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FOURTH ORDER ADVECTION SCHEMES FOR THE 10-LEVEL MODEL

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Fourth Order Advection Schemes for the 10 Level Model

by A. J.Gadd.

1. Introduction

As is the case with most numerical models of the atmosphere, the speed at which meteorological features move in 10 level model forecasts is noted to be slower in general than in the real atmosphere. A change in the method of calculating the static stability of the model atmosphere (Gadd 1973) has contributed significantly to a removal of this systematic trend. Efforts have also been made to reduce the phase lag errors through the reduction of truncation errors in finite difference approximations to horizontal derivatives. Reduction in truncation error has been achieved through the use of fourth order accurate approximations to derivatives in the horizontal advection terms. Fourth order approximations to pressure gradient and divergence terms would have a dramatic impact on the linear computational stability criterion and therefore have not been considered.

Two schemes have been developed. One is designed for use with the original explicit integration scheme of the 10 level model and has been used operationally in the 'Octagon' version of the model since September 1972. The second scheme is designed for use with the semi-implicit reformulation of the model (Burridge 1974) but has not been adopted for routine use. Both schemes are extensions of the basic two-step Lax-Wendroff scheme used in the original 10 level model and for advection terms in the semi- implicit model. One or other of the two steps is modified so as to reduce the truncation error implied by the complete two-step cycle. The principles of the schemes are best illustrated by their application to a highly simplified equation.

2. Simple advection

Following Morton (1971) we consider the properties of the difference schemes in advecting some property h in one space dimension with a uniform velocity (u_0) as described by the linear equation

$$\frac{\partial h}{\partial t} = -u_0 \frac{\partial h}{\partial x} \qquad (1)$$

Using the finite difference grid illustrated in Figure 1 and considering solutions h_j^n for h at grid point j, time step n, of the form

$$h_j^n = H_n \exp ikx = H_n \exp 2i\alpha j \quad \text{where } \alpha = \frac{k\delta x}{2}$$

we may show that

$$H_{n+1} = \mathcal{H}(\alpha, \mu)\, H_n$$

where $\mu = u_0 \delta t/\delta x$ and the function $\mathcal{H}$ depends on the particular finite difference scheme. α reflects the scale of the wave relative to the grid mesh, whilst μ is a measure of the advecting velocity.

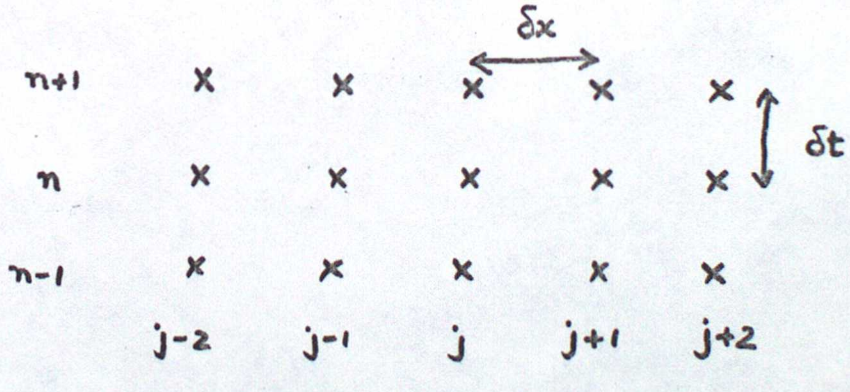

Figure 1   Finite difference grid for simple advection.

3. The Lax-Wendroff scheme and the fourth order modifications

(A) The two step Lax-Wendroff scheme applied to equation (1) is as follows

$$h_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \tfrac{1}{2}(h_j^n + h_{j+1}^n) - \tfrac{1}{2}\mu(h_{j+1}^n - h_j^n) \qquad (2)$$

$$h_{j+1}^n = h_j^n - \mu\left(h_{j+\frac{1}{2}}^{n+\frac{1}{2}} - h_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) \qquad (3)$$

On combining these two steps it may be shown that

$$K_A = 1 - 2\mu^2 \sin^2\alpha - 2i\mu \sin\alpha \cos\alpha$$

Now $D = |K|$ gives the damping per time step, whilst $P = \arg K / 2\mu\alpha$ is the relative phase speed, i.e. the ratio of the numerical to the analytical phase speed.

(B) If the second step of the Lax-Wendroff scheme is replaced by

$$h_j^{n+1} = h_j^n - \mu\left[\frac{3}{2}\left(h_{j+\frac{1}{2}}^{n+\frac{1}{2}} - h_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) - \frac{1}{2}\left(h_{j+\frac{3}{2}}^{n+\frac{1}{2}} - h_{j-\frac{3}{2}}^{n+\frac{1}{2}}\right)\right] \qquad (4)$$

and the first step is unchanged, it may be shown that the approximation to $\partial h/\partial x$ used in the complete integration cycle is of fourth order accuracy. (However the estimate of $\partial^2 h/\partial x^2$, which is also required, remains second order accurate). Equations (2) and (4) together lead to

$$K_B = 1 - 2\mu^2 \sin^2\alpha\left(1 + \tfrac{2}{3}\sin^2\alpha\right) - 2i\mu \sin\alpha \cos\alpha\left(1 + \tfrac{2}{3}\sin^2\alpha\right)$$

(C) Alternatively, if the <u>first</u> step of the Lax-Wendroff scheme is replaced by

$$\begin{aligned} h_{j+\frac{1}{2}}^{n+\frac{1}{2}} = &\left[\frac{7}{6}\left(\frac{h_j^n + h_{j+1}^n}{2}\right) - \frac{1}{6}\left(\frac{h_{j-1}^n + h_{j+2}^n}{2}\right)\right] - \\ &- \frac{1}{2}\mu\left[\frac{5}{4}\left(h_{j+1}^n - h_j^n\right) - \frac{1}{4}\left(\frac{h_{j+2}^n - h_{j-1}^n}{3}\right)\right] + \\ &+ \frac{1}{12}\mu^2\left[h_{j+2}^n - h_{j+1}^n - h_j^n + h_{j-1}^n\right] - \frac{1}{24}\mu^3\left[h_{j+2}^n - 3h_{j+1}^n + 3h_j^n - h_{j-1}^n\right] \end{aligned} \qquad (5)$$

and the <u>second</u> step is unchanged, it may be shown that the complete two step scheme is identical to Crowley's (1968) fourth order accurate scheme. Equations (5) and (3) together give

$$K_C = 1 - 2\mu^2 \sin^2\alpha\left[1 + \tfrac{1}{3}(1-\mu^2)\sin^2\alpha\right] - 2i\mu \sin\alpha \cos\alpha\left[1 + \tfrac{2}{3}(1-\mu^2)\sin^2\alpha\right]$$

4. Properties of the three schemes

Important properties of the Lax-Wendroff scheme and its two modifications as applied to the simple advection equation are revealed by $D = |K|$ and $P = \arg K / 2\mu\alpha$ which are illustrated in Figures 2 and 3 as functions of $|\mu|$ and α. D and P would both take the value unity in a perfect numerical simulation of simple advection.

It may be seen that the modified schemes B and C improve the damping in simple advection relative to the basic Lax-Wendroff scheme A. Schemes A and C are stable ($D \leq 1$) for $|\mu| \leq 1$, but scheme B is stable only if $|\mu| \leq 0.6$. The more significant feature in the present context is the significantly reduced relative phase errors of schemes B and C, reflected in the fact that the values of P are closer to unity for given values of $|\mu|$ and α.

5. Applications to the 10 level models

The schemes introduced above may easily be extended to two space directions and applied to the advection terms in the 10 level model equations. Of course, the staggered grid arrangements used in the models add some complications, but these in fact only amount of interpolation in the direction normal to the direction of differentiation before the application of equations (2), (3), (4) or (5). The single exception is the first term in equation (5) which in two dimensions becomes the following 16 point operator.

$$\frac{1}{144}\begin{pmatrix} 1 & -7 & -7 & 1 \\ -7 & 49 & 49 & -7 \\ -7 & 49 & 49 & -7 \\ 1 & -7 & -7 & 1 \end{pmatrix}$$

Computational economy must be kept in mind when choosing a higher order scheme for the 10 level model. Scheme B is well suited to the original explicit integration scheme, where the stability criterion is so dominated by the gravity wave terms that the adoption of more accurate approximations to the advection terms imposes no further significant restriction on the time step. In addition scheme B adds only about 5% to the computation required per time step.

Scheme B is however, quite inappropriate to the semi-implicit integration scheme, where advection imposes the most restrictive stability criterion. Scheme B would add 66% to the computation here because of the significantly reduced time step which would be required. Scheme C, on the other hand, allows the same time step as the basic Lax-Wendroff scheme, although unfortunately it adds almost 25% to the computation required per time step.

6. Note on the impact of the modified schemes on forecasts

No attempt is made in this note to illustrate the results of using the modified schemes in the 10 level models. It should be recorded, however, that the improvements produced have been generally disappointing. The speed of movement of systems is indeed improved compared with the basic Lax-Wendroff scheme, but major forecast errors are not accounted for, and much greater improvements have been achieved by changing the calculation of the static stability term.

The implication of such results is that truncation errors in horizontal advection terms are not a major problem in the 10 level model. This is at variance with the general climate of opinion among meteorologists. It may of course be the case that the more significant truncation errors are in the pressure gradient or divergence terms, although in the semi-implicit version these are already resolved with smaller truncation by the staggering of the grid. Alternatively, errors in vertical differencing might be dominant over those in horizontal differencing. This question is as yet unresolved.

References

Gadd	1973	Met.O.11 Technical Note No.42.
Burridge	1974	to be published
Morton	1971	Proc. Royal Soc. A323
Crowley	1968	M.W.R. 96

FIGURE 2   Damping ($D = |K|$) for the three schemes A, B, C.

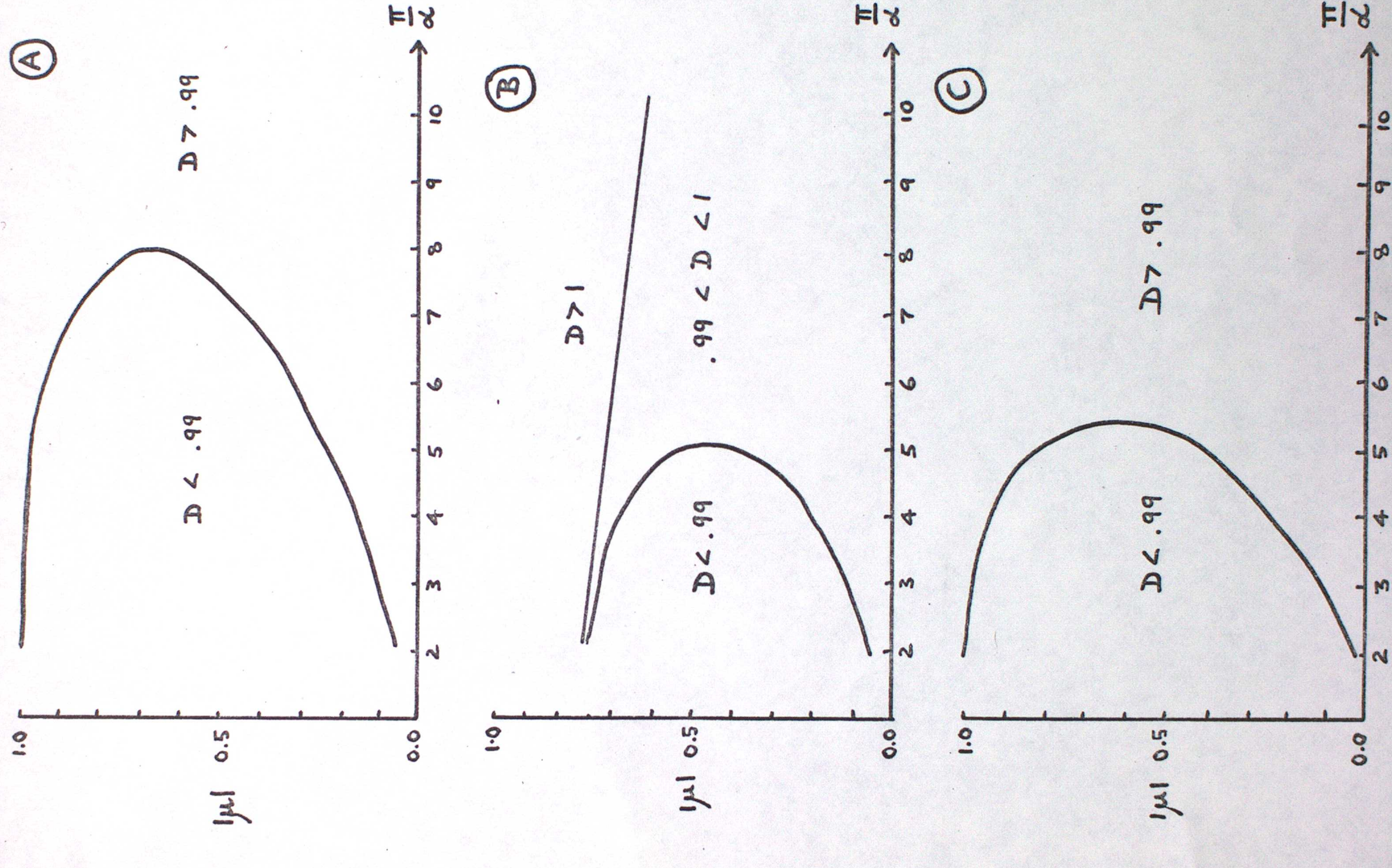

FIGURE 3 Relative phase speed ($P = \arg K/2\mu\alpha$) for the three schemes A, B, C.

Note. In all cases $P=0$ when $\pi/\alpha = 2$.