Met O 11 Technical Note No. 87

Finite element and finite difference schemes using local co-ordinates

M J P Cullen

1. Introduction.

This note describes an attempt to use finite difference and finite element schemes based on a locally defined co-ordinate system in a forecasting model. In such a scheme the co-ordinate system may be different on adjacent grid-boxes or finite elements and therefore to avoid transformations from one system to another it is necessary to work in terms of invariant quantities like div grad and curl. It is quite common for hemispheric and global models to use different co-ordinate systems on different parts of the globe, the very early model of Phillips (1959) using mixed map projections and the polyhedral grids used by Sadourny (1972) are examples. In these cases transformations between different co-ordinate systems had to be carried out. In this note an attempt is made to use schemes which avoid the need for co-ordinate transformations. However, it is found that this restricts the choice of numerical method severely and prevents the construction of a stable approximation to the forecasting equations on a moderately irregular grid with orography included. This approach therefore does not seem to be a competitor for operational forecast use, unless some new way of using it can be found.

2. Description of a local co-ordinate scheme for the shallow water equations.

We can write the shallow water equations as follows using standard notation:

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\zeta + f)\,\underline{\hat{k}} \times \underset{\sim}{u} + \nabla\left(\tfrac{1}{2}u^2 + \phi\right) = 0$$

$$\zeta \underline{\hat{k}} = \nabla \times \underset{\sim}{u} \qquad (1)$$

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \underset{\sim}{u}) = 0$$

As a first example we consider integrating (1) using a finite element approximation in which all the variables are defined on an irregular triangular mesh. All scalars, or vectors with a component only in the direction, are defined by piecewise linear interpolation between values at the vertices of the triangles. Even though we use a different co-ordinate system on each triangle, scalars can still be defined in this way without needing transformation. However vectors have to be defined independently on different triangles so we define them to be constants on the triangles. With these definitions it is easy to calculate all the quantities required to solve (1):

- ϕ is piecewise linear and can be written as $\sum \phi_n \chi_n$ where the ϕ_n are values at the vertices and the χ_n are piecewise linear basis functions
- $\underset{\sim}{u}$ is constant on each triangle with components defined by the local co-ordinates.
- $u^2 = \underset{\sim}{u} \cdot \underset{\sim}{u}$ constant on triangles but is independent of the co-ordinates, so a value can be assigned at each vertex.
- $\nabla\phi$ is constant on triangles with components $\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$ defined in the local co-ordinates.
- ζ can be approximated by the Galerkin method as follows:

Since ζ only has a component in the $\hat{\underline{k}}$ direction, write

$\zeta = \sum \zeta_n \chi_n$. Set

$$\int \sum \zeta_n \chi_n \cdot \chi_m \, d\Omega = \int \nabla \times u \cdot \chi_m \, d\Omega \tag{2}$$

where Ω is the domain over which we are solving (1). Since $\underset{\sim}{u}$ is piecewise constant, it cannot be differentiated, so we have to integrate the right-hand side of (2) by parts:

$$\int \sum \zeta_n \chi_n \cdot \chi_m \, d\Omega = \left[\nabla \times (u \chi_m) \right] - \int u \times \nabla \chi_m \, d\Omega$$

The boundary integral vanishes and the remaining terms can be calculated.

The divergence $\nabla \cdot \underset{\sim}{u}$ can be calculated in a similar way to the vorticity. There are no simple advection terms of the form $\underline{u} \cdot \nabla\phi$ in (1) but such terms can be calculated as the scalar product of the vectors $\underline{u}$ and $\nabla\phi$ on each triangle independently.

For clarity the approximations are set out below for a regular grid with triangles of unit side and points labelled as in Fig. 1. We use the Galerkin approximation (see Tech Note 82) to convert values of scalars calculated on each triangle independently to values at the vertices.

$$\frac{1}{2}(u^2)_4 + \frac{1}{12}((u^2)_1 + (u^2)_2 + (u^2)_3 + (u^2)_5 + (u^2)_6 + (u^2)_7) \tag{3}$$
$$= 1/6((u^2)_8 + (u^2)_9 + (u^2)_{10} + (u^2)_{11} + (u^2)_{12} + (u^2)_{13})$$

$(\nabla\phi)_8 = (\phi_4 - \phi_3, \frac{2}{\sqrt{3}}(\phi_1 - \frac{1}{2}(\phi_3 + \phi_4)))$, similar for other triangles.

$$\frac{1}{2}\zeta_4 + \frac{1}{12}(\zeta_1 + \zeta_2 + \zeta_3 + \zeta_5 + \zeta_6 + \zeta_7) =$$

$$\frac{1}{\sqrt{3}}(-u_9 - \frac{1}{2}(u_8 + \frac{\sqrt{3}}{2}v_8) + (\frac{1}{2}u_{11} - \frac{\sqrt{3}}{2}v_{11}) + u_{12} + \tag{4}$$
$$(\frac{1}{2}u_{13} + \frac{\sqrt{3}}{2}v_{13}) + (\frac{\sqrt{3}}{2}v_{10} - \frac{1}{2}u_{10}))$$

Writing $D = \nabla . \underline{u}$

$$\frac{1}{2}D_4 + \frac{1}{12}(D_1 + D_2 + D_3 + D_5 + D_6 + D_7) =$$

$$\frac{1}{\sqrt{3}}[v_9 + (\frac{1}{2}v_8 - \frac{\sqrt{3}}{2}u_8) - (\frac{\sqrt{3}}{2}u_{11} + \frac{1}{2}v_{11}) - v_{12} + \tag{5}$$
$$(\frac{\sqrt{3}}{2}u_{13} - \frac{1}{2}v_{13}) + (\frac{\sqrt{3}}{2}u_{10} + \frac{1}{2}v_{10})]$$

Then the full approximations to (1) become:

$$\frac{\partial u_8}{\partial t} - (\frac{1}{3}(\zeta_1 + \zeta_3 + \zeta_4) + f)v_8 + E_4 - E_3 = 0 \tag{6}$$

where $E = \frac{1}{2}u^2 + \phi$, u^2 defined by (3)

$$\frac{\partial v_8}{\partial t} + (\frac{1}{3}(\zeta_1 + \zeta_3 + \zeta_4) + f)u_8 + \frac{2}{\sqrt{3}}\{E_1 - \frac{1}{2}(E_3 + E_4)\} = 0 \tag{7}$$

$$\frac{1}{2}\frac{\partial\phi_4}{\partial t} + \frac{1}{12}\frac{\partial}{\partial t}(\phi_1 + \phi_2 + \phi_3 + \phi_5 + \phi_6 + \phi_7)$$
$$+ \frac{1}{\sqrt{3}}(\frac{1}{3}(\phi_1 + \phi_2 + \phi_4)v_9 + \frac{1}{3}(\phi_1 + \phi_3 + \phi_4)(-\frac{\sqrt{3}}{2}u_8 + \frac{1}{2}v_8) -$$
$$\frac{1}{3}(\phi_3 + \phi_4 + \phi_6)(\frac{\sqrt{3}}{2}u_{11} + \frac{1}{2}v_{11}) - \frac{1}{3}(\phi_4 + \phi_6 + \phi_7)v_{12} +$$
$$\frac{1}{3}(\phi_4 + \phi_5 + \phi_7)(\frac{\sqrt{3}}{2}u_{13} - \frac{1}{2}v_{13}) + 1/3(\phi_2 + \phi_4 + \phi_5)(\frac{\sqrt{3}}{2}u_{10} + \frac{1}{2}v_{10})) = 0$$

This scheme can be generalised using the same principles to use the vorticity and divergence form of the equations. Alternatively the Galerkin approximation for fitting piecewise linear functions can be dropped and simple averaging used instead, leading to a general finite difference approximation. This is obtained from (4) to (8) by replacing the terms of the form

$$\frac{1}{2}\phi_4 + \frac{1}{12}(\phi_1 + \phi_2 + \phi_3 + \phi_5 + \phi_6 + \phi_7)$$

by ϕ_4 . This makes the scheme entirely explicit.

3. Simple analysis of the scheme.

In many respects the scheme for the shallow water equations described in the previous section is like a finite difference scheme on a mesh staggered as shown in Fig. 2. The problem lies in the non-linear advective terms. To avoid co-ordinate transformations no vector quantities can be averaged over different triangles, only scalars. The scheme for the geopotential advection term

$$u\frac{\partial\phi}{\partial x}$$

is analogous to the finite difference scheme

$$\frac{\partial\phi}{\partial t} + \overline{u.\overline{\delta_x h}^{y}}^{xy} = 0$$

and the schemes for the velocity advection terms $u\frac{\partial u}{\partial x}$ are derived from either

$$u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right)$$

where the finite difference approximation is

$$\delta_{2x}\left(\overline{u^2}^{xy}\right) \qquad (9)$$

or

$$u\frac{\partial v}{\partial x}\left(-u\frac{\partial u}{\partial y}\right) = u\zeta \, .$$

where the finite difference approximation is

$$\zeta = \overline{v_x}^{y} - \overline{u_y}^{x}$$

$$u\zeta = \overline{u\left(\overline{v_x}^{y} - \overline{u_y}^{x}\right)}^{xy} \qquad (10)$$

Neither (9) or (10) are normal finite difference schemes for advection and the simple product in (10) is likely to lead to non-linear instability. Thus we would not expect a particularly good result.

Now consider the finite element version. Non-linear stability of finite element models is usually assured by the Galerkin approximation to products, for instance uv is approximated in one dimension by

$$\frac{2}{3}(uv) + \frac{1}{3}\overline{(uv)}^{2x} = \frac{1}{2}u.v + \frac{1}{6}u.\bar{v}^{2x} + \frac{1}{6}\bar{u}^{2x}.v + \frac{1}{6}\overline{uv}^{2x} \qquad (11)$$

Using the local co-ordinate scheme, however, most of the products in the advective terms are products of vectors with the gradients of scalars. Both these quantities are represented as constants on triangles, so the product scheme consists of single multiplication on triangles followed by fitting back to linear functions using the Galerkin method.

The analogous on dimensional scheme is thus:

$$\frac{2}{3}(uv) + \frac{1}{3}\overline{(uv)}^{2x} = \overline{u.v}^{x} \qquad (12)$$

where (uv) is defined at grid-points and u and v are defined at midpoints. This schemeis liable to aliasing, as can be seen at once if u and v are both two grid-length waves. The averaging to transfer from midpoints to grid-points does not help. Thus the extra stability of the finite element Galerkin scheme is lost.

4. Experiments and discussion.

Attempts were made to use the formulation described in Section 2 in the following ways:

a) On an icosahedral grid with the same co-ordinate system on each face of the icosahedron. The equations used on the projection are as set out in Sadourny (1972).

(i) With semi-implicit finite element scheme using velocity potential and stream function.

(ii) With explicit version of (i).

(iii) With explicit finite element scheme using velocity components.

(iv) With explicit finite difference scheme using velocity components.

b) On the same grid as (a) but using spherical polars and subdividing the icosahedron using lines in the latitude/longitude plane (as in Cullen (1976)). This was only done with method (iv).

In each case the integrations proved impossible to stabilise without doing unacceptable damage to the synoptic features. The blow-up occurred in each case where the maximum irregularity of the grid was close to the western end of the Himalayas; there were also problems over Greenland. Considerable effort was spent in checking for possible programming errors and eventually it was concluded that the failure was due to the inherent instability of the numerical scheme. The requirement that vector quantities cannot be averaged leads to a simple unambiguous formulation but it prevents the use of the techniques that have been found necessary in the past to stabilise forecast integrations.

References:

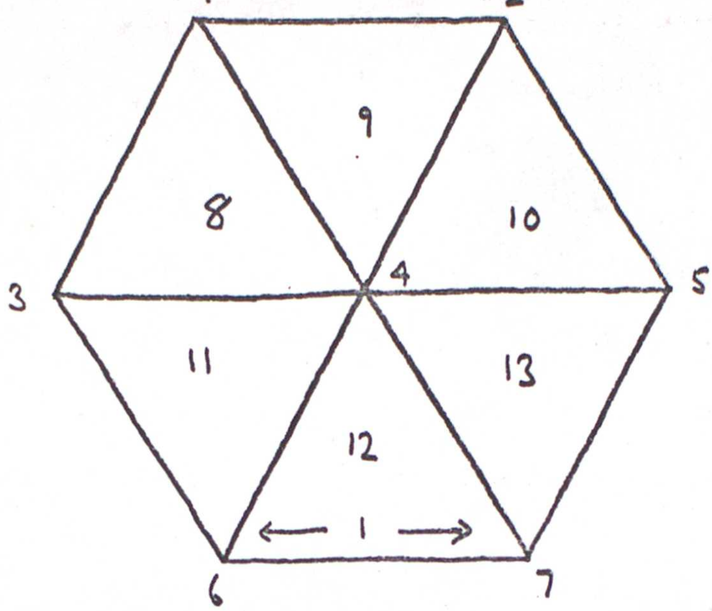
Cullen M.J.P. 1976 "Further forecast experiments with the finite element model". Met O 20 Tech Note II/68.

1977 "The use of finite element methods in atmospheric models" Met O 11 Tech Note 82.

Phillips N.A. 1959 "Numerical integration of the primitive equations on the hemisphere". Mon. Weather. Rev. 87, p 333.

Sadourny R. 1972 "Conservative finite-difference approximations of the primitive equations on quasi-uniform spherical grids". Mon. Weather. Rev. 100, p 136.

List of figures:



× h × h

o u,v o u,v

× h × h

o u,v o u,v

× h × h