma_r^{(n+1)} + \gamma_r^{(n-1)}\right] = 0$$

$$\frac{c_p \Theta_0}{2\delta t}\left[\gamma_r^{(n+1)} - \gamma_r^{(n-1)}\right] - \mu_r \frac{1}{2}\left[\beta_r^{(n+1)} + \beta_r^{(n-1)}\right] + c_s^2 \frac{1}{2}\left[\frac{\partial \alpha_r^{(n+1)}}{\partial x} + \frac{\partial \alpha_r^{n-1}}{\partial x}\right] = 0$$

b) partially implicit

$$\frac{1}{2\delta t}\left[\alpha_r^{(n+1)} - \alpha_r^{(n-1)}\right] + c_p \Theta_0 \frac{\partial \gamma_r^{n}}{\partial x} = 0$$

$$\frac{1}{2\delta t}\left[\beta_r^{(n+1)} - \beta_r^{(n-1)}\right] + c_p \Theta_0 \frac{1}{2}\left[\gamma_r^{(n+1)} + \gamma_r^{(n-1)}\right] = 0$$

$$\frac{c_p \Theta_0}{2\delta t}\left[\gamma_r^{(n+1)} - \gamma_r^{(n-1)}\right] - \mu_r \frac{1}{2}\left[\beta_r^{(n+1)} + \beta_r^{(n-1)}\right] + c_s^2 \frac{1}{2}\left[\frac{\partial \alpha_r^{(n+1)}}{\partial x} + \frac{\partial \alpha_r^{(n-1)}}{\partial x}\right] = 0$$

Eliminating $\alpha_r$ and $\beta_r$ from each of these sets of equations we obtain

a) $$\frac{\partial^2 \zeta_r}{\partial x^2} + \lambda_r \zeta_r = b_r$$

b) $$\lambda_r \zeta_r = b_r$$

where

$$\zeta_r = c_p \Theta_0 \left[\gamma_r^{(n+1)} - 2\gamma_r^{(n)} + \gamma_r^{(n-1)}\right]$$

$$\lambda_r = -\frac{1}{c_s^2}\left[\frac{1}{\delta t^2} + \mu_r\right]$$

$$b_r = \frac{2}{\delta t\, c_s^2}\left[\frac{c_p \Theta_0}{\delta t}\left(\gamma_r^{(n)} - \gamma_r^{(n-1)}\right) - \mu_r\left(\beta_r^{(n-1)} - \delta t\, c_p \Theta_0 \gamma_r^{(n)}\right) + c_s^2\left(\frac{\partial \alpha_r^{(n-1)}}{\partial x} - \delta t\, c_p \Theta_0 \frac{\partial^2 \gamma_r^{(n)}}{\partial x^2}\right)\right]$$

Comparing with the finite difference scheme discussed in section 3 we see that the method described uses the fully implicit scheme a) for the horizontally propagating sound waves, and scheme b) (which has an explicit representation for the horizontal gradient of pressure) for the vertically propagating modes.

The finite difference schemes are found to be stable if $|\varepsilon| \leq 1$ were $\varepsilon$ is given by

a) $$\varepsilon^4(1+p+q) - 2\varepsilon^2(1-p-q) + (1+p+q) = 0$$

b) $$\varepsilon^4(1+p) + 2q\varepsilon^3 - 2\varepsilon^2(1-p) + 2q\varepsilon + (1+p) = 0$$

and $p = \mu_r \delta t^2$ and $q = \frac{4c_s^2 \delta t^2}{\delta x^2} \sin^2(k\delta x/2)$. We deduce that

a) $|\varepsilon| \equiv 1$ , implying unconditional stability,

b) $|\varepsilon| = 1$ if $q < p$

Since

$$\frac{q}{p} = \frac{4c_s^2 \sin^2(k\delta x/2)}{\mu_r \delta x^2} \leq \frac{4c_s^2}{\mu_r \delta x^2} \approx \frac{4c_s^2}{\sigma_r^2 \delta x^2} \approx \frac{4h^2}{r^2 \pi^2 \delta x^2}$$

the scheme b) is stable provided that the depth of the model $h$ is sufficiently small compared with the horizontal grid length. For the model under consideration $q \ll p$ .

The frequency $\sigma^*$ of the sound waves in the finite difference model is given by

a) $$\cos 2\sigma^* \delta t = \frac{1-p-q}{1+p+q}$$

Since $p \approx 0$ for the horizontally propagating wave we deduce that

$$\sigma^* = \frac{1}{\delta t} \tan^{-1} \sqrt{q} = \frac{1}{\delta t} \tan^{-1}\left[\frac{2\sigma \delta t}{k\delta x} \sin(k\delta x/2)\right]$$

b) $$(1+p)\cos 2\sigma^* \delta t + 2q \cos \sigma^* \delta t - (1-p) = 0$$

i.e. $$\cos \sigma^* \delta t = \frac{q}{2(1+p)} \pm \left[\frac{q^2}{4(1+p)^2} + \frac{1}{1+p}\right]^{1/2}$$

Since, for the vertically propagating modes, $p \gg 1$ and $q \ll p$ we deduce that $|\cos(\sigma^* \delta t)| \ll 1$ , that is the internal waves (which in physical space have frequencies much higher than $1/\delta t$ ) have periods in the finite difference model of about $4\delta t$ .

## APPENDIX 3.

## 2-DIMENSIONAL REPRESENTATION OVER A UNIFORM GRID USING CHEBYSHEV POLYNOMIALS.

Let $T_n$ be the Chebyshev polynomial of order n. Then it is required to expand a function $h(x,y)$ in the form

$$h(x,y) = \sum_{r=0}^{n} \sum_{s=0}^{m} \alpha_{r,s} T_r(x) T_s(y)$$

The pressure heights $h$ are given at discrete points, i.e.

$$h = h(x_i, y_j) \; ; \quad 0 \leq i \leq N \quad , 0 \leq j \leq M$$

The conditions $n \leq N$ and $m \leq M$ are applied so that the problem is not over-determined. Strict inequality implies a least squares fit with a consequent smoothing of the data.

The problem of determining the coefficients $\alpha_{r,s}$ can be solved by utilising the orthogonality of the polynomials over a discrete set of points (see for example Ambramowitz and Stegun, 1965). It may be shown that the scalar product is given by

$$(T_p, T_q) \equiv \sum_{i=0}^{N} T_p(x_i) T_q(x_i) = \begin{cases} \psi(p,N) & p = q \\ 0 & p \neq q \end{cases}$$

where

$$\psi(p,N) = \frac{(N+p-1)!\;(N-p)!}{(2p+1)!\;(N!)^2}$$

The coefficients $\alpha_{r,s}$ are then calculated from

$$\alpha_{r,s} = \frac{\sum_{i=0}^{N} \sum_{j=0}^{M} h(x_i, y_j)\, T_r(x_i)\, T_s(y_j)}{\psi(r,N)\,\psi(s,M)}$$

The polynomials themselves are most conveniently found from the recurrence relation

$$T_0(x_i) = 1$$

$$T_1(x_i) = 1 - 2x_i/N$$

$$(r+1)(N-r)T_{r+1}(x_i) = (2r+1)(N-2x_i)T_r - r(N+r-1)T_{r-1}$$

## APPENDIX 4

### FILTERING VERTICALLY PROPAGATING SOUND WAVES IN THE INITIAL DATA

It may be shown (see for example Haltiner, 1971 ) that the hydrostatic approximation filters out vertically propagating sound waves. In order to ensure that spurious sound waves are not generated by inconsistencies in the initial data, the initial variables are adjusted so as to satisfy two constraints (i) the hydrostatic equation, and (ii) an equation which states that the total derivative with respect to time of the hydrostatic terms is zero. In the notation of the model these are

$$g\,\frac{\theta_1}{\theta_0} - c_p(\theta_0+\theta_1)\,\frac{\partial P_1}{\partial z} = 0 \qquad (4.1)$$

and

$$\frac{D}{Dt}\left[\,g\,\frac{\theta_1}{\theta_0} - c_p(\theta_0+\theta_1)\,\frac{\partial P_1}{\partial z}\right] = 0 \qquad (4.2)$$

Neglecting diffusion and non-adiabatic effects, the equations for $\theta_1$ and $P_1$ take the form

$$\frac{D\theta_1}{Dt} = 0 \qquad (4.3)$$

$$\frac{DP_1}{Dt} = \frac{w}{H} - (\gamma-1)(P_0+P_1)\,\nabla\cdot\underline{v} \qquad (4.4)$$

Using the adiabatic equation (4.3) the second of the initial constraints (4.2) becomes

$$c_p(\theta_0+\theta_1)\,\frac{D}{Dt}\left(\frac{\partial P_1}{\partial z}\right) = 0 \qquad (4.5)$$

Further, using the identity

$$\frac{D}{Dt}\left(\frac{\partial P_1}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{DP_1}{Dt}\right) - \frac{\partial \underline{v}}{\partial z}\cdot\nabla P_1$$

equation (4.5) can be written in the form

$$\frac{\partial}{\partial z}\left(\frac{DP_1}{Dt}\right) - \frac{\partial \underline{v}}{\partial z} \cdot \nabla P_1 = 0$$

Now substituting for $DP_1/Dt$ from the continuity equation (4.4) we obtain, after some algebraic re-organisation, the second order differential equation

$$(\gamma - 1)(P_0 + P_1)\frac{\partial^2 w}{\partial z^2} - \frac{\gamma}{H}\,\frac{\Theta_0}{\Theta_0 + \Theta_1}\,\frac{\partial w}{\partial z}$$

$$= -(\gamma - 1)\frac{\partial}{\partial z}\left[(P_0 + P_1)\nabla \cdot \underline{v}\right] - \frac{\partial \underline{v}_H}{\partial z} \cdot \nabla P_1$$

Given $u$ , $v$ , $P_1$ , and $\Theta_1$ , this equation can be solved for $w$. The boundary conditions used are $w = 0$ at $z = 0$ (in the absence of topography) and $w$ given by interpolation from the 10-level model at the top level .

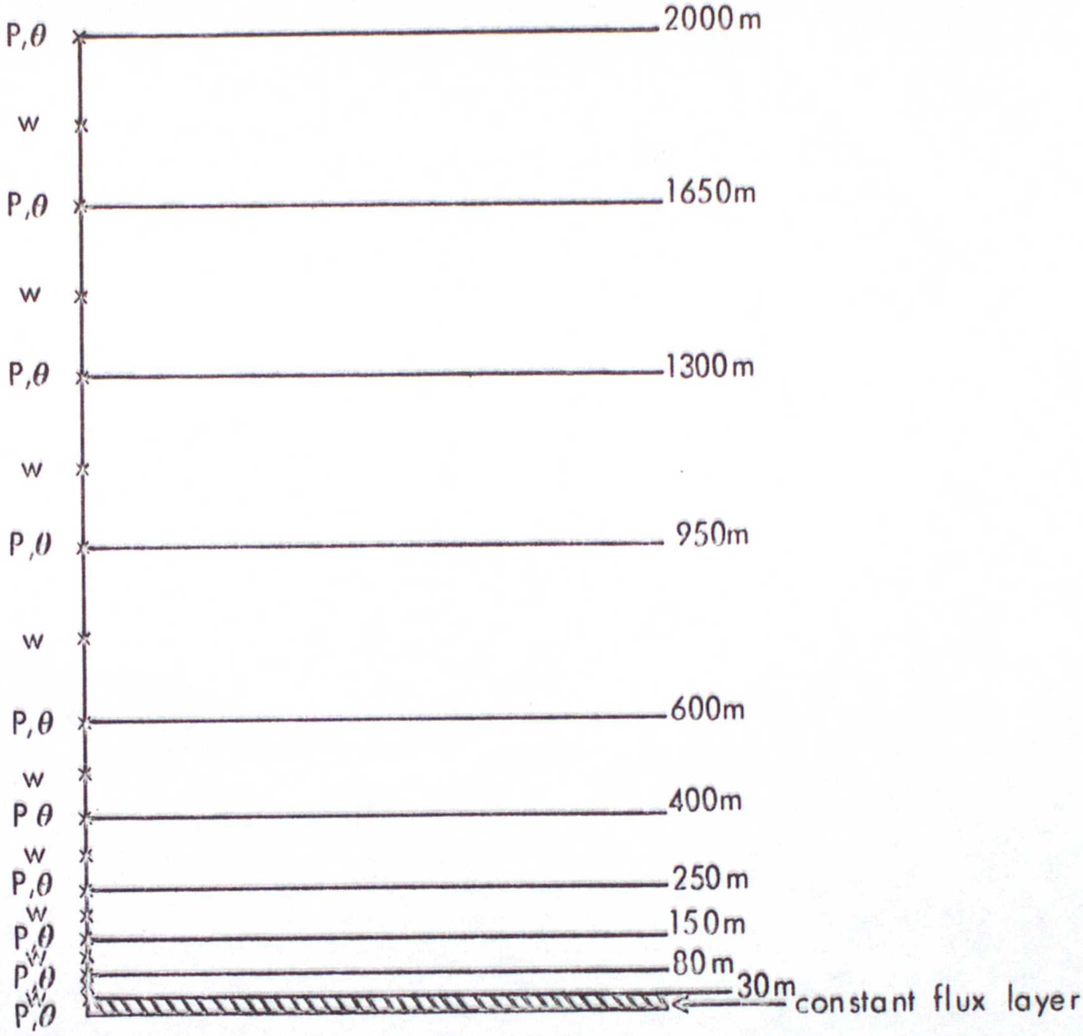


Figure 1 Grid structure in the vertical

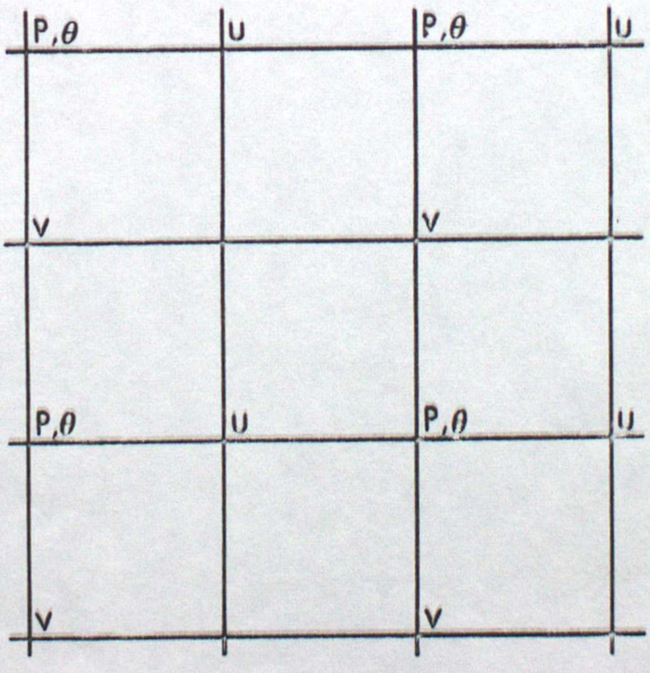


Figure 2 Grid structure in the horizontal


Fig 3

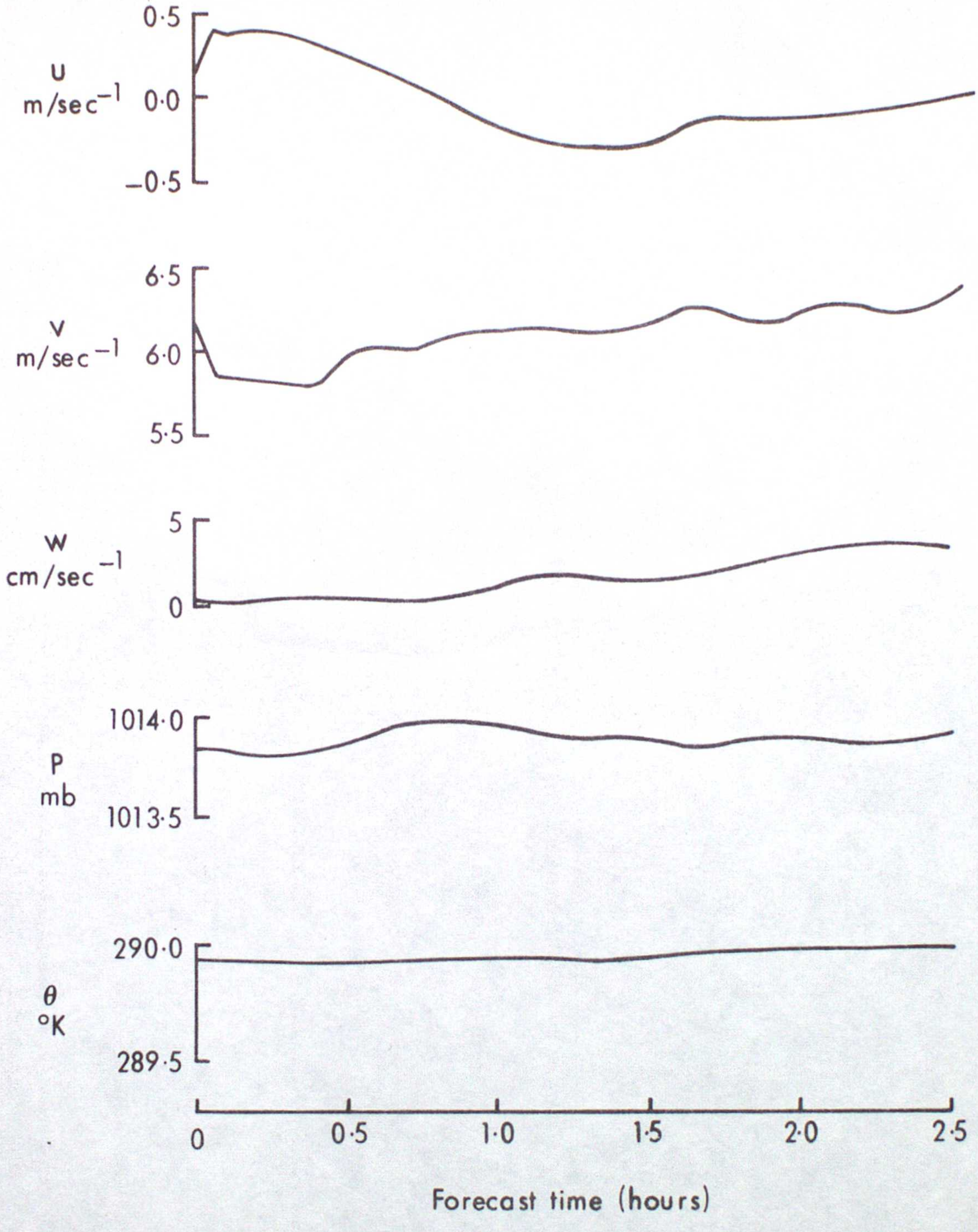


Figure 4 The time variation of parameters at an arbitrary gridpoint.

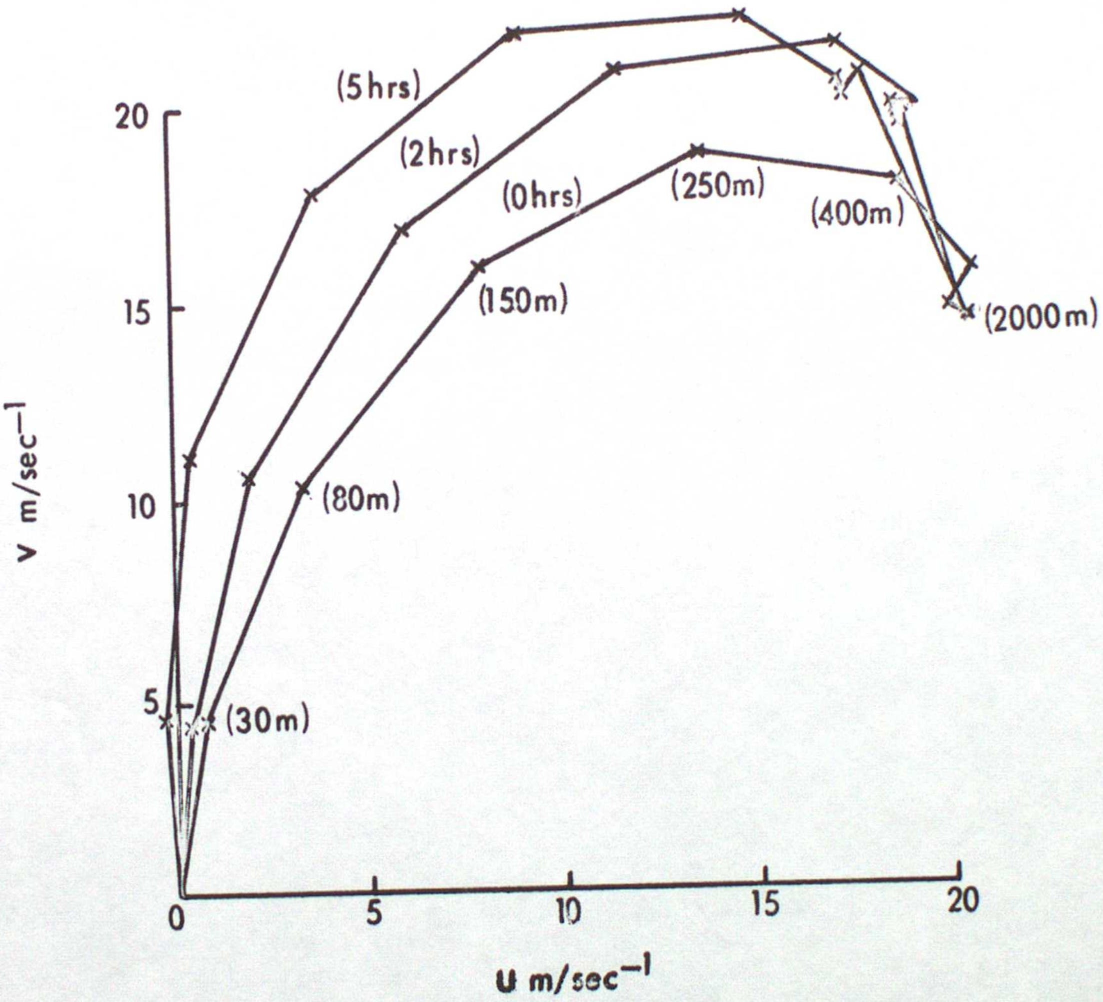


Figure 5 Variation with time of the wind hodograph for an arbitrary gridpoint

TOPOGRAPHY ON 10 KM SCALE
(HEIGHT INTERVAL 100 FT)

Fig 6