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MET O 11 TECHNICAL NOTE NO 77

An Interpretation of the Dishington Rate of Surface-Strain Tensor

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1. Introduction

In Met O 11 Technical Note No 76 Part 2 Dishington's Rate of Surface-Strain Tensor is introduced in connection with flow over a smooth mountain surface. As Dishington's tensor may well be unfamilar to many meteorologists this note provides an interpretation and also shows its connection with two long-standing basic results in fluid flow theory.

R H DISHINGTON:- "Rate of Surface-Strain Tensor"
Amer.J.Phys., 33, 10, Oct 1965, pp 827-831.

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2. Notation

$\delta\underline{s}$ is a fluid line element.

$\hat{\underline{s}}$ is the unit vector specifying the direction of $\delta\underline{s}$.

δs is the scalar length of $\delta\underline{s}$. Thus $\delta\underline{s} = (\delta s)\hat{\underline{s}}$.

$\delta\tau$ is an element of a fluid volume.

$\delta\underline{\sigma}$ is a fluid surface element.

$\hat{\underline{n}}$ is the unit normal vector to the element of surface.

$\delta\sigma$ is the scalar area of the fluid surface element.

Thus $\delta\underline{\sigma} = (\delta\sigma)\hat{\underline{n}}$

$\frac{1}{\delta s}\frac{d(\delta s)}{dt}$ is the proportional rate of increase of the length of the line element.

$\frac{1}{\delta\tau}\frac{d(\delta\tau)}{dt}$ is the proportional rate of increase of the volume of the element of fluid volume.

$\frac{1}{\delta\sigma}\frac{d(\delta\sigma)}{dt}$ is the proportional rate of increase of the element of fluid surface area.

D is the three-dimensional deformation dyadic.

I is the three-dimensional Idemfactor.

$\bullet$ denotes the Gibb's scalar product.

X denotes the Gibbs vector product.

The Scalar of any dyadic, say T, is obtained by inserting the $\bullet$ between the unit vectors in the nonion form of the dyadic and summing, and it is denoted by Sca T. In matrix notation it is the trace. In particular if $\underline{b}$ is any vector and the dyad $(\underline{bb})$ is formed then $\text{Sca}(\underline{bb}) = b^2$

$\tilde{T}$ denotes the transpose of T

$\underline{r}$ is the position vector

$\underline{V}$ is the three-dimensional velocity vector

3. The Dishington Rate of Surface-Strain Tensor

The rate of Surface-Strain Tensor introduced by Dishington in 1965 completed an important trilogy of fundamental results [1, 2, 3] in the kinematics of fluid flow

$$\frac{1}{\delta s}\frac{d}{dt}(\delta s) = \underline{\hat{s}}\cdot D\cdot\underline{\hat{s}} \tag{1}$$

$$\frac{1}{\delta\tau}\frac{d}{dt}(\delta\tau) = \underline{\hat{n}}\cdot(\mathrm{Sca}\,D)I\cdot\underline{\hat{n}} = \mathrm{Div}\,\underline{V} \tag{2}$$

$$\frac{1}{\delta\sigma}\frac{d}{dt}(\delta\sigma) = \underline{\hat{n}}\cdot{}^{s}D\cdot\underline{\hat{n}} \tag{3}$$

The tensor ${}^{s}D$ is defined as

$${}^{s}D = (\mathrm{Sca}\,D)I - D \tag{4}$$

Note that Div $\underline{V}$ = Sca D and that consequently (4) invites comparison with an inertia dyadic

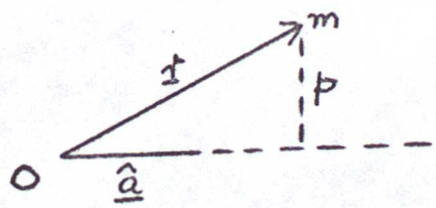

Fig 1

In the 3-space physical problem the second moment or moment of inertia of a particle of mass m relative to an axis defined by a unit vector $\underline{a}$, Fig 1, is given by mp^2 where

$$p^2 = (\underline{r}\times\underline{\hat{a}})^2 = r^2 - (\underline{r}\cdot\underline{\hat{a}})^2$$

ie

$$p^2 = \underline{\hat{a}}\cdot(r^2 I - \underline{r}\,\underline{r})\cdot\underline{\hat{a}} \tag{5}$$

and since r^2 = Sca ($\underline{rr}$) this is clearly of the form

$$p^2 = \underline{\hat{a}}\cdot{}^{s}\Psi\cdot\underline{\hat{a}} \tag{6}$$

where ${}^{s}\Psi$ is the inertia dyadic

$${}^{s}\Psi = {}^{s}(\underline{r}\,\underline{r}) = \mathrm{Sca}(\underline{r}\,\underline{r})I - \underline{r}\,\underline{r} \tag{7}$$

The formal correspondence between ${}^{s}D$ and ${}^{s}\Psi$ is evident from (4) and (7) but the analogy is drawn closer by noting that (5) may be rewritten as

$$p^2 = \underset{\sim}{r} \cdot (I - \hat{\underset{\sim}{a}}\hat{\underset{\sim}{a}}) \cdot \underset{\sim}{r} \qquad (8)$$

from which p^2 is seen as the square of the magnitude of the vector $\underline{p}$, the projection of $\underline{r}$ on to the space complementary to $\hat{\underline{a}}$, where

$$\underline{p} = (I - \hat{\underset{\sim}{a}}\hat{\underset{\sim}{a}}) \cdot \underset{\sim}{r} \qquad (9)$$

The advantage of (8), which expresses the inertia in terms of a complementary projection, is that it generalizes readily if the vector $\underline{r}$ is replaced by a dyadic T. The complementary projection is then the dyadic

$$P = (I - \hat{\underset{\sim}{a}}\hat{\underset{\sim}{a}}) \cdot T \qquad (10)$$

The square of the magnitude of this projection is then Sca $(\tilde{P} \cdot P)$.
Now

$$\tilde{P} \cdot P = \tilde{T} \cdot (I - \hat{\underset{\sim}{a}}\hat{\underset{\sim}{a}}) \cdot T$$

$$= \tilde{T} \cdot T - (\tilde{T} \cdot \hat{\underset{\sim}{a}})(\hat{\underset{\sim}{a}} \cdot T)$$

$$= \tilde{T} \cdot T - (\hat{\underset{\sim}{a}} \cdot T)(\tilde{T} \cdot \hat{\underset{\sim}{a}})$$

so that

$$Sca(\tilde{P} \cdot P) = Sca(\tilde{T} \cdot T) - \hat{\underset{\sim}{a}} \cdot (T \cdot \tilde{T}) \cdot \hat{\underset{\sim}{a}} \qquad (11)$$

ie

$$Sca(\tilde{P} \cdot P) = \hat{\underset{\sim}{a}} \cdot \left[Sca(\tilde{T} \cdot T) I - \tilde{T} \cdot T \right] \cdot \hat{\underset{\sim}{a}}$$

and now putting $\hat{\underset{\sim}{a}} = \hat{\underset{\sim}{n}}$ and $T = D^{\frac{1}{2}}$ it is seen that (3) may be rewritten as

$$\frac{1}{\delta\sigma} \frac{d}{dt}(\delta\sigma) = \hat{\underset{\sim}{n}} \cdot \left[Sca(D^{\frac{1}{2}} \cdot D^{\frac{1}{2}}) I - D^{\frac{1}{2}} \cdot D^{\frac{1}{2}} \right] \cdot \hat{\underset{\sim}{n}} \qquad (12)$$

so that the proportional rate of change of area of a fluid surface element is determined by the relationship of the unit surface normal to a deformation inertia dyadic. Dishington's tensor ${}^{S}D$ may be regarded as an inertia dyadic in the deformation space of the problem.

1. H Durrande; C R Acad. Sci. Para 73, 1871, pp 736-788

2. L Euler; Hist. Acad. Berlin, 1755, pp 217-273

3. R H Dishington, Amer. J. Phys., 33, 10, Oct. 1965 pp 827-831

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