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EXTREME VALUE ANALYSIS IN METEOROLOGY

by

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Summary

The theory of extreme values requires that the maxima (or minima) are drawn from the tail of a distribution of observations which belong to a single population. Failure to satisfy the theory can therefore be due to the inclusion of observations which are insufficiently extreme or which belong to more than one population. It is demonstrated that in meteorology these reasons are often alternative inter-expressions of the same problem, namely lack of data.

A series of extremes may be regarded as belonging to the same population if a single forcing factor is responsible for the whole range of extremes encountered. The seasonal variation of meteorological variables means that a complete set of daily values do not belong to a single population, but it is shown that analyses of annual extremes are still to be preferred to those based on monthly data. Topography often acts as a second forcing factor by introducing fohns and standing waves, and by encouraging the development of stationary storms.

The return period at which extremes approach a physically imposed upper limit varies widely with element and location. When the observed extremes fall well short of their upper bound, it is suggested that they can appear to be unbounded above. In the case of short duration rainfall, this can be interpreted as being due to changes in the organizational structure of convective storms as we pass from the lesser to the greater extremes.

1. Introduction

A knowledge of the highest and lowest values which meteorological variables are likely to attain in a given number of years is important to many aspects of engineering design. The analysis of extreme values is therefore a topic of great importance in meteorology, and very good general accounts of the subject are given by Gumbel (1958) and Jenkinson (1977).

Many extreme value analyses of meteorological variables have been undertaken in the past. In the UK for instance, temperature has been analysed by Hopkins and Whyte (1975), wind by Hardman et al (1973), and rainfall by Jenkinson in the

Flood Studies Report (NERC, 1975). The application of extreme value analysis to meteorological data is seldom without its problems. Hopkins and Whyte, for example, found that the predicted upper bound of temperature was too low, while Jenkinson found that rainfall extremes appeared unbounded above. Hardman et al encountered problems with outliers, ie observations which, when plotted on extreme value probability paper, did not lie on the same general curve as the remainder of the data. An example of an outlier is shown in fig 1, which displays maximum temperature in June at Ivigtut on the south west coast of Greenland. Most of the observations lie between 13°C and 23°C, but the highest recorded temperature is 30°C.

In this paper, the assumptions behind the theory of extreme values are examined, and the difficulty of meteorological observations in meeting them is discussed. Suggestions are made as to how the various problems posed by analyses of meteorological extremes may best be interpreted. All the data used are tabulated in Appendix 2. They were either held in manuscript form within the Meteorological Office or extracted from the year books of the appropriate country.

2. Theory

Consider a series of independent observations belonging to the same population and divided into samples each containing N observations. The series of extreme values is constructed by selecting the highest (or lowest) observation from each sample. The results of this procedure are illustrated in fig 2, which displays schematically the probability density functions f(x) of the parent and extreme value distributions. The extreme value distribution is usually expressed in terms of the cumulative distribution function F(x), which represents the probability that an extreme value is less than x, and is given by

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = 1 - \int_{x}^{\infty} f(x)\,dx$$

Let the average number of observations in each sample greater than x = n.

Then the probability that an observation is greater than x $= n/N$

and the probability that an observation is less than x $= 1-n/N$.

Hence F(x) = probability that a maximum value is less than x $= (1-n/N)^N$

$$= e^{-n} \text{ if } p = n/N \text{ is small.} \qquad (1)$$

Exactly how small p must be is discussed in the next section following an argument put foreward by Jenkinson (1969).

When the cumulative probability F(x) is plotted against x, an S-shaped curve is obtained. It is usual to transform F(x) to a new variable y, known as the reduced variate, in which the cumulative probability distribution is represented by a straight line when plotted against x. (see fig 3). For the extreme value distribution, a common procedure is to set

$$y = -\log(u)$$

so that

$$F(x) = \exp[-e^{-y}]$$

The variable y can be related to the return period T. The value x which has the probability 1/T of being exceeded in any one sample is said to have a return period of T. The valve y = 0 corresponds to the mode of the probability density function of the extreme value distribution.

Jenkinson (1955) obtained a general solution for x in the form

$$x = x_0 + \alpha\left(\frac{1-e^{-ky}}{k}\right)$$

On a graph of x against y, x_o is the value at y = 0, is the slope at y = 0, and k is a curvature parameter. The solution may be categorised into 3 types corresponding to separate solutions previously obtained by Fisher and Tippett (1928). They have come to be known as Fisher - Tippett types I, II, and III and are characteristed by their different shapes when plotted on a graph of x against y (see fig 4)

(i) Type I corresponds to K = 0 and forms a straight line. It is the solution popularised by Gumbel (1958) and is unbounded above and below.

(ii) Type II corresponds to K $<$ 0 and is bounded below but not above.

(iii) Type III corresponds to K $>$ 0 and is bounded above but not below.

3. <u>Small samples</u>

The theory of extreme values requires that the extremes should be drawn from the tail of the parent distribution. This can be achieved by selecting from a large sample, ie by making N large, and this is reflected in equation (1) by the condition that p = n/N should be small. If N is insufficiently large then a number of the lesser extremes will not be drawn from the tail of the distribution.

Consider an analysis of 50 years of maximum temperatures for July. The number of observations from which extremes may be extracted is 31, but serial correlation reduces the number of independent values N to around 10. The highest temperature observed in 50 years will be associated with a value of n of around 0.02. In this case, e^{-n} and $(1-R/N)^N$, the latter term subsequently being denoted by Z, both equal 0.980, and so the theory of extreme values is satisfied. Suppose, however, we conside a modest extreme for which n = 2. This is associated with a temperature observation which, although the highest in a given July, has been exceeded 100 (independent) times in 50 years of record. In this case e^{-n} equals 0.135, but Z is only 0.107. It is clear, then, that this case fails to satisfy the theory of extreme values. If N is increased to 30, a value more appropriate to annual maximum temperatures, Z is increased to 0.126, while if N equals 150, corresponding approximately to 5 year extremes, then Z equals 0.135. Thus it can be seen that when temperature extremes are sampled once every 5 years, nearly all the maxima satisfy the theory of extreme vales, but that if the sampling period is as short as one month, only the more extreme maxima do so.

In any extreme value analysis, the greater extremes will always satisfy the theory more than the lesser extremes. Thus, in any given extreme value plot, a failure to draw observations from the tail of the parent distribution will be most readily apparent in the less extreme observations. There the influence of the parent distribution may be expected. Since the type I distribution has a skewness of 1.14, these effects will be most evident when the parent distribution has negative or large positive skewness. This is illustrated in fig 5 for maximum temperature in January at Oxford. The general extreme value distribution has been fitted by simulating 5 year maxima using a computer program designed by Jenkinson (1977). Although the general curve is clearly bounded above, and may be fitted by a type III distribution, the lowest 4 points clearly reflect the negative skewness of the parent distribution. (para cont. overleaf)

The contrast with the effects of a positively skewed parent distribution is evident in fig 6, which displays maximum temperatures for August at Santander, on the north coast of Spain.

Points drawn from a normal distribution are plotted on extreme value probability paper in fig 7. It can be seen that a sample of normally distributed observations could easily be accepted as belonging to an extreme value type III distribution. In practice, the only criterion for obtaining a set of data which displays linearity on extreme value probability paper is that it should have a skewness close to 1.14. Since positively skew distributions are common in meteorology, this explains why many sets of 'extreme values' appear to be well fitted by the type I distribution even though N falls short of that required by theory.

4. Mixed distributions

The discussion in section 3 assumed that the extremes were drawn from a single population. Where the observations are derived from several independent populations, each may be treated separately. One area where this approach has been adopted is in the analysis of winds in regions affected by tropical storms (ie hurricanes, typhoons). The general methodology is well described by Gomes and Vickery (1977).

If the data contain samples from Q populations, and the distribution of extremes of the qth population are denoted by $F_q(x)$, then the distribution of extremes associated with the mixed distribution if given by

$$F(x) = \prod_{q=1}^{Q} F_q(x)$$

A simple example is illustrated schematically in fig 8. The extreme winds are assumed to belong to two populations, those due to hurricanes, and those due to other causes. Each set of extremes is assumed to belong to a type I distribution. The combined probability distribution will then appear to be unbounded above, and to be similar to a type II distribution. Fig 9 presents an extreme value plot of winds for Progreso, in the Yucatan peninsula of Mexico. In this case, the discontinuity in the data is exceptionally well marked.

A problem with analyses involving hurricanes is that for many locations the storms themselves are rare while instrumental records are brief. The data are therefore subject to considerable sampling error. The problem has been tackled by using a statistical model to simulate hurricane winds. This involves expressing the wind in terms of the intensity of the storm and the distance from its centre, and then taking into account the frequency of hurricanes in the general vicinity of the station. A type I distribution is then fitted to the simulated series of extreme winds. Provided the fit is good, this is a perfectly reasonable practical step to take. The distribution may be regarded as an adequate description of the synthetic data, and may be usedto estimate return values within the range of simulated extremes. The relation has, however, no theoretical basis. For the data to satisfy the theory of extreme values, any given location would need to be visited by a large number of hurricanes each year. Extrapolation of the type I distribution beyond the range of simulated extremes therefore lacks any theoretical justification.

5. Seasonal variations

Most meteorological variables undergo a pronounced seasonal variation, and consequently the observations cannot be regarded as coming from the same population. By extracting annual maxima, therefore, the theory of extreme values is not being properly satisfied.

Consider a variable (eg wind) whose values rise to a seasonal maximum in (say) December. Suppose that a type I distribution is fitted to maximum values for December and the year. The results are illustrated schematically in fig 10. The slope for the monthly maxima will always be steeper than that for the annual. This is because the highest maximum for December is likely to be close to that for the year (if not equal to it) while in a season in which no strong winds are recorded in December, the annual maximum will occur in another month.

Fig 10 shows that linear extrapolation of the monthly relation will always lead to higher estimates than those for the year. Clearly, linear extrapolations of both lines is not possible. Either the slope of the annual fit has to increase towards that of the monthly, or the slope of the monthly relation has to decrease towards that of the annual.

Carter and Challenor (1981), analysing winds and wave height, obtain return values from linear extrapolation of the monthly extremes. The annual maxima are taken to represent a mixed distribution in which the data for each month are regarded as belonging to different populations.

An alternative view is as follows. Consider a variable which is identically distributed throughout the year. By extracting annual maxima, N is 12 times as large as for monthly maxima. By imposing a seasonal variation, N is reduced to one quarter (say) of its previous value, but this is still larger than that for monthly maxima. Accordingly, extreme value theory is more nearly satisfied for annual than for monthly maxima.

In order to decide which argument is correct, consider first the case of wind. One can easily imagine a month which is mainly anticyclonic, and in which no deep depressions pass close to a station. A low maximum wind is then recorded for the month. Similarly, for the case of short duration convective rainfall, one can easily imagine a summer month in which very little thunderstorm activity takes place. A low maximum rainfall is then recorded. It is these low maximum values which are responsible for the steep slope of the plots of monthly maxima. These 'extremes' are clearly not being drawn from the tail of the parent distribution. In other words, for monthly maxima, N is too small. The independence of monthly maxima, as required by the mixed distribution approach, may also be questioned. It follows that extrapolations of analyses of annual maxima are more firmly based than those of monthly maxima. It also follows that analyses of m year maxima (for $m > 1$) will be even more firmly based than those of annual maxima.

6. Populations and forcing factors

The theory outlined in section 2 required that the extremes be drawn from a series of observations which belonged to a single population. This condition may be relaxed to enable the original observations to belong to several populations. As long as the extremes are drawn from just one of these populations, and the sample of original data belonging to that population is large enough, the theory will be satisfied.

Consider a series of daily temperatures in January to be constructed from observations made at two stations, one in the arctic and the other in the tropics.

Suppose that for each year of data, 31 observations from the arctic station are followed by 31 observations from the tropical station. If the complete data series is then divided into samples of size 62, and all the maxima were observed (say) at the tropical station, then all the extremes will come from the same population. The appropriate sample size, however, will be only 31, and not 62.

Thus extreme value theory does not require that the original observations be exclusively drawn from a single population. Provided all the extremes belong to one population, and the associated sample size is large enough, the original data series may be comprised of observations from a large number of sources.

Meteorological observations may be assigned to different populations according to the external mechanism or forcing factor chiefly responsible for producing the observation. A series of extremes may then be regarded as belonging to the same population if a single forcing factor is responsible for the whole range of extremes examined. In meteorology there are so many degrees of freedom that this condition is rarely completely satisfied. In practice it is reasonable to regard the observations as belonging to the same population if one forcing factor is dominant.

Consider maximum temperatures in summer at a place like Oxford. The mean temperature (thickness) of the lower half of the troposphere may be regarded as the dominant forcing factor. As the thickness is determined by advection and dynamical subsidence, each may strictly be regarded as a separate variable, but on the hottest days they are highly correlated. Sunshine is another possible external variable, but at Oxford nearly all hot days in summer are sunny.

Suppose we could find a location which was almost permanently overcast. Then the extremes of maximum temperature would follow a type III distribution with thickness as the dominant external factor, but the observations would be drawn from cloudy days instead of sunny ones. If there was then one sunny day, the maximum temperature would be higher than before, and the type III curve would no longer provide a good fit to the observations. This would be because sunshine had become a second forcing factor.

In their analyses of annual maximum temperatures over the UK, Hopkins and Whyte (1975) found that a type III curve fitted to all the data produced an upper limit which was close to the highest recorded temperature, ie too low. This would be because on the days which produced the lower maxima, a combination of high thickness and prolonged sunshine may not have been achieved, and so these observations may be regarded as belonging to a different population from the majority of the maxima. This form of heterogeneity in annual extremes has previously been suggested by Jenkinson (1969).

In some locations, especially near coasts, wind direction is another possible forcing factor. We could imagine a coastal resort where the maximum temperature was almost always limited by a sea breeze. If on rare occasions a sea breeze failed to develop, much higher temperatures than usual would be observed, and a type III distribution would not provide a good fit to the observations.

In Britain the best examples of the effect of a second forcing factor on temperature are found in places affected by the fohn. Fig 11 displays a plot of maximum temperature for January at Aber in North Wales. The majority of observations may be fitted by a type III curve where thickness is the dominant factor, but the more extreme points may be regarded as lying on another curve in which the fohn is a forcing factor. The fohn is quite capable of producing the highest temperatures observed (18°C) since on those occasions temperatures at 900 mb were around 11°C. As fohns are rare, the events plotted in fig 11 will not represent extremes selected from a large sample, and there is no question of their satisfying the theory of extreme values. The dotted lines sketched through the fohn events in fig 9 are therefore purely empirical, and have no theoretical basis.

Another example of topography introducing additional forcing factors concerns the case of the Sheffield gales of 16th February 1962. These are described by Aanensen (1965) and were caused by standing waves set up by the Pennines. Much stronger winds were observed than if topographic effects had been absent, and so this event belongs to a different population from the majority of gales at Sheffield.

The outlier in the data for Ivigtut presented in fig 1 was probably caused by a fohn. Temperatures of 30°C around Southern Greenland are quite possible as is evidenced in fig 12 by a plot of maximum temperatures for June at Teigarhorn in south east Iceland. There the appearance of a type I curve is caused by the absence of a dominant forcing factor for many contiguous points, with thickness, wind direction, sunshine and fohn all playing a part.

7. Populations and sample sizes

In general, any set of data can be divided into a number (Q) of populations. As Q increases, the number N of independent cases from which the extremes are selected decreases. Thus while the aim of dividing data into separate populations is to provide a firmer foundation for extreme value analysis, this aim is negated by a decrease in N. The failure to satisfy extreme value theory is really due to one cause - lack of data. The two reasons advocated thus far, namely mixed populations and insufficiently large N, are merely different ways of expressing the same problem.

Consider the example of maximum temperature for January at Oxford (fig 5). In section 3, the lack of homogeneity in the data was expressed by saying that N was too small. The lowest values of the maxima were clearly not being drawn from the tail of the complete distribution. There is another way of looking at the problem. The majority of observations in fig 5 will be associated with incursions of tropical maritime air across the country. These can be regarded as constituting one population. The lowest January maxima in fig 5 are clearly not associated with tropical maritime air, and can therefore be regarded as belonging to a different population.

In section 6, Hopkins and Whytes' (1975) analysis of annual maximum temperature was discussed. The failure of the type III distribution to provide sensible extrapolations was attributed to the lower maxima belonging to a different population from the majority. An alternative way of expressing the problem is as follows. Hopkins and Whyte show that the lowest annual maximum at Oxford is 23°C. Since the average maximum in July is around 22°C, an observation of 23°C is clearly not being drawn from the tail of the parent distribution. Hence the failure to obtain a good extreme value analysis can be ascribed to insufficiently large N.

Similarly, consider a plot of extreme winds at a place like Progreso (fig 9) where hurricanes are a feature of the climate. Ones' immediate reaction is to ascribe the lack of a good fit to a general extreme value distribution as due to the presence of two populations, ie those winds due to hurricanes, and those due to other causes. Suppose, however, that sufficient data were available for a long series of century (as opposed to annual) maxima to be extracted. Then all the extreme events would be caused by hurricanes, and a good extreme value analysis for a single population could be obtained. It follows that the plot in fig 9 may be regarded as an inadequate sampling of hurricanes, ie the lack of fit to a general extreme value distribution is caused by too small a value of N.

The formation of a combined probability distribution from a number of populations (as indicated in section 4) is only valid if the populations are independent. Determining the independence of populations may not be easy, and many of the different populations described above probably could not be considered independent. In any analysis, therefore, it is sensible to keep the number of populations to a minimum, and to regard the data as belonging to a single population where possible. In general, this aim will be furthered by the choice of as large a value of N as possible (eg 5 year maxima instead of annual maxima).

8. Rainfall and the type II distribution

Annual extremes of daily and hourly rainfall are frequently best fitted by a type II distribution, and this has always caused problems of interpretation. In the Flood Studies Report (NERC, 1975). Jenkinson groups observations according to the magnitude of the fall which has a return period of 5 years. He shows that the greatest departures from a type I distribution occur for 5 year falls of 20 mm in England and Wales and 15 mm in Scotland and Northern Ireland. These falls correspond to a duration of around an hour, which is a typical duration of thunderstorms. The departure from a type I distribution is also greater in England and Wales than in Scotland or Northern Ireland. These facts suggest that the type II appearance of the observations may be related to the behaviour of individual convective storms.

Warrilow (1981) has shown that by taking the distribution of storm movement into account, modest rainfall extremes which follow a type III distribution following the storm (Lagrangian) are converted to a type II distribution when observed at a point (Eulerian). This is likely to account for most of the type II behaviour of observed rainfall extremes. Other possible contributory factors are as follows:-

(i) The complete distribution of short duration rainfall displays large positive skewness, so any failure to satisfy extreme value theory due to insufficiently large N will result in a concave upward distribution of the lesser extremes.

(ii) Most of the larger extremes will be due to thunderstorms, but in many places, some of the lesser extremes may be due to frontal rainfall. The mixture of populations would then give rise to a type II appearance to the observations. As convection will be dominant in the heaviest frontal rainfalls as well as in thunderstorms, however, the distinction between the two may not be as great as at first appears. Some of the heaviest rainfalls have been frontal in origin, but have contained embedded thunderstorms. If we restrict consideration to convective storms, however, it is true that as we pass from the lesser to the greater extremes, the organizational structure of storms also changes, from single cell through multicell to supercell. Hence the increasing organizational structure of storms as we pass from small to large return periods is likely to contribute to the type II distribution of rainfall extremes.

(iii) In certain areas, topography may encourage the development of stationary storms and give rise to a distribution of storm movements different from that considered by Warrilow. In districts thus affected, the conversion from the Lagrangian to Eulerian frame of reference will result in some spectacular type

II curves, and very large point rainfalls may have relatively modest return periods. In the UK, some of the large storms that have occurred in South West England, together with the Hampstead storm of 1975, may enter this category.

9. The Upper Bound

Many sets of meteorological extremes are well fitted by the type I distribution, but this is unbounded above. If physical considerations impose an upper bound,then eventually the type I distribution must give way to the type III. For events related to the duration of a single physical entity, eg a thunderstorm, gale, or an afternoon maximum temperature, it is clear than an upper limit to extremes must exist. For longer duration events, eg monthly rainfall, which involve a succession of physical entities, a realistic physically imposed upper limit is more difficult to visualize and evaluate. Most practical applications are covered with short duration events for which the concept of an upper limit is valid, and we now restrict consideration to these cases.

The return period at which extremes approach the upper limit varies widely with element. In the UK, for instance, maximum temperatures approach their upper boundfor return periods around 100 years, while for rainfall Jackson (1979) shows this does not happen until return periods of the order of a million years are reached. For a given element, the return period at which the upper bound is approached will also vary from place to place.

Consider the case of maximum temperatures. First compare typical inland and coastal sites. For any given return period, the maximum temperature on the coast will be lower than that inland. The upper limit on the coast, however, may be the same, or nearly the same, as inland. Although optimum conditions will be rarer, it is still possible to imagine a set of conditions in which the highest coastal temperatures would be (nearly) the same as those inland. The situation is illustrated schematically in fig 13. The inland station is represented by curve A, and the coastal resort by curve B. The upper limits at both locations are similar, but the return period at which this is approached is larger on the coast.

Suppose we now move from a linear stretch of coastline to a headland. For any given return period, the headland will experience lower temperatures than the remainder of the coast, and the highest temperatures experienced inland may never be reached. The slope of the extreme value analysis, although smaller than that for other inland or coastal locations, may well remain linear for longer return periods than in the previous two cases. This is illustrated by curve C in fig 13. Over the open sea, however (curve D), the probable maximum temperature will be much lower than over the land, and will probably be approached at similar (modest) return periods.

The same arrangements may be extended to other elements. In the case of wind, for instance, curve A may represent places like the Faeroes, where intense depressions are frequent. Curve B may then represent places further South, eg Valentia, where deep depressions are less frequent but still possible. In the case of rainfall, districts with frequent thunderstorms will be represented by curve A, while curve B represents places where they are less frequent, but still possible.

In general, the return period at which the upper limit is approached is inversely proportional to the frequency of the physical conditions which give rise to the probable maximum values. The modest return periods at which this occurs for temperature may be related to its more 'continuous' nature when compared for instance, to the 'discrete' nature of gales and rainfall.

When a set of extremes fall well short of their upper limit, a highly skewed parent distribution and an inadequate sampling of limiting physical conditions are indicated. The lack of data for extreme value analysis is then acute. If the observed extremes are believed to belong to the same population as those near the upper limit, it can be argued that the observed extremes are not being drawn from the tail of the single population. Alternatively, if it is argued that the observed extremes are being drawn from the tail of one population, then there are grounds for thinking that the highest possible extremes belong to another population. Under either argument, the observed extremes are likely to display the appearance of being unbounded above.

10. Conclusions

The theory of extreme values assumes that the maxima (or minima) are drawn from the tail of a distribution of observations which belong to a single population. Failure to satisfy the theory may therefore be caused by the inclusion of observations which are insufficiently extreme or which belong to more than one population. In meteorology, these reasons are alternative ways of expressing the same problem, namely lack of data.

Most meteorological variables undergo a pronounced seasonal variation, and consequently the observations cannot be regarded as coming from the same population. The problem may be tackled by regarding monthly maxima as belonging to separate populations, and then combining them to obtain a distribution of annual extremes. This approach, however, is compromised by the inclusion of insufficiently extreme monthly observations, and it is better to perform a straight-forward analysis of annual maxima.

In meteorology there are so many physical processes involved in the creation of a series of observations that defining a single population is difficult. In practice, a set of data may be regarded as belonging to the same population if a single forcing factor is primarily responsible for the range of extremes encountered. Topography can often act as a second forcing factor. It can cause high temperatures or strong winds through fohns and standing waves, and produce high point rainfalls by encouraging the development of stationary storms.

The return period at which extremes approach a physically imposed upper limit varies widely with element and location. When a set of maxima lie close to the upper bound, a type III distribution of the extremes may be expected. When the observed extremes fall well short of their upper limit, however, they may appear to be unbounded above. In the case of short duration rainfall, this may be interpreted as being due to changes in the structure of convective storms as we pass from the lesser to the greater extremes.

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Appendix 1 - Plotting Positions on Extreme Value Probability Paper

Intuitively, one expects a return period of around M to be associated with the largest of a series of M observations. This thinking is expressed by ascribing a cumulative probability p to the mth ranking observation of

$$p = \frac{m}{M+1} \tag{A1}$$

This formula was first suggested by Weibull (1939) and was popularised by Gumbel (1958). The data used in this paper were plotted according to the relation

$$p = \frac{m - 0.31}{M + 0.38} \tag{A2}$$

This equation was first proposed by Beard (1943) and has been widely used in the UK Meteorological Office following its adoption by Jenkinson (1969).

If 100 years of data are available then the first ranking observation is attributed a return period of 101 years by equation (A1) but 145 years by equation (A2). The difference between the two is essentially the difference between the mean and the median.

If 1000 years of data are available, then the event with a return period of 100 years may be approximated to the value of the 10th ranking observation. The distribution in time of these 10 largest events is not uniform. Their separation is bounded below by one and so a positively skew distribution emerges in which the median separation is less than the mean. Thus while the mean separation of the 10 largest events will be 100 years, the median separation will be less than this. Now the largest event in 100 years of data lies close to that whose median recurrence interval is 100 years. It can be shown that the mean return period of such an event is 145 years, and this is the result given by equation (A2).

It can now be seen that it is the skewed distribution of the separation of the most extreme events which causes the largest observation in M to have a return period greater than M. The Weibull formula will only be correct if the largest events are uniformly distributed in time. An excellent review of plotting positions in general is given by Cunnane (1978), who recommends the use of the relation

$$p = \frac{m - 0.4}{M + 0.2}$$

The difference in the plotting positions given by the above equation is generally small except for the largest extreme. In the case of a set of extremes which fitted the type I distribution, use of the Weibull plotting positions result in a slight 'type II' appearance of the observations, with the position of the largest event having the greatest error.

Appendix 2 - Data

A = Maximum temperatures in June at Ivigtut (^{o}C)

B = Maximum temperatures in January at Oxford (^{o}F)

C = Maximum temperatures in August at Santander (^{o}C)

D = Annual maximum gusts at Progreso (m/sec)

E = Maximum temperatures in January at Aber (^{o}F)

F = Maximum temperatures in June at Teigarhorn (^{o}C)

Year	1872	1872	1873	1874	1875	1876	1877	1878	1879	1880	1881	1882	1883	1884	188
A					17.4	19.6	21.5	20.0	19.0	18.8	18.0	19.2	18.5	13.5	18.
B	46	53	54	54	55	53	56	55	47	53	51	55	55	55	52

Year	1886	1887	1888	1889	1890	1891	1892	1893	1894	1895	1896	1897	1898	1899	190
A	16.6	18.3	20.5	15.2	21.3	21.2	20.7	17.4	16.1	21.9	18.0	23.0	20.0	16.9	19.
B	52	52	54	51	56	52	51	53	54	51	53	50	55	54	53

Year	1901	1902	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	1913	1914	191
A	21.0	20.6	20.6	19.1	18.4	19.4	17.4	17.8	22.2	17.7	17.3	19.9	18.9	14.1	30.
B	52	53	54	54	55	56	52	55	50	53	55	51	52	55	52

Year	1916	1917	1918	1919	1920	1921	1922	192	1924	1925	1926	1927	1928	1928	193
A	-	-	16.6	-	19.4	15.2	13.7	18.6	20.2	17.4	21.4	-	20.0	21.7	16.
B	57	52	55	52	55	55	57	54	52	55	53	55	56	53	58
C												30	31	27	38
D						17.2	25.6	17.8	24.4	19.4	19.4	16.1	19.4	18.9	23.
E										60	60	54	56	63	61
F							16.8	20.3	17.8	23.3	25.2	19.9	17.5	21.5	20.

Year	1931	1932	1933	1934	1935	1936	1937	1938	1939	1940	1941	1942	1943	1944	194
A	19.4	20.2	18.2	16.5	19.4	21.1	16.0	17.4	18.4	16.1	18.6	17.3	17.6	18.0	19.
B	52	55	53	54	54	57	53	54	55	51	48	50	55	56	50
C	26	32	28	26	26	27	35	23	24	25	24	23	40	30	26
D	19.4	22.2	27.8	17.8	23.3	22.2	22.2	33.3	20.6	22.8	22.8	19.4	21.1	20.0	18.
E	51	57	56	55	53	57	58	57	57	56	51	52	57	57	52
F	20.3	23.6	22.1	23.9	20.6	27.8	20.9	22.7	30.5	22.2	25.7	18.7	19.7	23.0	18.

Year	1946	1947	1948	1949	1950	1951	1952	1953	1954	1955	1956	1957	1958	1959	1960
A	16.2	23.1	19.6	19.2	21.0	21.7	18.6	20.0	16.5	20.5	19.8	19.6	17.8	18.0	18.8
B	57	56	57	53	54	53	52	55	56	53	54	57	55	52	54
C	30	31	29	24	27	30	29	35	29	29	25	30	26	28	34
D	22.2	21.1	34.0	32.0	48.0	42.0	38.0	17.0	41.0	32.0	44.0	36.0	36.0	31.0	19.4
E	55	56	56	56	61	53	53	54	57	57	56	58	65	55	58
F	17.3	17.0	18.1	22.0	16.0	17.5	16.6	18.6	18.7	18.1	19.2	17.7	16.2	17.5	24.1

Year	1961	1962	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972	1973	1974	1975
B	53	55	41	54	53	54	55	55	56	51					
C	30	25	23	34	27	27	25	30	24	25	28	28	32	27	28
D	16.7	18.9	19.4	18.3	22.8										
E	55	55	48	57	55	57	57	55	58	58	65	56	56	58	58
F	17.0	17.9	22.2	18.0	17.0	14.3	16.5	18.0	15.2	16.2	15.5	19.5	15.0	16.2	15.5

Year	1976	1977	1978	1979	1980
C	30	24	24	27	30
E	55	54	52		

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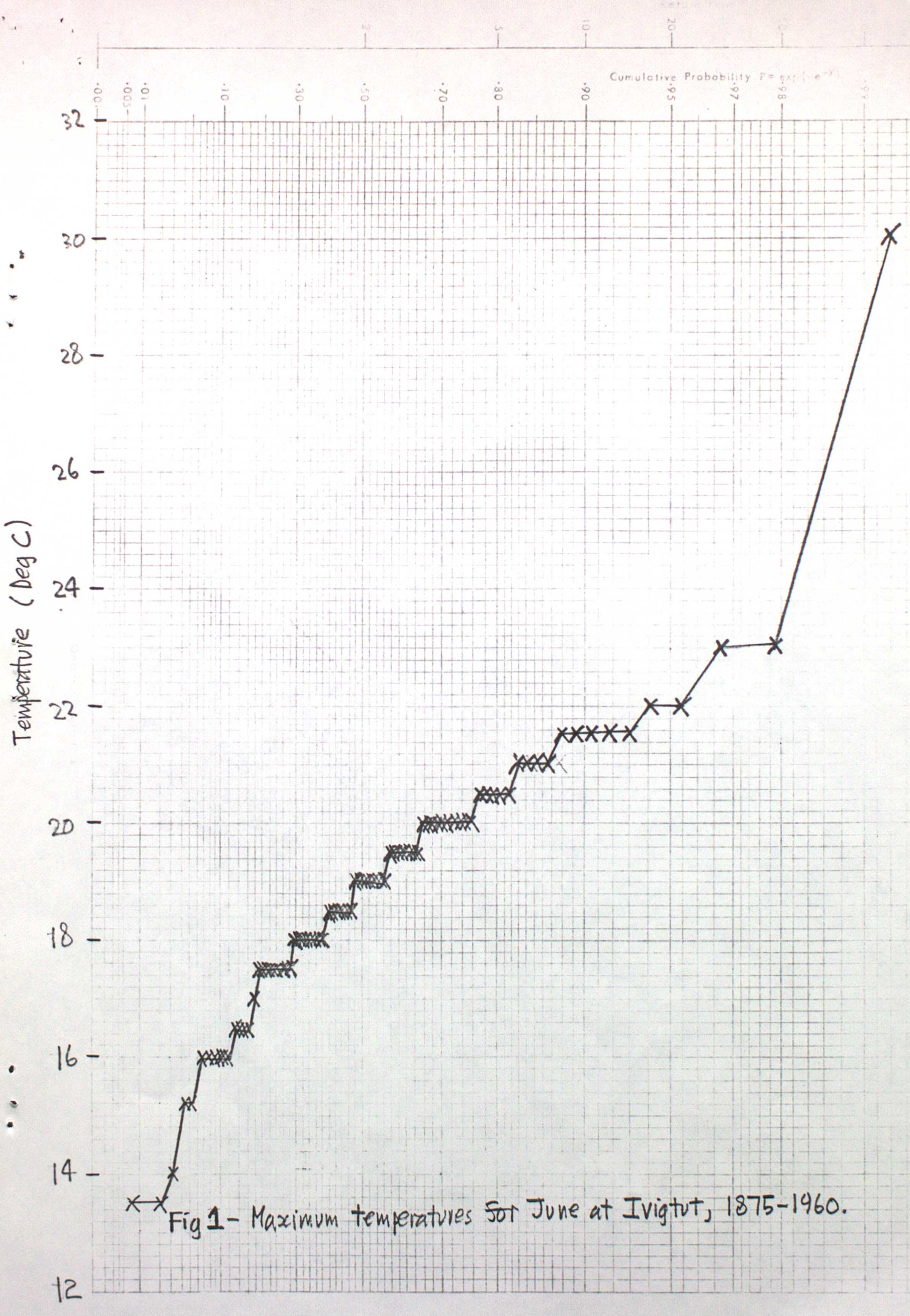


Fig 1- Maximum temperatures for June at Ivigtut, 1875-1960.

Fig 2 - Probability density functions of parent and extreme value distributions.

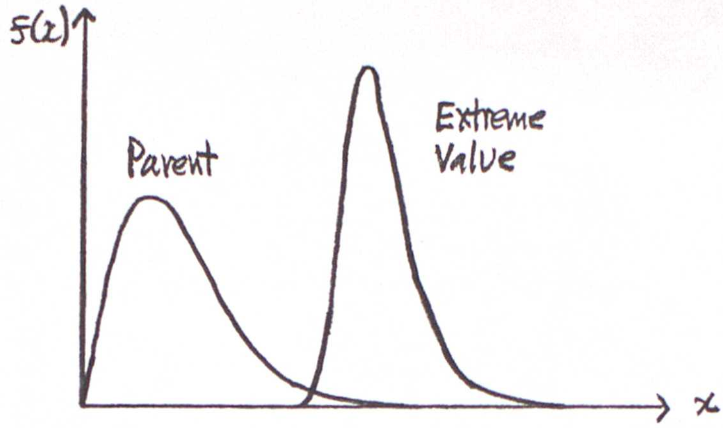
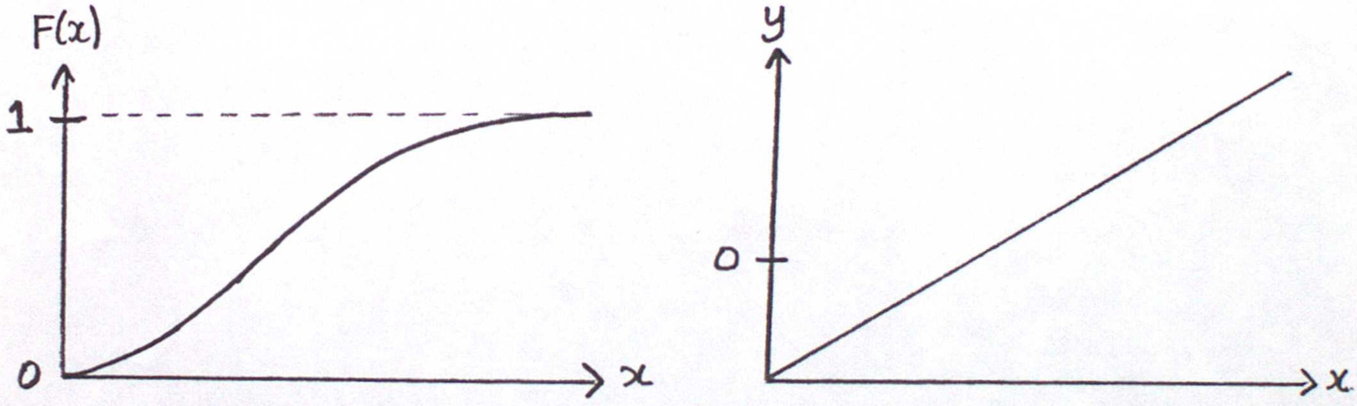


Fig 3 - Cumulative probability function of extreme value distribution



(a) F(x) plotted against x

(b) Reduced variate y plotted against x.

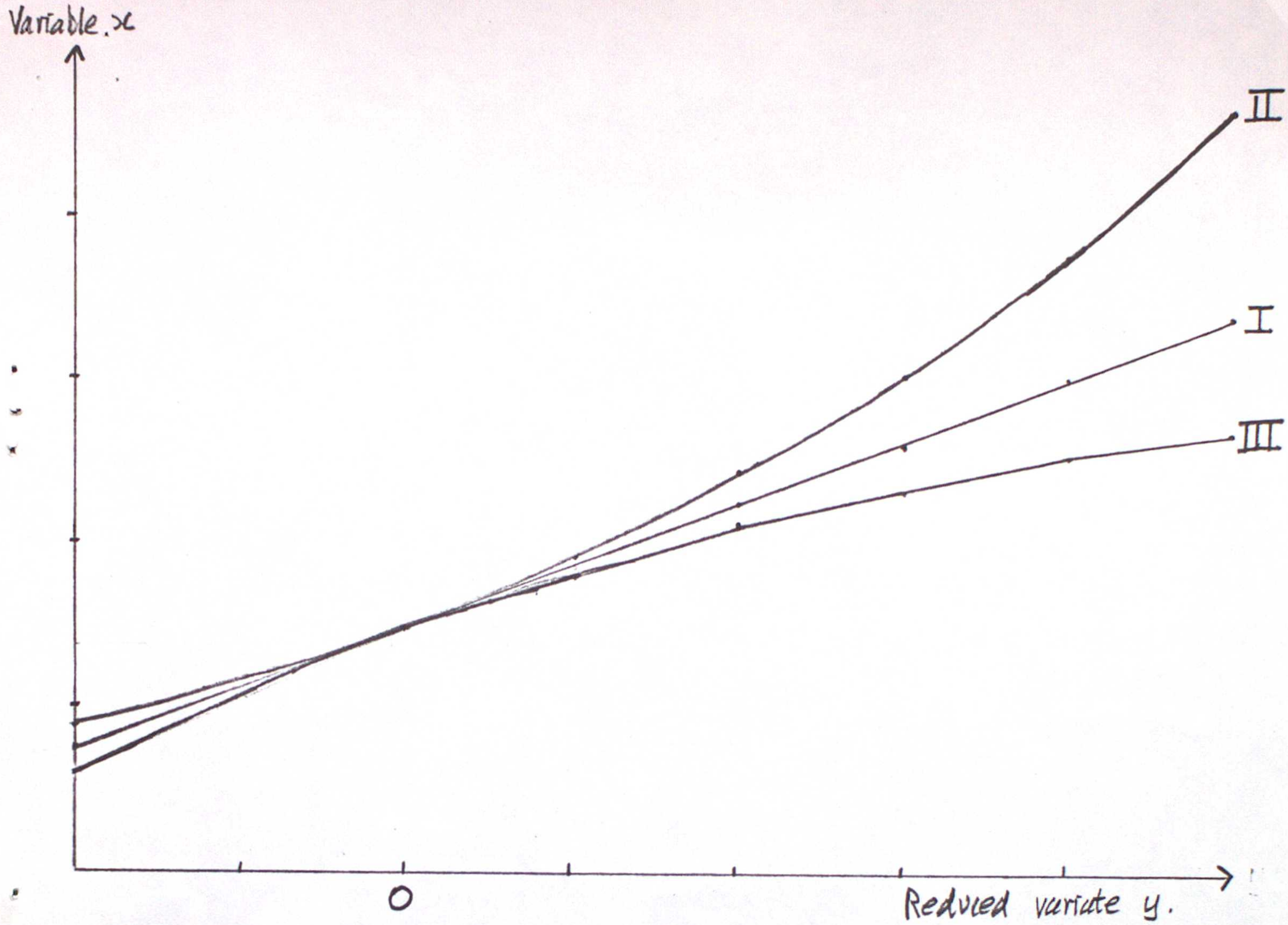


Fig 4 - Fisher - Tippett Distributions Types I, II, and III.

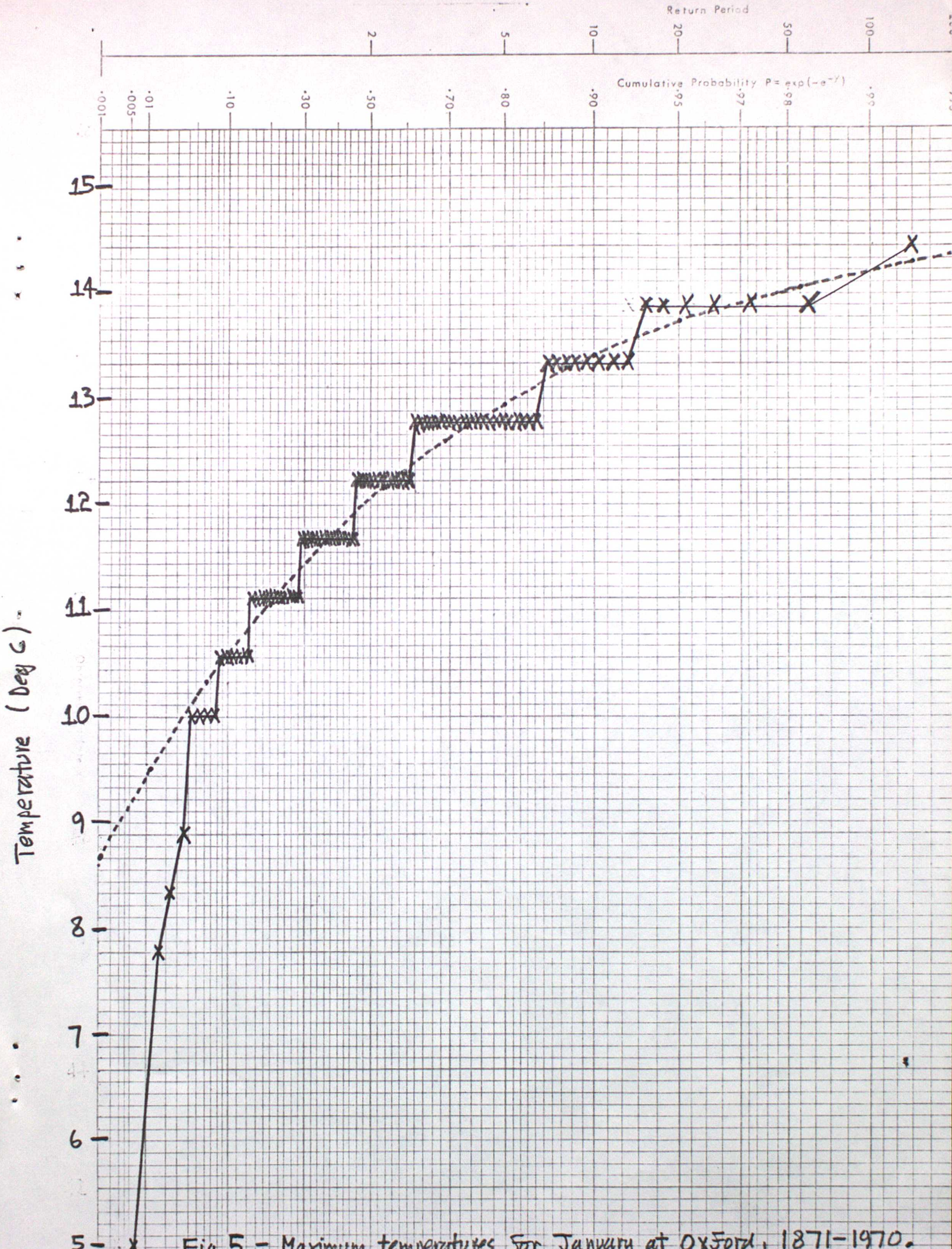


Fig 5 – Maximum temperatures for January at Oxford, 1871–1970.
The general extreme value distribution is represented by the dotted line.

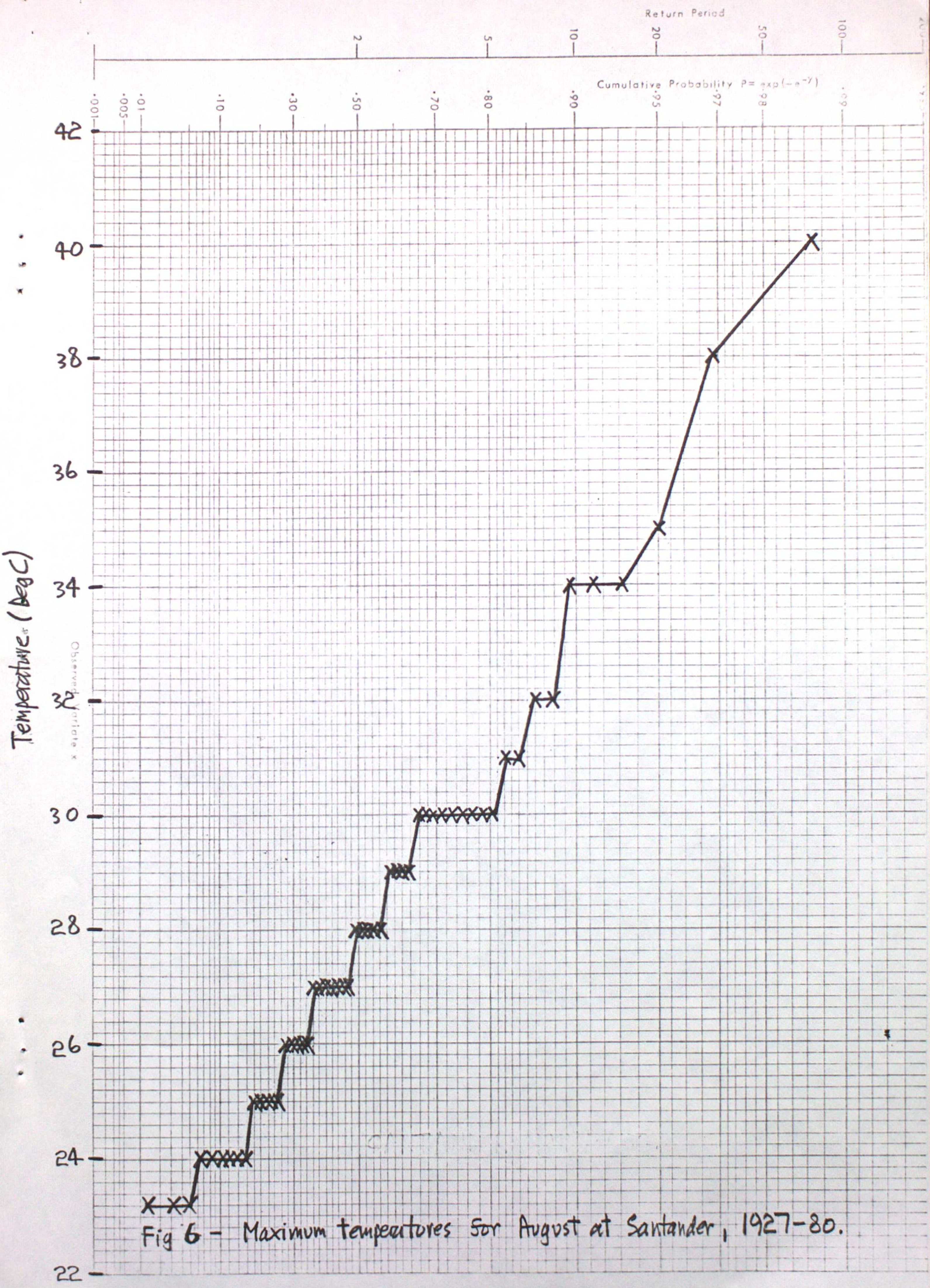


Fig 6 - Maximum temperatures for August at Santander, 1927-80.

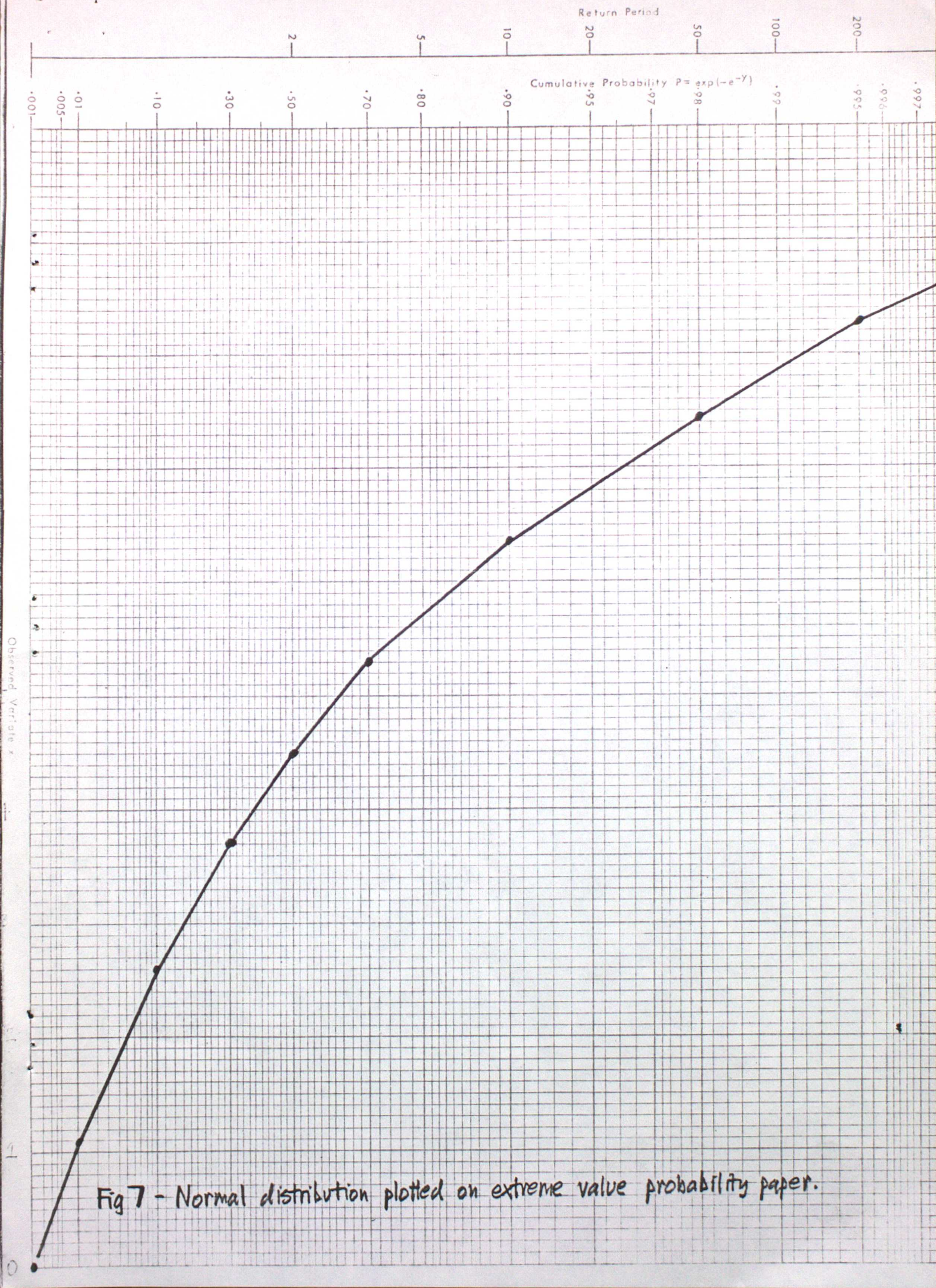


Fig 7 - Normal distribution plotted on extreme value probability paper.

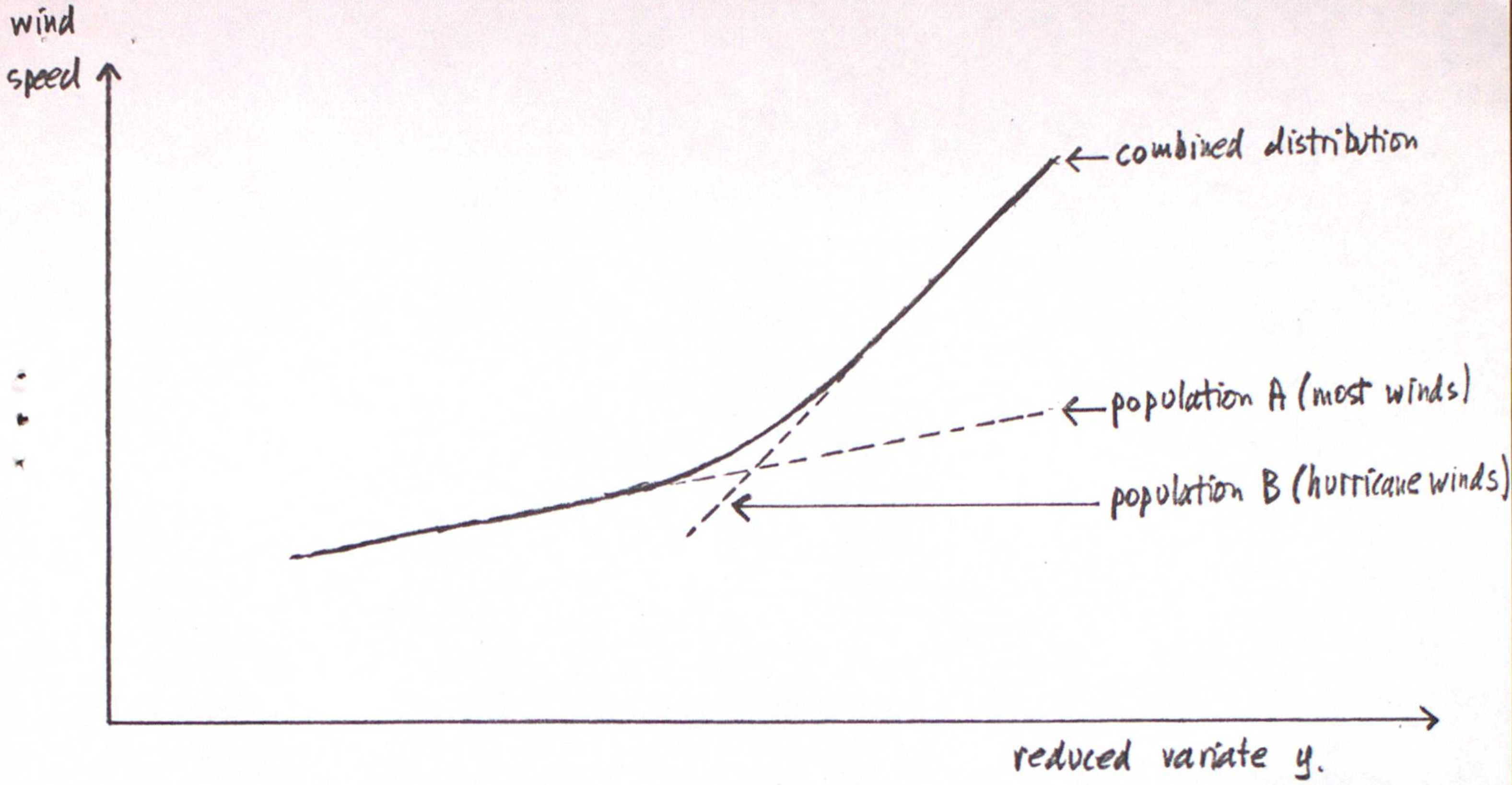


Fig 8 – Extreme value analysis of observations drawn from two populations.

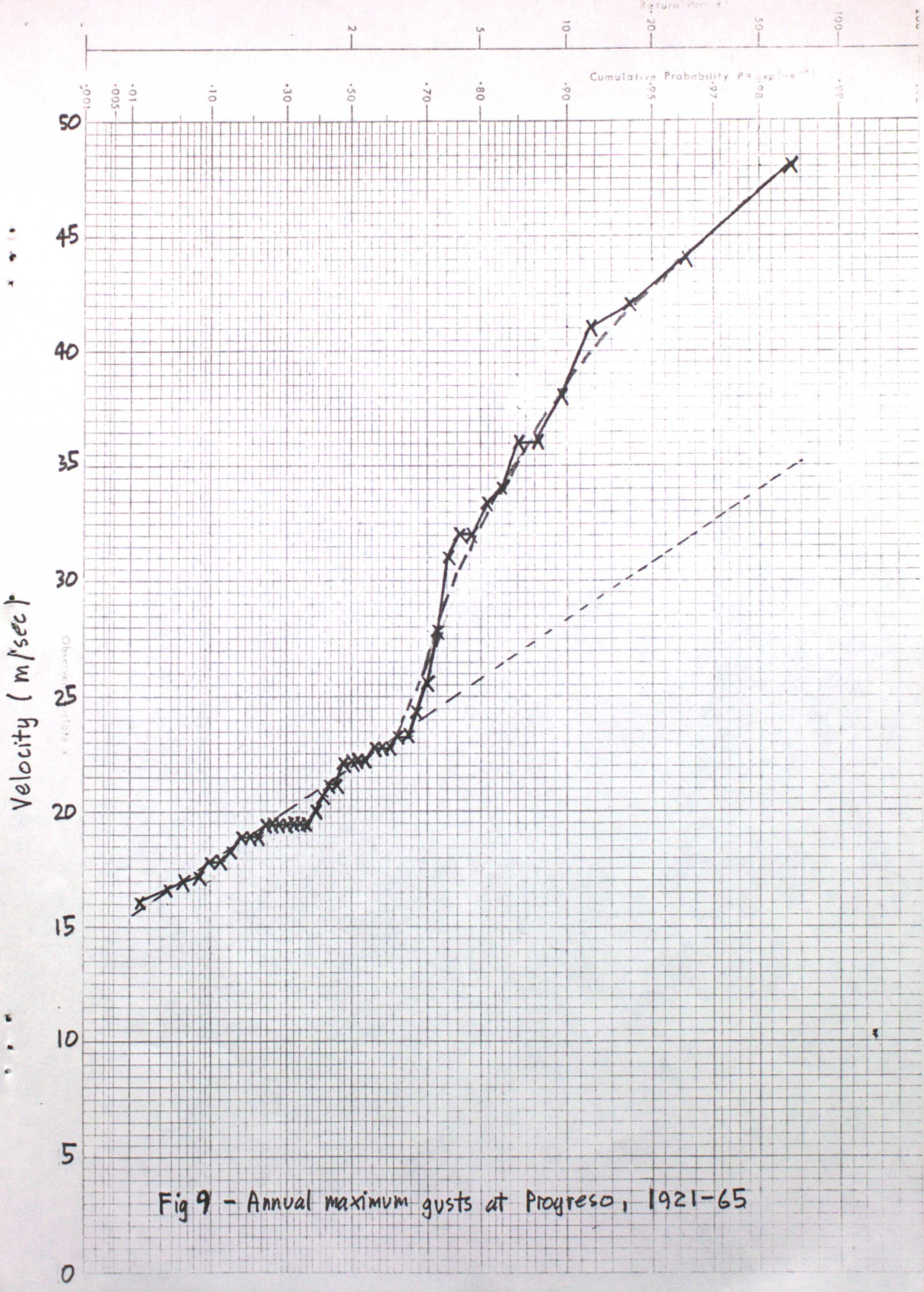


Fig 9 - Annual maximum gusts at Progreso, 1921-65

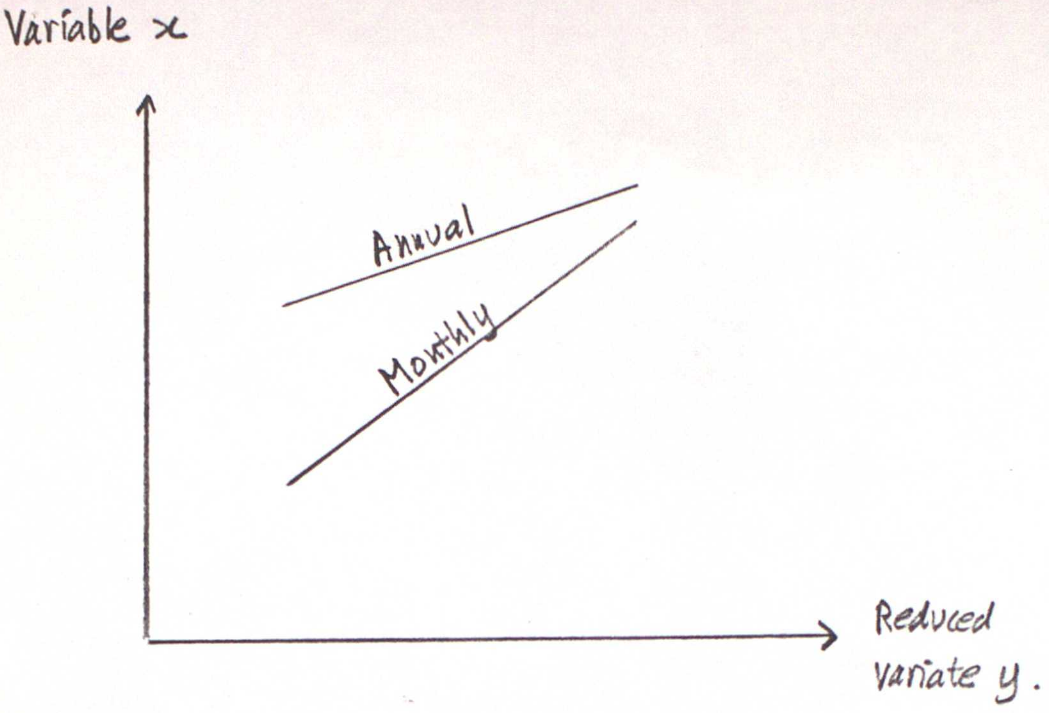


Fig 10 - Analyses of monthly and annual extremes.

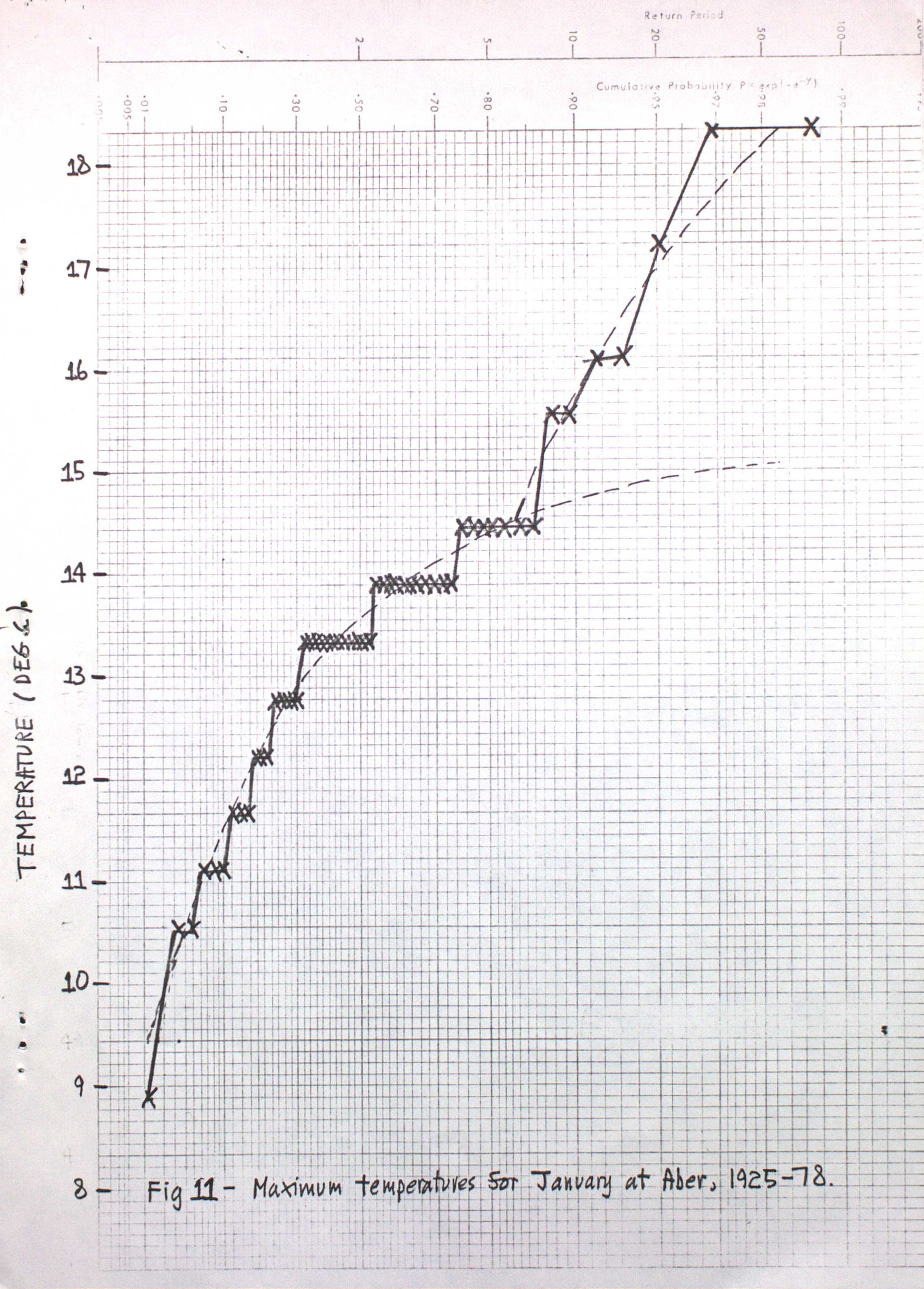


Fig 11 - Maximum temperatures for January at Aber, 1925-78.

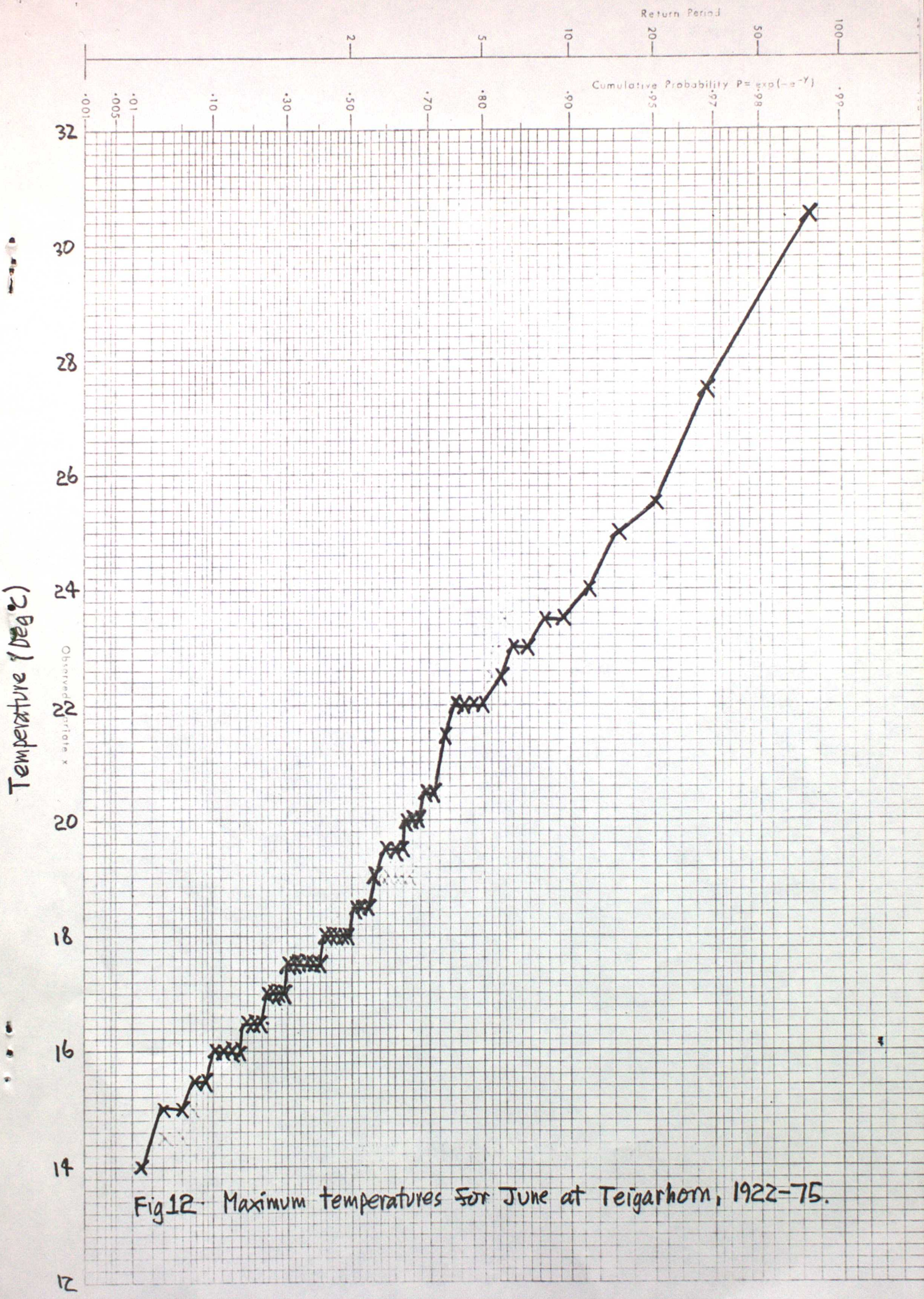


Fig 12 Maximum temperatures for June at Teigarhorn, 1922-75.

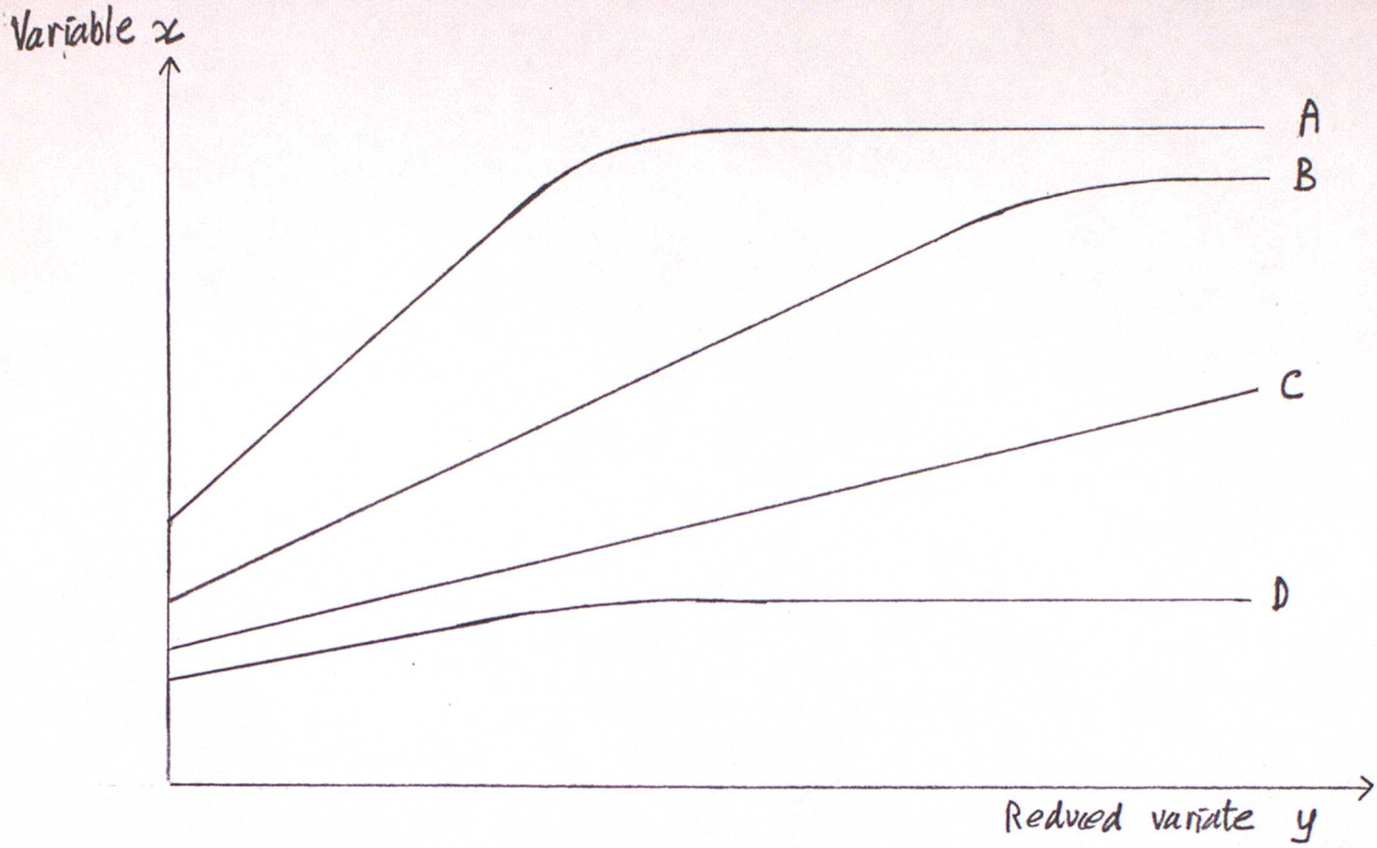


Fig 13 – Schematic extreme value analyses of temperature
(A) Inland (B) Coast (C) Headland (D) Open sea.