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SHAPE PRESERVING ADVECTION SCHEMES FOR METEOROLOGICAL MODELS

by

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Traditionally the accuracy of a finite difference approximation for the advective terms in the equations of motion has been examined by comparing the damping and phase speed characteristics of the scheme with the analytic behaviour of all resolvable waves, e.g. Gadd (1978). The errors are usually largest for the shortest waves. The constraints imposed by modern computers ensure that most meteorological models will contain some poorly resolved features. In practice anomalous dispersion leads to ripples of significant amplitude on the edges of such disturbances. Even a scheme with no phase errors must suffer in this way due to the discretisation of the mesh (Boris and Book 1976). In this note the ability of several schemes to preserve the shape of a disturbance is examined. Shape preservation may be crucial in some models to retain the qualitative nature of the solution and also to ensure that a variable remains bounded by its initial extrema. In cloud models with an explicit moisture parametrisation the failure of water variables to remain positive can be a serious source of error (Clark 1979).

The linear advection equation in one dimension, for some variable q,

$$\frac{\partial q}{\partial t} + \frac{\partial (cq)}{\partial x} = 0 \qquad (1)$$

where c is the constant advection speed, has been solved numerically to illustrate these points. Figure 1 shows the results of advecting a poorly resolved disturbance (described over seven grid points) for 200 time steps with a Courant number, ($c\Delta t/\Delta x$; where Δx, Δt are the space and time increments respectively), of 0.5, using several advection schemes. All the schemes are written in flux form to conserve the nomalised linear integral

$$I_1 = \int_{-\infty}^{\infty} q(x,t)\,dx \Big/ \int_{-\infty}^{\infty} q(x,t=0)\,dx$$

but all exhibit a slow decay of the quadratic integral

$$I_2 = \int_{-\infty}^{\infty} \{q(x,t)\}^2\,dx \Big/ \int_{-\infty}^{\infty} \{q(x,t=0)\}^2\,dx$$

For the leapfrog schemes most of the loss in I_2 is due to the start up procedure chosen. The figure shows the resulting disturbance shape, quadratic conservation (I_2) and the time taken to compute the solution for each of the schemes. (The times were obtained using vector FORTRAN on a CDC Cyber 205 computer).

The leapfrog schemes are best for maintaining I_2 but produce oscillations where the gradient changes rapidly. The Gadd (1978) scheme has a larger phase error than the 4th order leapfrog scheme and, as pointed out by Gadd (1980), also produces significant oscillations. Clearly such schemes are extremely bad at shape preservation for poorly resolved features despite their formal accuracy.

It is well known that such oscillations can be eliminated by using upstream differencing but that this is very dissipative, Zalesak (1979). Recently more complicated schemes have been introduced, (Boris and Book (1976), Harten (1978)), that eliminate or greatly reduce oscillations in the solution without excessive damping. Good shape preservation is achieved by a careful combination of a low order scheme (such as upstream differencing) with a high order scheme. For example, let

$$q^{n+1} = q^{n} - \delta_x F \qquad (2a)$$

define a finite difference approximation to the advection equation, (1), where the flux F (= cq) is defined at points between q values and the notation is conventional. The flux is represented by the expression

$$F = (1-S)H + SL \qquad 0 \le S \le 1 \qquad (2b)$$

where H is the flux calculated using some high order scheme, (order 2 or above, such as Gadd (1978)), and L is the flux from a low order scheme (order one or zero, such as upstream differencing). S is a function of q designed to reduce the formation of oscillations. The quantity determines the weighting of the schemes at each point.

$$S = \overline{\left| \frac{\delta_x |\delta_x q|}{2\, \overline{|\delta_x q|}^{x}} \right|^{p}}^{x} \cdot \Delta x \qquad (3)$$

Three possible methods of evaluating S have been suggested. Harten (1978) uses (3) and p is chosen empirically to optimise the results.

Clark (1979) describes a function which acts selectively when the variable approaches some critical value taken here as

$$S = \left| \frac{\delta_x q}{2\overline{|q|}^x + \Delta_0} \right|^2 \cdot \Delta x \qquad (4)$$

where Δ_0 is an extremely small positive number required to prevent possible division by zero but otherwise irrelevant to the scheme. The lower bound, $q = 0$, is appropriate if q is a water variable in a cloud model.

Finally S can be evaluated by a flux corrected transport (FCT) algorithm as described by Boris and Book (1976) and Zalesak (1979). In this S is an approximation to the largest values for S that can be tolerated in (2) at each point without producing new maxima or minima. S is expressed as a function of q, H and L, and is described in Appendix A.

In Figure 2 a comparison is made between a number of hybrid schemes using these S functions. All of the schemes use upstream differencing for the low order flux. The Clark scheme uses the specification (4) with the Crowley scheme (Crowley (1968)) for the high order part. The Harten-Crowley scheme also uses the Crowley scheme but S is given by (3) with $p = 1$, (this was found to be optimal value in the cases considered below). The form (4) is also used in the Hybrid Gadd scheme while the high order flux is that given by the Gadd scheme. The FCT scheme uses

$$H = c\overline{q}^x$$

and would be unstable if $S = 0$ but the algorithm used for calculating S prevents this. Only the FCT algorithm has this property and if other methods are used for determining S the constituent schemes are required to be separately stable.

Clearly all the hybrid schemes are worse than the non-hybrid schemes (Figure 1) for maintaining quadratic conservation (I_2) and considerably more time consuming. However shape preservation is much better. The result of the FCT scheme shows the least damping and most accurate position of those obtained using the hybrid schemes. The tendency for FCT algorithms to produce a plateau instead of a peak, evident in this case, can be reduced following Zalesak (1979) with some additional computation.

The advection of a poorly resolved single cycle of a wave is shown in Figure 3. It is apparent that the non-hybrid schemes grossly distort the function shape. The Hybrid Gadd scheme is extremely dissipative but the FCT performs well. The behaviour of all the schemes is much improved for a well resolved disturbance. Figure 4 shows the advection of a single cycle wave resolved over 14 points and a Courant number of 0.1 by several of the schemes. As expected the differences between the results obtained using the different schemes are much reduced compared with those in Fig 3.

The Clark formulation for S (equation (4)) has been used in a hybrid scheme for the water variables in the cumulonimbus model of Miller & Pearce (1974) in which the Miller alternating timestep algorithm (Thorpe (1982)) is used as the high order scheme. Previously the negative values produced by the advection scheme were set to zero producing a spurious generation of water in the model. Although this was small compared to the physical terms, it was comparable with the small residual between production and evaporation of rain water. Predicted rainfall with the new scheme was changed by several hundred per cent.

In conclusion it is apparent that schemes exhibiting high formal accuracy are not good at preserving the shape of poorly resolved disturbances. There is substantial computation involved with schemes such as FCT but when localisation and boundedness are essential they may offer the best rational choice.

Acknowledgement

We would like to thank Mr J R Lavery who performed much of the programming for this work.

Appendix A The FCT Algorithm

The FCT algorithm used for this work uses the original flux limiter introduced by Boris and Book (1973).

Equation (2a) is solved in two steps. An intermediate field of q_*^{n+1} is calculated as

$$q_*^{n+1} = q^n - \delta_x L \tag{A1}$$

where L is the flux given by upstream differencing. Then q^{n+1} is calculated by

$$q^{n+1} = q_*^{n+1} - \delta_x \left[(1-S)(H-L)\right] \tag{A2}$$

where H is shown in equation (5), and (1–S) is the flux-limiter defined by

$$(1-S)_{i+\frac{1}{2}} = I_{i+\frac{1}{2}} \max\Big\{0, \min\Big[|H-L|_{i+\frac{1}{2}},\ I_{i+\frac{1}{2}}(\delta_x q_*^{n+1})_{i+3/2}\Delta x,\ I_{i+\frac{1}{2}}(\delta_x q_*^{n+1})_{i-\frac{1}{2}}\Delta x\Big]\Big\} \tag{A3}$$

and

$$I_{i+\frac{1}{2}} = \begin{cases} +1 & \text{if } (H-L)_{i+\frac{1}{2}} \geq 0 \\ -1 & \text{if } (H-L)_{i+\frac{1}{2}} < 0 \end{cases}$$

where $(i + \frac{1}{2})$ as a subscript indicates an expression evaluated at the mid point between grid points labelled i and (i + 1).

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Figure Captions

Figure 1 Comparison of solutions from several advection schemes after 200 timesteps with Courant number 0.5. Horizontal axis tick marks denote grid points from a domain of 600 points. Timings are quoted as multiples of the 2nd order leapfrog time.

Figure 2 As in figure 1 for the hybrid schemes. (Harten Crowley estimated on an IBM 360/195).

Figure 3 As in figure 1 for advection of a single cycle wave with comparable resolution. Tick marks on the horizontal axis denote every 4th grid point.

Figure 4 As in figure 3 but for a Courant number of 0.1 and a well-resolved initial wave.

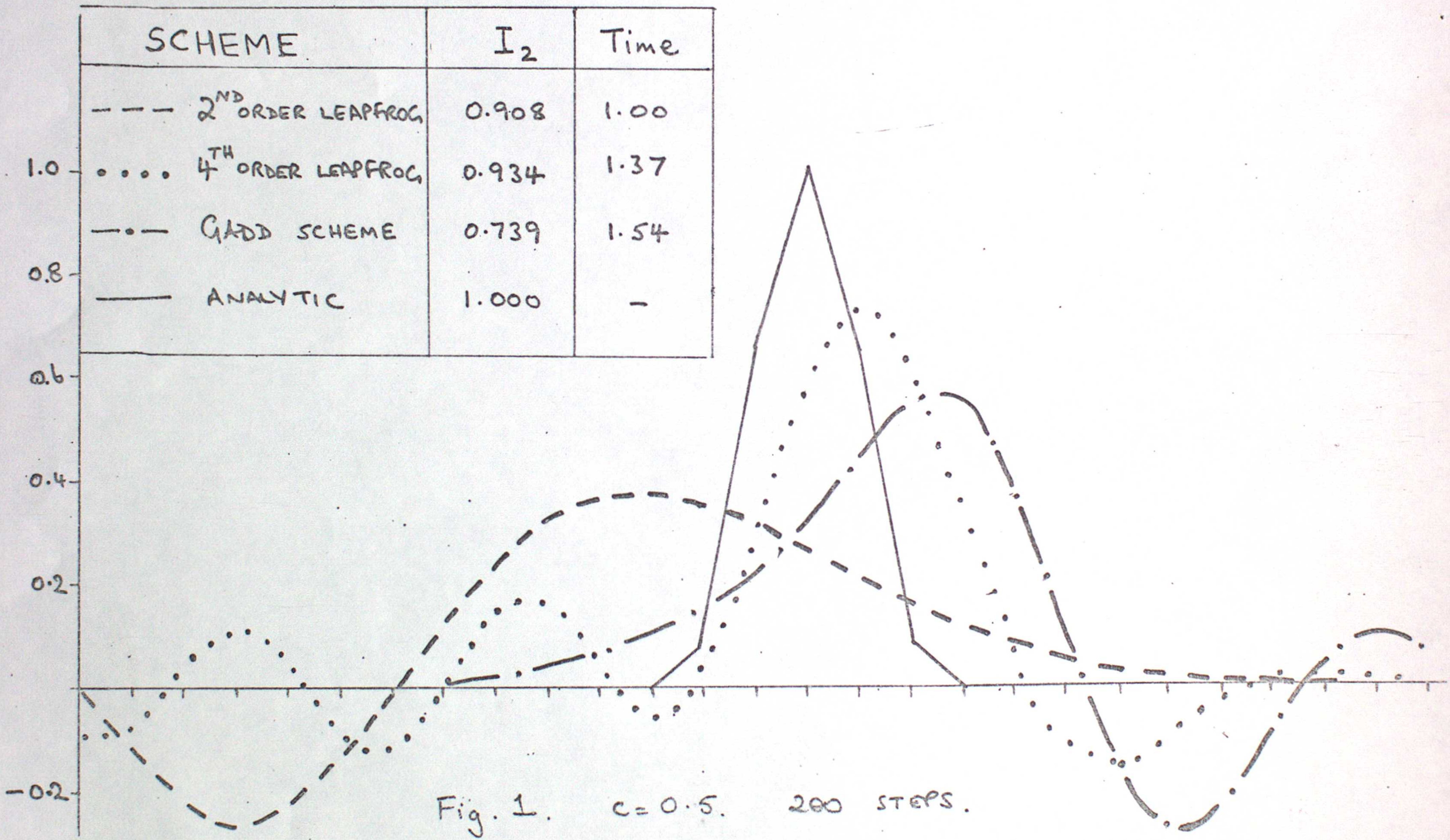


Fig. 1. $c = 0.5$. 200 STEPS.

Fig. 2. $c = 0.5$ 200 STEPS

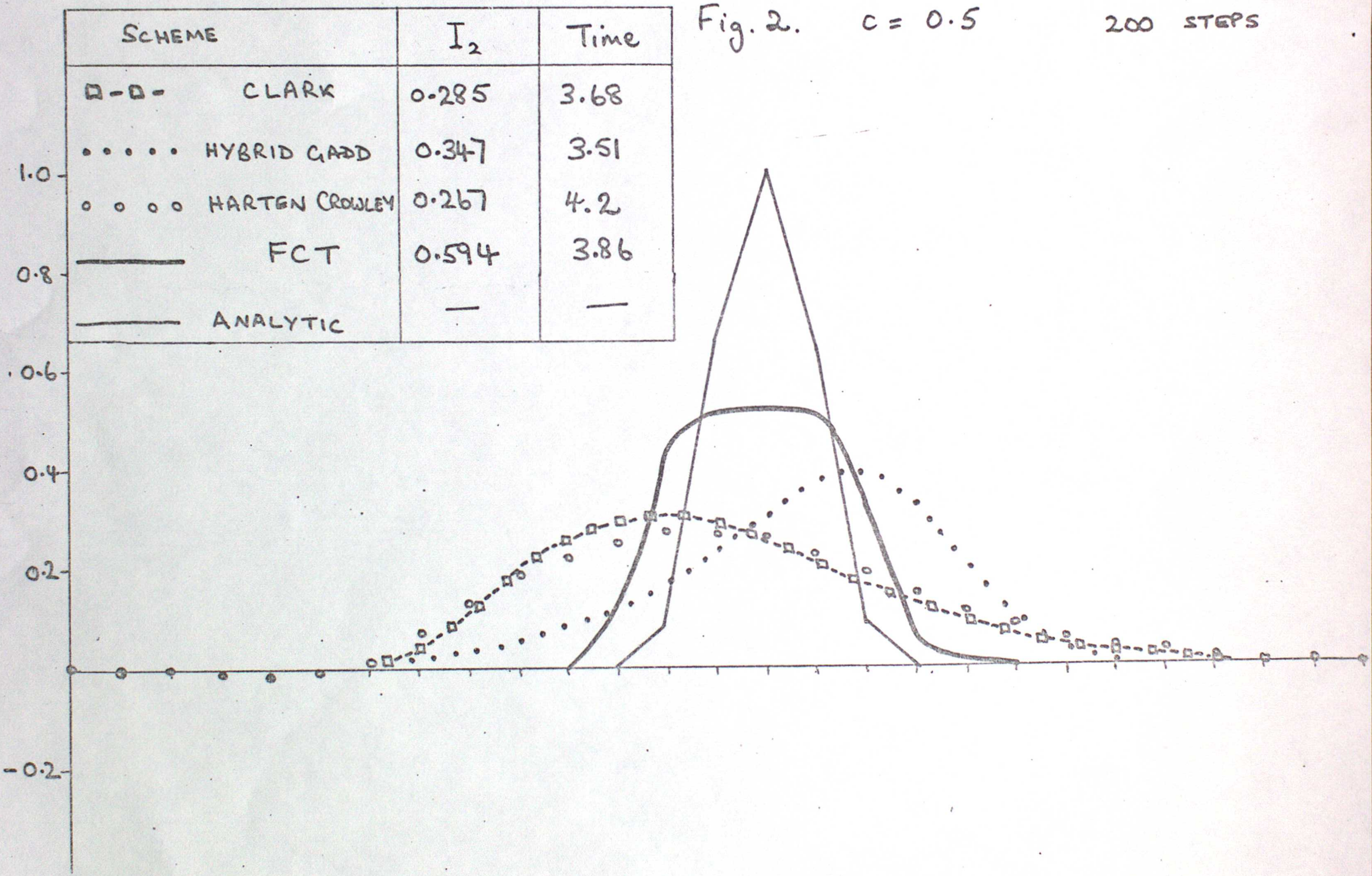


Fig.3. $c = 0.5$ 200 STEPS

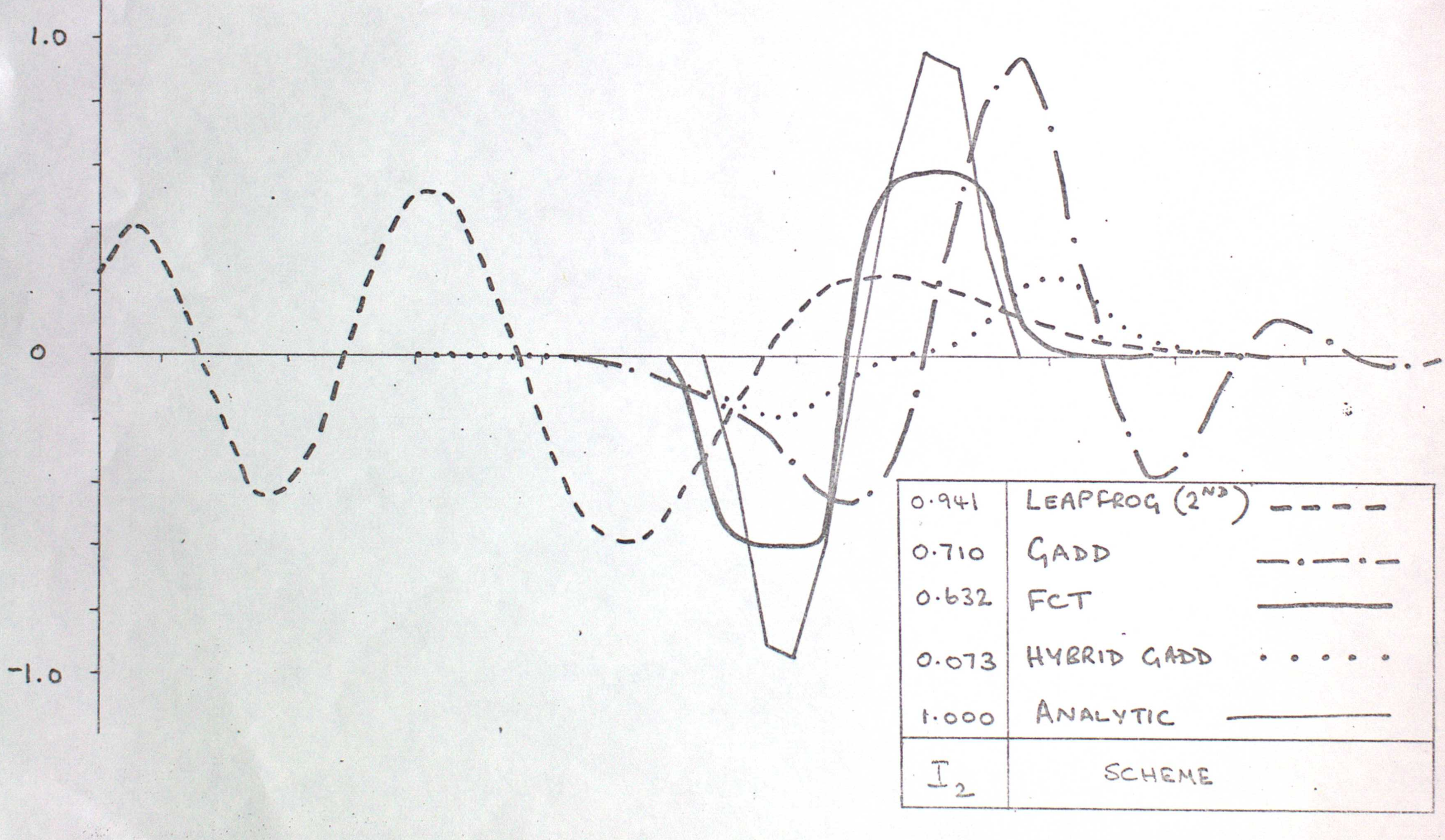
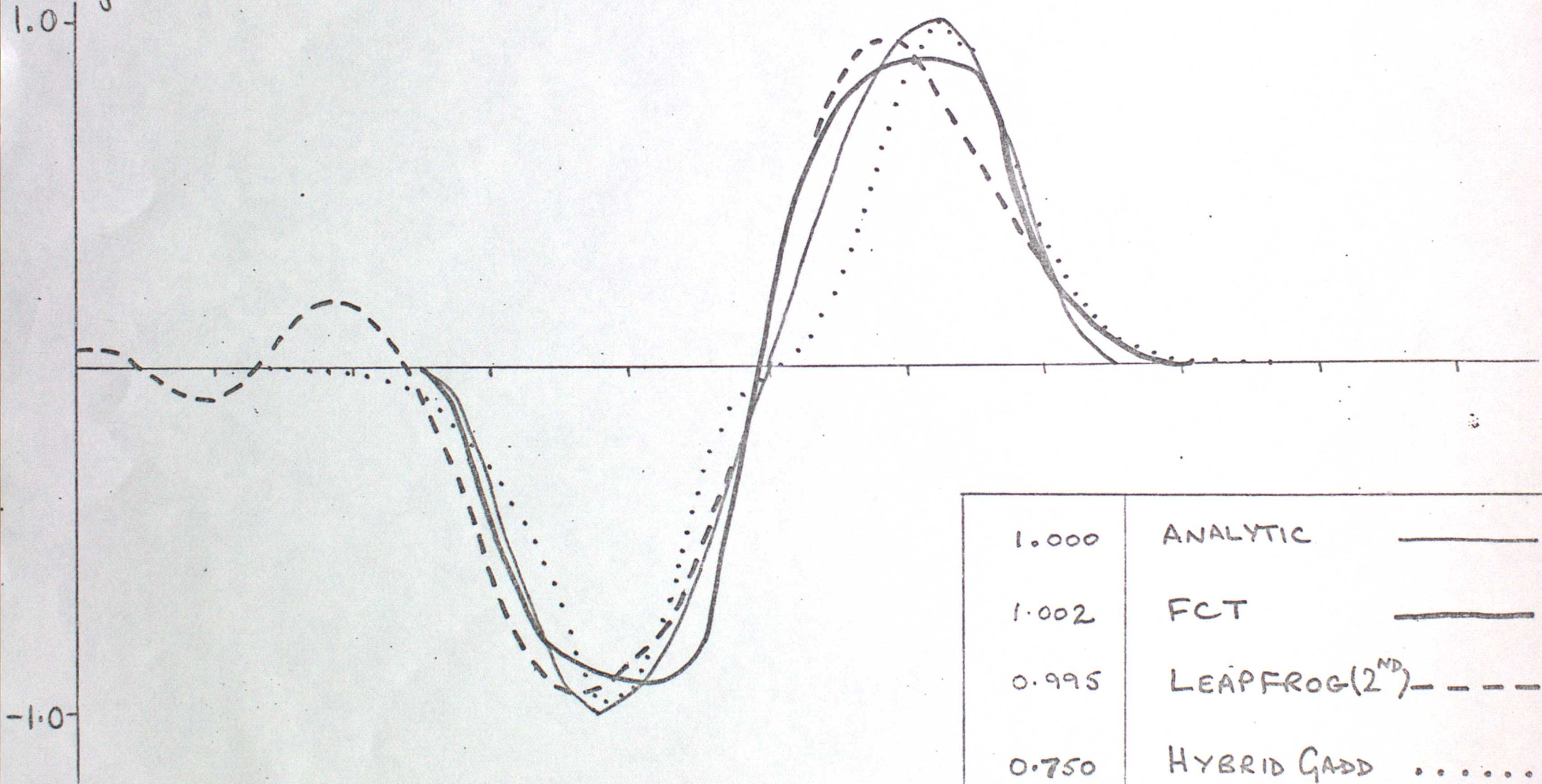


Fig. 4. $c = 0.1$ 200 STEPS



I_2	SCHEME
1.000	ANALYTIC ———
1.002	FCT ———
0.995	LEAPFROG(2^{ND}) - - - -
0.750	HYBRID GADD . . . . . .