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Data-base for meteorological filters.
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MET O 16 BRANCH MEMORANDUM NO 2

DATA-BASE FOR METEOROLOGICAL FILTERS

(A data-base of technical data essential for the implementation of hardware or software filters)

by A C L LEE

DATA-BASE FOR METEOROLOGICAL FILTERS

1. Introduction

Met O 16 Branch Memo 1 describes how filters, particularly Polynomial filters of the Bessel type, can be advantageous for Meteorological use. Met O 16 Branch Memo 3 describes a method of implementing desired linear transfer functions, that can be described by Pole-Zero patterns in a Complex Plane, using computationally economical software. This document attempts to give technical data essential for the implementation of filters in both hardware and software. Emphasis is laid on filters that can be described in terms of poles and zeroes, as these filters can be made equivalent in hardware and software. Certain references are available within Met O 16, and these are noted.

2. Low Pass Filters

2.1 Met O 16 Branch Memo 1 gives some discussion of Low-Pass filters, and the trade-off between performance in the time and frequency domain.

2.2 Other forms of filter (High-Pass, Band-Pass, Band-Stop) can often be described mathematically by suitable manipulation of the mathematical description of a Low-Pass.

2.3 We will consider only filters that can be described in terms of the complex frequency s $(s \equiv j\omega)$. The simplest class of Low-Pass filter that can be described in this way is the Polynomial Filter:

$$F(s) = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + 1} \qquad (1)$$

where F(s) is the Transfer Function of the filter, being the ratio of output signal to input signal as a function of ω for the signal $e^{j\omega t}$

It can be seen that (1) describes a Low-Pass Filter because the transfer function is around unity for $s \to 0$, and decreases to 0 as s becomes very large.

2.4 The order of a Polynomial Filter is the highest value of n. Once the order has been chosen, the individual coefficients can be chosen to give a suitable Transfer Function, often maximising some required property such as sharpness of cut-off.

2.5 The polynomial in s can be factorised to find the roots of the polynomial. These are the "poles" of the Transfer Function:

$$F(s) = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + 1}$$

$$\text{becomes} = \frac{1/a_n}{(s+\sigma_1+j\omega_1)(s+\sigma_1-j\omega_1)\cdots\cdots(s+\sigma_m+j\omega_m)(s+\sigma_m-j\omega_m)}$$

where m = n/2 if n is even (2)

$$\text{or} = \frac{1/a_n}{(s+\sigma_0)(s+\sigma_1+j\omega_1)(s+\sigma_1-j\omega_1)\cdots\cdots(s+\sigma_m+j\omega_m)(s+\sigma_m-j\omega_m)}$$

where $m = \frac{n-1}{2}$ if n is odd.

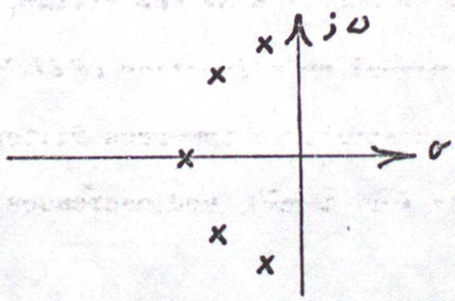
The poles are the values of s for which F(s) becomes infinite,

ie

Poles are: s = $-\sigma_0$ (if n is even)

$$\left.\begin{array}{l} (-\sigma_1 + j\,\omega_1) \\ (-\sigma_1 - j\,\omega_1) \\ \quad\vdots \\ (-\sigma_m + j\,\omega_m) \\ (-\sigma_m - j\,\omega_m) \end{array}\right\} \quad (3)$$

Poles are often shown as crosses on the "Complex Plane", their position denoting the values of σ and ω:



A particular pattern of crosses (poles) denotes a particular filter.

Non-Polynomial Transfer Functions may have a numerator polynomial in s. The roots of this polynomial are termed "Zeros" These are usually shown as small circles on a complex plane.

2.6 Once a pattern of poles has been established, and gives some suitable frequency performance, this frequency performance can be scaled. For example, if a pole pattern gives 3 dB attenuation at $\omega = \omega_1$ radian/sec, it can be scaled to give 3 dB attenuation at ω_2 radians/sec by shifting poles (and zeros) radially from (0,0) by a factor ω_2/ω_1. This is equivalent to replacing s by $\left(\frac{s\,\omega_1}{\omega_2}\right)$ in eqns (1) and (2).

2.7 The Transfer Function of any given pattern of poles, or set of coefficients, can be calculated as a function of ω by using eqn (2), or eqn (1). In practice, however, data for the most useful patterns can usually be obtained as curves from reference books such as Zverev (1967), or Christian (1966). The former is held in Met 0 16f. The latter is out of print, but can be obtained through a library.

2.8 For Meteorological Purposes the Bessel Filter is important. This series of pole patterns is designed to have a maximally flat group delay, as discussed in Met 0 16 Branch Memo No 1. For convenience, the first few orders of Bessel Filter are given here, normalised to give 3 dB attenuation at 1 radian/sec:

Order	Bessel Filter Poles (σ, ω) (Normalised to 3 dB at 1 radian/sec)
2	$(-1.10349, \pm 0.63710)$;
3	$(-1.32475, 0)$; $(-1.04909, \pm 1.00085)$
4	$(-1.37222, \pm 0.41085)$; $(-0.99681, \pm 1.25913)$
5	$(-1.50470, 0)$; $(-1.38301, \pm 0.71904)$ $(-0.95913, \pm 1.47345)$;
6	$(-1.57404, \pm 0.32145)$; $(-1.38405, \pm 0.97304)$ $(-0.93219, \pm 1.66457)$;
7	$(-1.68713, 0)$; $(-1.61464, \pm 0.59014)$ $(-1.38113, \pm 1.19353)$; $(-0.91130, \pm 1.83942)$
8	$(-1.76034, \pm 0.27330)$; $(-1.63966, \pm 0.82417)$ $(-1.37613, \pm 1.39067)$; $(-0.89433, \pm 2.00166)$

2.9 It is sometimes convenient to express poles and complex conjugate pole-pairs in terms of the "natural frequency" fn, and the "Quality Factor" Q of a resonant circuit. It is possible to calculate (σ, ω) from such data by using the relations:

$$\sigma = -\frac{fn}{2Q} \qquad\qquad \omega = \pm\sqrt{f_n^2 - \sigma^2} \qquad (4)$$

Table 1 reproduces a sheet of such data taken from the Burr-Brown data sheet on the UAF41 Universal Active Filter, giving details on Bessel and Butterworth Filters, and also some forms of Chebyshev filter.

2.10 The Chebyshev type of Polynomial Filter (discussed in Met O 16 Branch Memo No 1) has an extra degree of freedom in that the pass-band ripple can be chosen. This means that a concise listing can only be given for selected examples. Two examples, for 0.5 dB and 2 dB ripple, are given in Table 1. Some examples are given in Zverev (1967) which gives nomograms for selecting Chebyshev filters according to their frequency domain properties (Chapter 5), and also shows various response curves for selecting filters. Pole-patterns are listed for Chebyshev filters having pass-band ripple values of 0.0004 dB, 0.0017 dB, 0.0039 dB, 0.007 dB, 0.011 dB, 0.028 dB, 0.044 dB, 0.099 dB, 0.18 dB, 0.28 dB, and 1.25 dB for orders $n = 3$ to 7. These are shown as degenerate forms of Cauer -Chebyshev filters. Fig 6.5 of Zverev will be found useful. It should be noted that Zverev defines "Poles" and "Zeros" in the opposite way to the above convention. Further examples (with pass-band ripple values of 0.1 dB, 0.25 dB, 0.5 dB, 1 dB, 2 dB and 3 dB for filter orders 2 to 10) are given in Shepard (1969). This latter reference is really a listing of capacitor values for hardware implementation of filters, but for even order filters the poles are easily extracted from the data

TABLE I

NUMBER OF POLES	BUTTERWORTH		BESSEL		CHEBYSCHEV 0.5dB RIPPLE		CHEBYSCHEV 2dB RIPPLE	
	fn(1)	Q	fn(1)	Q	fn(2)	Q	fn(2)	Q
2	1.0	0.70711	1.2742	0.57735	1.23134	0.86372	0.907227	1.1286
3	1.0	----	1.32475	----	0.626456	----	0.368911	----
	1.0	1.0	1.44993	0.69104	1.068853	1.7062	0.941326	2.5516
4	1.0	0.54118	1.43241	0.52193	0.597002	0.70511	0.470711	0.9294
	1.0	1.3065	1.60594	0.80554	1.031270	2.9406	0.963678	4.59388
5	1.0	----	1.50470	----	0.362320	----	0.218308	----
	1.0	0.61805	1.55876	0.56354	0.690483	1.1778	0.627017	1.77509
	1.0	1.61812	1.75812	0.91652	1.017735	4.5450	0.97579	7.23228
6	1.0	0.51763	1.60653	0.51032	0.396229	0.68364	0.31611	0.9016
	1.0	0.70711	1.69186	0.61120	0.768121	1.8104	0.730027	2.84426
	1.0	1.93349	1.90782	1.0233	1.011446	6.5128	0.982828	10.4616
7	1.0	----	1.68713	----	0.256170	----	0.155410	----
	1.0	0.55497	1.71911	0.53235	0.503863	1.0916	0.460853	1.64642
	1.0	0.80192	1.82539	0.66083	0.822729	2.5755	0.797114	4.11507
	1.0	2.2472	2.05279	1.1263	1.008022	8.8418	0.987226	14.2802
8	1.0	0.50980	1.78143	0.50599	0.296736	0.67657	0.237699	0.89236
	1.0	0.60134	1.83514	0.55961	0.598874	1.6107	0.571925	2.5327
	1.0	0.89998	1.95645	0.71085	0.861007	3.4657	0.842486	5.58354
	1.0	2.5629	2.19237	1.2257	1.005984	11.5305	0.990142	18.6873

(1) -3 dB Frequency
(2) Frequency at which amplitude response passes through the ripple band.

by using the expressions:

$$\sigma = -\frac{1}{C_1} \qquad \omega = \pm\sqrt{\left\{\frac{1}{C_1 C_2} - \frac{1}{C_1^2}\right\}} \tag{5}$$

A more complete listing is in Christian (1966).

2.11 For specialised applications it is possible to determine the poles of Chebyshev filters with any value of pass-band ripple by manipulating the basic Cheyshev Polynomials. The method is explained in Zverev (1967) Chapter 3.5, where Chebyshev Polynomials up to order 12 (corresponding to order 12 filters) are listed. The method involves extracting complex roots of real polynomials, for which Hewlett-Packard calculator library programs exist, or else the I.B.M. Scientific Subroutine Package can be used. An example of the method is given in D/Met O 16f/88 E5.

2.12 The Chebyshev form of filter gives the sharpest rate of increase of attenuation outside the pass-band for any polynomial filter (implemented with poles only). By allowing ripple in both pass-band and stop-band (implying a numerator polynomial and the existence of filter zeros) the rate of increase of attenuation can be considerably increased. These "Cauer -Chebyshev" filters are listed in Zverev (1967) Chapter 5, and also in Christian (1966). The time-domain responses are, of course, considerably inferior to Chebyshev, Butterworth, or Bessel filters.

3. Transformation of Low-Pass to High-Pass, Band-Pass, or Band-Stop Filters

Method of performing the above transformations are discussed in Zverev(1967) Chapter 5. Transfer Functions $F\left(\frac{s}{\omega_c}\right)$ representing Low-Pass filters can be transformed to other filter forms by substituting for $\frac{s}{\omega_c}$ in $F\left(\frac{s}{\omega_c}\right)$:

High-Pass: $H\left(\frac{s}{\omega_c}\right) = F\left(\frac{\omega_c}{s}\right)$

Band-Pass: $BP(\frac{s}{\alpha}) = F(\frac{s}{\omega_c} - \frac{\omega_c}{s})$

where $a = \frac{fm}{\Delta f}$

fm = mid-band frequency

Δf = band width

Band-Stop: $BS(\frac{s}{a}) = F\left(\frac{s\,\omega_c}{\alpha(s^2 + \omega_c^2)}\right)$

where $a = \frac{f_0}{f_1 - f_2}$

f_0 = geometric mid-frequency

f_2 = low cut-off frequency

f_1 = upper cut-off frequency

Equivalent transformations can be performed on poles and zeros in the Complex Plane (which must include any zeros at infinity) to produce new Pole-Zero patterns. These can be implemented as shown below.
The transformations shown above are not the only ones possible. If special conditions are required, eg precise arithmetic symmetry in a band-pass or band-stop filter, then other transformations are possible.
See D/Met O 16f/88 E2 and E3.

4. Hardware Implementation of Filters

4.1 While designing a system (usually analogue) that has a filtering function, it is usually best to consider the system as a whole, and so design the optimum pole-zero pattern (or filter type) for the application. Implementation is then usually performed by designing a passive network, or by cascading active stages which each each implement a small number of poles and zeros. It is possible to implement a larger number of poles, and even incorporate gain, in one stage. However, in practice this puts severe demands on the performance of amplifiers if near ideal design performance is to be obtained. Required performance demand increases as the Q of individual stages is increased. This increases as one goes through the filter types: Bessel, Butterworth, Chebyshev, Cauer-Chebyshev; and increases with filter order number. It also increases with transformation to band-pass or band-stop, especially if the overall band-widths are narrow. Slew-rates of amplifiers are likely to be important as well as gain-bandwidth products.

4.2 Passive networks are listed, as normalised component values, for various types of filter in Zverev (1967).

4.3 Various types of Active RC filters are discussed and explained in a collection of important papers collected together in a book edited by L P Huelsman (1976). This book is available in Met O 16. This discusses Sallen and Key Elements, State Variable Filters, Bi-Quadratic Building Blocks and other topics. It also discusses methods of simulating inductances, such as gyrators. Using such methods "passive" circuits can be implemented using the data from Zverev (1967) with certain advantages in stability.

4.4 The simplest form of active RC filter is the Sallen and Key. This requires one capacitor and one resistor per pole, and a unity gain amplifier for every two poles. Zeros at the origin are accommodated without any extra components. These can be designed using Pole-Zero data, together with the relevant paper from Huelsman (1976). Alternatively the normalised component values are listed in Shepard (1969) for various filters. These are:
Bessel, Butterworth, 0.1 dB, 0.25 dB, 0.5 dB, 1 dB, 2 dB and 3 dB Chebyshev for order n = 2 to 10. Frequency scaling, and High-Pass Transformation are explained in terms of component values. Suitable printed circuit boards for Sallen and Key filter elements, with layout carefully designed to minimise the critical stray conductances, are held in the Met O 16 Drawing Office under Drawing List 13757.

4.5 Sallen and Key elements are highly economical in hardware, but start to place excessive demands on their unity gain amplifiers where the Q of a stage approaches 10. This does not happen even for relatively high order Bessel or Butterworth Low-Pass filters, but occurs above a 7th order 0.5 dB Chebyshev, or above a 5th order 2 dB Chebyshev Low-Pass filter. At about this level of performance or above, State Variable Filters should be used. For each pole-pair, combined with one or two zeros at the origin, this configuration requires three amplifiers, two capacitors, and six resistors. In return for this extra complexity, Q values approaching 1000 can be relatively easily achieved, although at the higher end demands are placed on the gain-bandwidth product of the amplifier as high overall gains are implied.

5. Software Implementation of Filters

5.1 Software (or digital hardware) methods of implementing filters usually involves sampling a digital representation of the input signal on a regular basis, manipulating the signal to filter it, and then presenting a sampled version of the resultant signal to the outside world.

5.2 It must be realised that the sampling is a significant part of the overall process. If the sampling rate is N samples/second, then frequencies above the Nyquist frequency $\frac{N}{2}$ Hz cannot be adequately represented, and will be aliased into lower frequencies. Conversely, if no energy in the input signal exists above the Nyquist frequency, then the sampling produces no loss of information, and the original continuous signal can be reconstructed exactly from the sampled data. This is discussed in Met O 16 Branch Memo No 3, Bell (1968), and more physically using an ideal sampler in Tou (1959). This can imply that when using digital forms of filtering, the sampling rate must be high; or else the samples must be preceded by an analogue filter, which could be the first filter stage, or else a pre-filter having high-frequency poles whose specification need not be critical.

5.3 It is possible to filter in software by weighting the input data stream with the (time reversed) impulse response of the required filter. This type of method is well documented (eg Pesaressi, 1971; Craddock, 1968). It has the advantage that filters other than pole-zero types can be used (finite impulse duration filters). However, if the ratio of the highest frequency present in the input data stream: significant filter frequencies (eg pole frequencies) is large, then this method implies a large number of multiplications per sampling period. Because the filter is often present because of high-frequency noise, around one hundred multiplications per timestep is not unusual.

5.4 Met O 16 Branch Memo No 3 describes a digital recursive method of implementing pole-zero filters. In software form, the computation can amount to much less than one multiplication per timestep. This economy of computing effort, combined with the fact that little storage space is required, is likely to be significant, particularly in microprocessor applications.

6. Other Applications of Linear Transfer Functions

6.1 Linear transfer functions are used in all forms of analogue circuit, and in Control applications. Control Theory is a methodology enabling pole-zero patterns to be specified for networks (or filters) used in various control applications, which usually involve feedback. Such pole-zero patterns can be implemented using the techniques described above. Di Stefano (1967) is a useful introduction to Feedback and Control Systems. A tool in Control Theory is Root Locus Analysis. The Met O 16 program RTLOCUS is useful in this application.

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