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A PARAMETERISATION OF DEEP PRECIPITATING CUMULUS FOR USE IN THE METEOROLOGICAL OFFICE OPERATIONAL 10 LEVEL MODEL

by

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1. Introduction

The importance of cumulus convection in large-scale disturbances is now widely recognised, particularly for large-scale tropical disturbances, through the fact that cumulus clouds are the major sources of latent heat release in the tropics, and also through the interactive mechanism between large and small scales, of conditional instability of the second kind (CISK), postulated by Charney and Eliassen (1964), and Ooyama (1964). Much effort has gone into parameterising the effects of cumuli, that is, modelling the effects of a population of cumuli rather than the individual clouds themselves. In particular, because of its importance in this area, most of the work has concentrated on developing parameterisations for inclusion in tropical prediction models, eg Krishnamurti et al (1974), and general circulation models, eg Arakawa (1971). Little research has been performed on the application of these parameterisations to temperate latitude forecasting models; typically they are inserted without major modification. This report describes a parameterisation scheme which is based on ideas from several parameterisations, chiefly those of Arakawa and Schubert (1974) and Kuo (1965).

Most cumulus parameterisation schemes fall into two classes: convective adjustment and penetrative convection. The former usually involves some form of mixing of adjacent layers of the atmosphere according to whether certain criteria of conditional instability are satisfied or not, eg Krishnamurti and Moxim (1971). A convective adjustment scheme developed for use in the Meteorological Office 10-level model is described by Benwell et al (1971). This scheme has been in operational use since August 1972. However there are certain defects in the scheme, the primary one being that it predicts too little convective rainfall. Examination of model behaviour indicates that the convective adjustment scheme described by Benwell et al handles the heat and moisture transports of shallow non-precipitating clouds only, and not shower-type clouds. The latter class of parameterisation, that of penetrating convection, is usually based on a parcel-type theory. Parcels of saturated cloud air are raised until thermal buoyancy carries them to higher layers of the atmosphere. Heat and moisture are transported in the clouds, and some schemes also model these transports in the air between clouds. Such models are based on mass conservation, where cumulus updraught is balanced by a compensating downdraught in the environmental air (the clear air between the clouds). They are based not so much on the conditional instability between adjacent atmospheric layers, but more on the conditional instability of many atmospheric layers. These models require closure through some estimate of the total updraught mass or moisture flux or the fraction of the grid square covered by cumulus clouds. This is usually handled by making an assumption about the kinetic energy generation (Arakawa and Schubert, 1974) or the grid-scale moisture convergence (Kuo, 1965).

The role of cumulus convection in the 10-level model is to provide a) rainfall rate information, b) areal accumulations of rainfall, c) interaction with large-scale systems. Parts a) and b) are satisfied only when both cumulus and layer-type clouds are present in the model. This is a different situation from the tropics, where the majority of rainfall is convective-type. Since the convective adjustment scheme alone does not appear to satisfy a) and b), it was decided to develop a parameterisation of cumulus based on penetrative convection. This was designed to work in conjunction with the convective adjustment scheme, modified to model only non-precipitating shallow cumulus. The scheme was designed for use in both temperate and tropical latitudes and finally, constrained to be used in an operational forecast model. This final condition imposes restrictions on the complexity of the model, while at the same time encouraging efficient use of data available to the scheme. At present the additional computer time required accounts for less than 5 per cent of the forecast.

2. The Model

2.1 The predictive equations

The model is designed as a two regime set, namely saturated cumulus updraught and a compensating unsaturated downdraught of 'environmental' air in the spaces between the cloud updraughts. "Updraught" and "downdraught" refer to deviations from the mean vertical ascent in the grid-square. The scheme models a population of clouds and so it is assumed that clouds are already in existence at the beginning of the time-step. In addition it is assumed that all clouds are independent of each other, and do not interact directly. The scheme is comprised of two parts: the diagnostic part, in which values of parameters in the clouds and in environmental air are derived from consideration of the grid square mean profiles of temperature and humidity, and the prognostic part, in which these parameters are used in a simple forward time stepping of the predictive parameterisation equations. The parameters diagnosed are: temperature (T), humidity mixing ratio (r), cloud water mixing ratio (l), large water drops mixing ratio (p), local vertical velocity (ω) and fractional cloud cover (α).

The prediction equations are basically of the same form as those quoted by Kuo (1965) and Arakawa and Schubert (1974). Their derivation is given in Appendix I. The difference between this parameterisation and others lies partly in the way the prediction equations are used and partly in the assumptions made to provide a closure to the problem. Kuo has an implicit form of the simplified equations, while Arakawa and Schubert integrate the equations over a spectrum of cloud sizes, and make their 'quasi-equilibrium' assumption to provide the closure. In this model the assumption is made that the clouds may be represented by one cloud type. This remains a common but not completely justified assumption. The prediction equations are then integrated using a simple forward time step. The equations become:

$$\frac{\partial \bar{s}}{\partial t} + \nabla.(\bar{\underline{v}}\bar{s}) + \frac{\partial}{\partial p}(\bar{\omega}\bar{s}) + \overline{\frac{\partial(\omega' s)}{\partial p}} = \bar{Q} + L\bar{W} \qquad (2.1)$$

$$\frac{\partial \bar{r}}{\partial t} + \nabla.(\bar{\underline{v}}\bar{r}) + \frac{\partial}{\partial p}(\bar{\omega}\bar{r}) + \overline{\frac{\partial(\omega' r)}{\partial p}} = \bar{W} + W_{cb} \qquad (2.2)$$

The symbols are defined in Appendix I. Barred parameters refer to grid mean values, unbarred refer to local values, and dashed parameters are deviations from the grid mean value due to cumulus clouds alone. The moist static energy s, is defined by:

$$s = C_p T + gz + Lr \tag{2.3}$$

It can be seen that the effects of cumulus clouds on the grid-mean values of s and r are restricted to two vertical flux terms and a moisture sink term (representing rainfall). All the other terms are handled by the integration of the large-scale equations. Thus the parameterisation consists of integrating forward in time the set:

$$\left.\frac{\partial \bar{s}}{\partial t}\right]_{\text{DUE TO CUMULUS}} = -\frac{\partial \overline{(\omega' s)}}{\partial p} \tag{2.4}$$

$$\left.\frac{\partial \bar{r}}{\partial t}\right]_{\text{DUE TO CUMULUS}} = -\frac{\partial \overline{(\omega' r)}}{\partial p} + W_{CB} \tag{2.5}$$

Integrating forward in time yields changes in the grid mean values of s and r, which may be suitably transformed back to changes of the model parameters of thickness (h') and r. In computing the changes, it is assumed that the percentage variation of z with time is sufficiently small to allow constant values derived from the I.C.A.O. standard atmosphere to be used.

Since cumulus clouds are assumed to be already in existence at the beginning of the time-step, the grid mean of any parameter X may be simply expressed by the formula:

$$\bar{X} = \alpha X_C + (1-\alpha) X_E \tag{2.6}$$

where subscripts C and E refer to values in the cloud and environment air respectively, and α is the fraction of the grid square covered by cumulus updraught at the level under consideration. The diagnostic part of the model computes cloud parameters, then environmental values are computed using equation (2.6). For computations of ω, only local deviations from the grid mean are required, and these may be computed using:

$$\overline{\omega'} = \alpha\omega'_c + (1-\alpha)\omega'_E = 0 \qquad (2.7)$$

since by definition the mean of the deviations of ω is zero.

It will be noted that in equation (2.4) and (2.5) there are no horizontal flux terms. This is the consequence of an assumption that, from the point of view of the parameterisation, the grid box is surrounded by solid walls. The lack of horizontal terms implies that fluxes due to entrainment and detrainment are not used in the computations. However, both entrainment and detrainment rates are used in the diagnostic determination of cloud parameters and the fluxes are used in the computation of rainfall. Because of the need for economy, the rain producing mechanism involves only a simple parameterisation of the conversion from cloud to precipitable water. The assumption is made that during a time step the temperature and water contents inside the clouds remain constant. This assumption seems reasonable since the model is intended to represent a population already in existence, which tends to modify the environment air rather than conditions inside the clouds. Any moisture pumped into a layer of cloud, either through vertical advection or through entrainment and detrainment must therefore be realised as rainfall waterdrops. This moisture is allowed to fall instantaneously to the ground, and is equated to the moisture sink term W_{CE} in equation (2.5). Since entrainment and detrainment are taken into account, horizontal fluxes of moisture must be included in the computation of rainfall.

The rates of entrainment of environmental air into the clouds and detrainment of cloudy air to the environment are based on data given by Yanai et al (1973) and Ogura et al (1973), and are constant values representing population means. Since the values are derived from the Marshal Islands data (Nitta, 1972), giving conditions in tropical latitudes, the rates have been modified by exaggerating the vertical scale so that the tropopause is moved from 225 mb to 300 mb. In this way it is hoped to produce values applicable to middle latitudes. The values are shown in Fig. 1. It can be seen that at low levels there is a net detrainment, representing mainly the small shallow cumuli. At mid-levels there is a net entrainment, suggesting a jet-type regime, and finally at high levels close to the tropopause, a net detrainment at the maximum height of the clouds. In this way it is intended that conditions in the population of clouds are modelled closer than if a representative cloud with a 'mean cloud radius' were chosen, with entrainment derived, for example, from Simpson and Wiggert's (1969) formula:

$$\text{Entrainment rate,} \quad \frac{1}{m}\frac{dm}{dz} \cong \frac{0.2}{\text{Cloud Radius (Km)}} \tag{2.8}$$

Writing the equation in terms of s and r renders the predictive equations in a particularly simple form. Since the moist static energy is conserved in both lifting of air and condensation of moisture, there is no source/sink term; s is only advected. The moisture sink can only represent the loss of moisture due to rain fallout, since clouds are not of themselves a source of moisture into the grid square. The actual source of moisture is either large-scale convergence or transfer by evaporation from the earth's surface, and these processes are already dealt with by the large scale equations. Finally it can be seen that, since the assumption is made that s and r remain constant in the cloud, the effects of the cumulus population on grid-scale parameters are due to 1) subsidence in the environment, warming and drying the atmosphere 2) entrainment of environmental air and detrainment of cloudy air. This will have the effect of warming and moistening higher levels, and drying out lower levels. The net effect will be a redistribution of moist static energy throughout the depth of convection.

2.2 Criteria for convection

In order to initiate convection in a grid square several criteria derived from the large scale parameters must be satisfied. Firstly and most importantly there must be conditional instability present, that is, a parcel with a certain defined temperature and humidity when raised from one level to the next will have a net positive buoyancy with respect to the surrounding air. The buoyancy is tested by comparing the virtual temperatures of cloud and environment air. The second criterion is based on the low level moisture convergence (c.f. Kuo 1965). The initiating disturbance for deep convection is related to the transfer of moisture from the earth's surface, and also the low level large scale upward forcing of moisture. The parameter I is defined as:

$$I = -\bar{\omega}_{900}\bar{r}_{900} + \left.\frac{\partial \bar{r}}{\partial t}\right]_{SE} \Delta p \qquad (2.9)$$

where the upward flux of moisture $\bar{\omega}\bar{r}$ is computed at 900 mb, $\left.\frac{\partial \bar{r}}{\partial t}\right]_{SE}$ is the flux of moisture from the earth's surface, and Δp is the depth of a layer in the 10-level model. In Kuo's paper, a variable similar to I is equated to the flux of moisture needed to produce all the latent heating and increase in water content required to produce clouds. This is acceptable in tropical regions, where the major precipitation mechanism is cumulus overturning. However in temperate latitudes, rain producing layer clouds also exist. Thus I cannot be entirely devoted to producing cumulus clouds; the same upward forcing can also produce layer clouds. Experiment has shown that merely partitioning I into these two mechanisms has a detrimental effect on the total rainfall, since cumulus clouds also redistribute moisture to dryer upper layers. The solution is found by regarding I as the initiating impulse which triggers off convection. Clouds will continue to exist subsequently because the conditional instability has been released. I is thus equated to the flux of moisture required to initiate convection. It may be seen from equation (2.9) that the flux of moisture is enhanced or damped by the presence of uplift or subsidence on the grid-scale. Thus the second criterion for convection is:

$$\text{if } I \leqslant 0, \text{ no convection can occur} \tag{2.10}$$

A third criterion for convection is mentioned in the next section.

2.3 Cloud Base Values

In order to derive values of temperature, humidity, etc for inclusion in equations (2.4) and (2.5), a one dimensional diagnostic model is used, based on a parcel of air rising through and entraining the environmental air. The level defined as the cloud base level is taken to be the cloud base level of existing shallow cumulus (convective adjustment) or of pre-existing deep cumulus, whichever is the lower. If the cloud base level cannot be found in this way it is taken as 900 mb. The air fed into the cloud base originates from the layer below the cloud base level. At this point, a third criterion for convection is applied. In order for a cloud to be formed, a parcel of air from the layer below the cloud base level must achieve saturation when raised to the next layer. The criterion is applied approximately by testing if the initial relative humidity of the parcel exceeds a certain minimum value. The minimum values for different layers were read off a tephigram, and are 60, 57, 52, 45, 37 and 26 percents respectively for the layers centred on 950 mb to 450 mb. Cloud bases above 400 mb are not included in the model. This criterion arises from the need to prevent spurious rainfall over semi-arid areas of the model.

Nearly all the convection models based on penetrative convection require some initiating disturbance to release conditional instability, and it has been the subject of much experiment to determine a satisfactory disturbance for this model. The most consistent and seasonally independent results have been obtained by an increase $\Delta\theta_{\omega}$ in the wet bulb potential temperature of the parcel after it has been raised to the first layer. This has the advantage that the internal energy of the parcel is perturbed, but neither the temperature or humidity increment of the original parcel is explicitly defined. A value of 2.0°C for $\Delta\theta_w$ has been demonstrated to be satisfactory.

The procedure adopted is to take a parcel with the grid-mean values of temperature and humidity, and to raise it adiabatically, as described in the next section. At the layer above cloud base level, the parcel is given increments in both temperature and humidity mixing ratio corresponding to the increment $\Delta\theta_w$. The increment in humidity mixing ratio is computed from $\Delta\theta_w$ via the Clausius-Clapeyron equation. The parcel is mixed with entrained air, then moisture is allowed to condense or evaporate to bring the parcel to saturation. The virtual temperature of the parcel (inversely proportional to the density) is then compared with the virtual temperature of the environmental air. If the parcel is warmer, and therefore buoyant, convection proceeds to the next layer.

2.4 Raising the Parcel

An economical and simple way of raising the parcel adiabatically is to raise dry air and moisture content independently to the new layer, then allow moisture to condense, with appropriate latent heating until the parcel becomes saturated.

Let p_0 and p_1 be the lower and upper pressure levels involved (see Fig. 2). After lifting, but before adjusting to saturation, the temperature T, humidity mixing ratio r, and cloud water content W of the parcel will be

$$T = \left(\frac{p_1}{p_0}\right)^{\kappa} T_0 \quad , \quad r = r_0 \quad , \quad W = W_0 \qquad (2.11)$$

Writing the temperature, humidity mixing ratio and total water content after adjustment as T^*, $r_s(T^*)$, W^*, then

$$L(r - r_s(T^*)) = c_p(T^* - T) \qquad \text{– latent heating due to condensation} \qquad (2.12)$$

$$r - r_s(T^*) = W^* - W \qquad \text{— conservation of moisture} \qquad (2.13)$$

$$r_s(T^*) - r_s(T) = (T^* - T)\frac{\epsilon L r_s(T)}{RT^2} \qquad \text{— Clausius-Clapeyron} \qquad (2.14)$$

These equations may be solved iteratively to provide values of T^*, $r_s(T^*)$ and W^*, but in practice just one iteration gives sufficient accuracy for the model. When T^* is computed to be less than -10°C, the iteration is repeated using the latent heat of sublimation rather than condensation. In this way the ice phase is incorporated into the model.

After adiabatic lifting the parcel is mixed with entrained environmental air. After mixing, the parcel temperature, humidity mixing ratio and water content will be

$$\frac{T_C + \gamma T_E}{1+\gamma} \quad , \quad \frac{r_s(T_C) + \gamma r_E}{1+\gamma} \quad , \quad \frac{W_C}{1+\gamma} \qquad (2.15)$$

where subscripts C and E refer to cloud and environment values, and γ is the rate of entrainment per 100 mb.

According to Ludlam (1966) deep cumulus towers are formed by bubbles of air ascending through stagnant cloudy air in lower layers. This enables bubbles to ascend higher since the decelerating effect of entraining unsaturated environmental air is reduced. In this model the surrounding air which is entrained is cloud air if there is shallow cumulus - defined by a convective adjustment in the layer under consideration. The cloud air is assumed to be at the temperature of the environmental air, but saturated. If there was no convective adjustment, true environmental air is entrained.

Finally equations (2.12) to (2.14) are solved again to bring the parcel back to saturation. The final values of T and r are taken to be the cloud values at the new level. The model needs to be modified above 350 mb since no moisture is held in the 10-level model above this level. A method of approximating the moist adiabatic lifting from 350 to 250 mb is given in Appendix II. Above 250 mb the parcel is raised along the dry adiabatic lapse rate. Above 350 mb the cloud moisture variables are set to zero. However, in order to conserve moisture, the environmental downard moisture flux at 300 mb is set equal to the upward cloud moisture flux at that level.

In response to experiments which have shown that values of W computed in this way are too large for cloud liquid water content, they are split into two parts: large raindrop p, and small cloud droplets l , according to a crude parameterisation of the coalescence mechanism, similar in form to that suggested by Kessler (1965),

$$p = A(W - B) \;,\; \ell = W - p \tag{2.16}$$

Values of A and B for the liquid and ice phases are set out in figure 3. In this model it is assumed that small cloud droplets are advected in the updraught, while large raindrops fall against the updraught.

2.5 Vertical Acceleration

Vertical velocities in the cloud are computed by integrating Newton's second law:

$$\frac{d}{dt}(m\dot{z}) = F \tag{2.17}$$

over a slab of air 100 mb thick. The body forces acting on the parcel are:

a. Thermal buoyancy, $g^* \dfrac{(T_{v_c} - T_{v_E})}{T_{v_E}}$

b. Condensate drag, $-g^*w$

c. Aerodynamic drag $-\dfrac{\gamma \omega^2}{\rho g}$ (see Perkey and Kreitzburg, 1972)

where g^* is the gravitational constant modified to include the effect of the virtual mass of the accelerating bubbles of air. The equation derived from the integration of equation (2.17) is of the form:

$$\omega_2^2 = A_1 \omega_1^2 + B_1 \bar{F} \tag{2.18}$$

where A_1 and B_1 are constants. This equation is derived in Appendix III. This is a relatively simple form for programming, and the vertical velocity at each level is easily computed using an initial value, w = 0 in the layer below cloud base level. The cloud tops are defined to be the level at which the updraught velocity returns to zero.

2.6 Cloud Area

A cloud area varying with height is assumed. Following Perkey and Kreitzburg (1972), the mass entrainment and detrainment coefficients, $\frac{1}{m}\frac{dm}{dp}$ are equated to the mass flux entrainment and detrainment coefficients $\frac{1}{M_F}\frac{dM_F}{dp}$

Thus: rate of entrainment + detrainment = $\gamma + \delta = -\frac{1}{M_F}\frac{dM_F}{dp}$ (2.19)

where $M_F = \alpha\,\omega_c$. Integrating between two pressure levels p_o and p_1, ($p_o > p_1$)

$$\left[dM_F\right]_{p_o}^{p_1} = \left[-(\gamma+\delta)\bar{M}_F\,dp\right]_{p_o}^{p_1}$$

ie
$$M_{F_0} - M_{F_1} = \tfrac{1}{2}(M_{F_0} + M_{F_1})(\gamma+\delta)\Delta p$$

where γ and δ are constant in the layer.

so
$$M_{F1} = M_{Fo} \times \frac{\left[1 + \frac{1}{2}(\gamma+\delta)\Delta p\right]}{\left[1 - \frac{1}{2}(\gamma+\delta)\Delta p\right]} \qquad (2.20)$$

Since the total mass flux in the grid square is $A\alpha\omega_c$ where A is the grid area, and α the fractional cloud (ie updraught) cover, equation (2.20) becomes

$$\alpha_1 = \alpha_o \frac{\omega_o}{\omega_1}\frac{\left[1 + \frac{1}{2}(\gamma+\delta)\Delta p\right]}{\left[1 - \frac{1}{2}(\gamma+\delta)\Delta p\right]} \qquad (2.21)$$

This equation is used to evaluate the fractional cloud cover at all levels in the model. However, there remains the problem of evaluating the fractional cloud cover at the cloud base level. As described earlier, the upward available moisture flux I is seen as an initiating moisture flux, and is equated to the initiating impulse given to clouds in the grid area. The wet-bulb potential temperature increment $\Delta\theta_\omega$ is equivalent to temperature and humidity mixing ratio increments ΔT and Δr_s where $\Delta T = \Delta\theta_\omega$ and Δr_s maybe computed from the Clausius-Clapeyron equation. (See equation (2.14)). The moisture producing these increments is $\left(\frac{c_p}{L}\Delta T + \Delta r_s\right)\Delta p$.
I is equated tp the total moisture flux required to produce initiation of all the clouds:

$$I = \alpha_{CBL}\left(\frac{c_p}{L}\Delta T + \Delta r_s\right)\frac{\Delta p}{\Delta t}$$

ie
$$\alpha_{CBL} = \frac{I\Delta t}{\left(\frac{c_p}{L}\Delta T + \Delta r_s\right)\Delta p} \qquad (2.22)$$

where Δt is the time step.
The equation is applied at the cloud base level.

Finally, values of parameters in the environmental air are computed using equation (2.6).

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Appendix I   Derivation of the Parameterisation prognostic equations

The thermodynamic and water balance equations are:

$$C_p \frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} = Q \qquad (I.1)$$

$$\frac{dr}{dt} = W \qquad (I.2)$$

with the notation of Scientific Paper No 32 (Benwell, et al 1971)

Using the hydrostatic approximation equation (I.1) becomes

$$\frac{d}{dt}(C_pT + gz) = Q \qquad (I.3)$$

Define the moist static energy s

$$s = C_pT + gz + Lr$$

where L is the latent heat of evaporation.

Then, from equations (I.2) and (I.3)

$$\frac{ds}{dt} = Q + LW \qquad (I.4)$$

That is

$$\frac{\partial s}{\partial t} + \underline{v}.\nabla s + \omega\frac{\partial s}{\partial p} = Q + LW \qquad (I.5)$$

Use of the equation of continuity of mass renders the equation in flux form

$$\frac{\partial s}{\partial t} + \nabla.(\underline{v}s) + \frac{\partial(\omega s)}{\partial p} = Q + LW \qquad (I.6)$$

Similarly equation (I.2) becomes

$$\frac{\partial r}{\partial t} + \nabla.(\underline{v}r) + \frac{\partial}{\partial p}(\omega r) = W \qquad (I.7)$$

Now write $\underline{v} = \underline{\bar{v}} + v'$, $w = \bar{w} + w'$, etc where the barred quantity is the grid mean value, and the dashed quantity refers to deviations about the grid mean due solely to the presence of cumulus clouds. Thus $\int_A s' \, dA = 0$, etc where A is the grid area. Substituting into equation (I.6), integrating over the grid area, and making the assumption that the grid area has impermeable boundaries, we get

$$\frac{\partial \bar{s}}{\partial t} + \nabla.(\underline{\bar{v}}\,\bar{s}) + \frac{\partial}{\partial p}(\bar{\omega}\,\bar{s}) + \overline{\frac{\partial}{\partial p}(\omega' s)} = \bar{Q} + L\bar{W} + \overline{Q'} + \overline{LW'} \qquad \text{(I.8)}$$

Note that since the moist static energy is conserved in evaporation or condensation, the clouds are not a source or sink of s in the grid square, and the last term of equation (I.8) vanishes. Finally, with equation (I.7) treated in the same way, we get

$$\frac{\partial \bar{s}}{\partial t} + \nabla.(\underline{\bar{v}}\,\bar{s}) + \frac{\partial}{\partial p}(\bar{\omega}\,\bar{s}) + \overline{\frac{\partial}{\partial p}(\omega' s)} = \bar{Q} + L\bar{W} \qquad \text{(I.9)}$$

$$\frac{\partial \bar{r}}{\partial t} + \nabla.(\underline{\bar{v}}\,\bar{r}) + \frac{\partial}{\partial p}(\bar{\omega}\bar{r}) + \overline{\frac{\partial}{\partial p}(\omega' r)} = \bar{W} + W_{CB} \qquad \text{(I.10)}$$

where W_{CB} is the water sink term due to rainfall from clouds.

Appendix II   Raising the Cloud Parcel from 350 mb to 250 mb

Above 350 mb no moisture is stored in the 10-level model. However, in order to raise the parcel above this level, a lapse rate other than the dry adiabat must be used for a moist parcel. Above 250 mb, however, the dry adiabat is sufficiently accurate.

In ascending, the saturated humidity mixing ratio of the parcel changes by dr_s:

$$dr_s = \left.\frac{dr_s}{dp}\right)_T dp + \left.\frac{dr_s}{dT}\right)_p dT$$

$$r_s(p_{350}, T_{350}) - r_s(p_{250}, T_{250}) = \left.\frac{dr_s}{dp}\right)_T dp + \left.\frac{dr_s}{dT}\right)_p dT$$

so $$r_s(p_{250}, T_{250}) = -\left.\frac{dr_s}{dp}\right)_T dp + r_s(p_{350}, T_{250})$$ approximately.

Now $$r_s = \frac{\epsilon e_s}{p}, \text{ so } \left.\frac{dr_s}{dp}\right)_T = -\frac{\epsilon e_s}{p^2} = -\frac{r_s}{p}$$

Thus $$r_s(p_{250}, T_{250}) = r_s(p_{350}, T_{250})\frac{\Delta p}{p} + r_s(p_{350}, T_{250})$$ approximately.

$$r_s(p_{250}, T_{250}) = \left(1 + \frac{\Delta p}{p}\right) r_s(p_{350}, T_{250}) \qquad \text{(II.1)}$$

The procedure adopted in raising the parcel from 350 mb to 250 mb is as follows:

a) Raise the parcel along the dry adiabat to obtain T_{250} before condensation.

b) Adjust T_{250} to include the latent heating due to the condensation of moisture, dr_s

where $$dr_s = r_s(p_{350}, T_{350}) - r_s(p_{250}, T_{250})$$

ie $$dr_s = r_s(p_{350}, T_{350}) - \left(1 + \frac{\Delta p}{p}\right) r_s(p_{350}, T_{250}) \qquad \text{(II.2)}$$

This liquid condensate, dr_s is also used in the computation of vertical velocity at 250 mb.

The approximation:

$$\left.\frac{dr_s}{dT}\right)_p dT \cong \left. dr_s \right)_p$$

used in obtaining equation (II.1) is not very accurate in view of the size of dT. However, the formula is included in the model since in practice it is more accurate than simply using the dry adiabatic lapse rate, and also because it is computationally very economical.

Appendix III <u>Derivation of Cloud Vertical Velocities</u>

Newtons 2nd law:

$$\frac{d}{dt}(m\dot{z}) = \sum_i F_i \qquad \text{(III.1)}$$

where $\sum_i F_i$ is the sum of body forces on the mass m

Using the hydrostatic relation,

$$\frac{dp}{dt} = -\rho g \frac{dz}{dt}, \quad \text{so} \quad \dot{z} = -\frac{\omega}{\rho g}$$

Hence

$$\frac{d}{dt}\left(-\frac{m\omega}{\rho g}\right) = \sum_i F_i$$

ie

$$\frac{1}{m} \cdot \frac{dp}{dt} \cdot \frac{d}{dp}\left(-\frac{m\omega}{\rho g}\right) = \frac{\sum_i F_i}{m} = F\text{, say} \qquad \text{(III.2)}$$

The body forces per unit mass are, as described in the text:

$$F = \frac{g^* (T_{V_c} - T_{V_E})}{T_{V_E}} - g^*(\ell + p)$$

= Thermal buoyancy − Liquid condensate drag

Also an aerodynamic drag $-\dfrac{\gamma\omega^2}{\rho g}$ is applied (Perkey and Kreitzburg, 1972)

g^* is defined by: $g^* = \dfrac{g}{1+\nu}$ where ν is a virtual mass coefficient (Batchelor, p 404) for a sphere moving through a fluid. It has a value of 0.5. γ is the fractional rate of entrainment for the parcel.

Thus equation (III.2) becomes

$$\frac{\omega}{m}\frac{d}{dp}\left(-\frac{m\omega}{\rho g}\right) = F - \frac{\gamma\omega^2}{\rho g} \qquad \text{(III.3)}$$

Expanding and rewriting:

$$\frac{\omega}{\rho g}\frac{d}{dp}\left(\frac{\omega}{\rho g}\right) = 2\gamma\left(\frac{\omega^2}{\rho^2 g^2} - \frac{F}{2\gamma\rho g}\right)$$

Writing $a = \dfrac{\omega}{\rho g}$ and $X = \dfrac{F}{2\gamma\rho g}$, we get

$$\frac{2a\,da}{a^2 - X} = 4\gamma\,dp$$

Integrating between two pressure levels p_1 and p_2 ($p_1 > p_2$), and replacing X by $\overline{X}$, the mean value in the interval (p_1, p_2), we get

$$\ln\left[\frac{a_2^2 - \overline{X}}{a_1^2 - \overline{X}}\right] = -4\gamma\Delta p \qquad (\Delta p = p_1 - p_2)$$

ie $$a_2^2 = a_1^2 \exp(-4\gamma\Delta p) + \overline{X}(1 - \exp(-4\gamma\Delta p))$$

Substituting back and rearranging:

$$\omega_2^2 = \omega_1^2\left(\frac{\rho_2}{\rho_1}\right)^2 \exp(-4\gamma\Delta p) + \overline{\left(\frac{F}{\rho g}\right)}\,\frac{\rho_2^2 g}{2\gamma}\,(1 - \exp(-4\gamma\Delta p)) \qquad \text{(III.4)}$$

By using ICAO standard values of density and constant entrainment rates, it can be seen that this equation is of the form

$\omega_2^2 = A_1\omega_1^2 + B_1\dfrac{\overline{F}}{g}$, where A_1 and B_1 are constants for each layer in the model.

Pressure (mb)	Entrainment, γ $(100mb)^{-1}$	Detrainment, δ $(100mb)^{-1}$
200	0·20	-1·00
300	0·20	-0·48
400	0·20	-0·23
500	0·27	-0·16
600	0·19	-0·16
700	0·19	-0·22
800	0·29	-0·46
900	0·77	-0·96

FIGURE 1

Entrainment and Detrainment Rates used in the Parametrisation

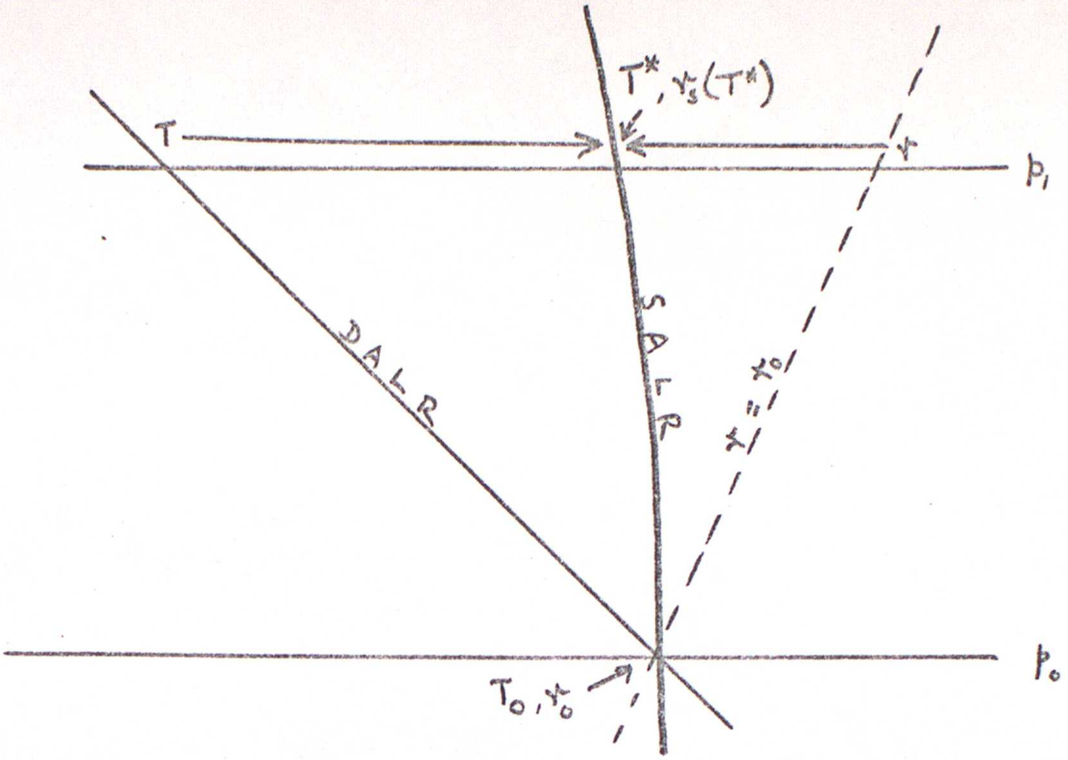

FIGURE 2

Raising the Parcel

	A	B
$T \geqslant -10°C$	0·5	0·5
$T < -10°C$	0·7	0·2

FIGURE 3

Constants used in Parametrisation of Microphysics