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An Economical Explicit Integration Scheme

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1. Introduction

Burridge (1974) has described a semi-implicit reformulation of the 10 level primitive equation model first described by Bushby and Timpson (1967). The semi-implicit integration scheme achieves a saving of a factor of four in the computation required for a numerical weather prediction with the 10 level model. The scheme described by Burridge makes use of a splitting technique in which the terms in the primitive equations describing advection and the terms describing gravity-inertia oscillations (called the adjustment terms) are treated separately in successive stages of the integration cycle. This paper describes an alternative integration scheme which has been tested within the context of the 10 level model. The alternative scheme is also based on a splitting technique and the treatment of the advection terms is identical to that described by Burridge. However, the gravity-inertia terms are treated by an explicit integration scheme which nontheless offers computational economies similar to those of the semi-implicit scheme.

Other workers have given attention to the development of economical explicit schemes. Brown and Campana (unpublished) have described a technique which offers a factor of two saving in an explicit leapfrog scheme. Magazenkov et al (1971) have described a scheme for a barotropic model similar in principle to that described in this paper.

2. The splitting technique

The governing equations of the 10 level model are split into advection and adjustment stages as described by Burridge (1974). The advection terms are treated by a two-step Lax-Wendroff integration scheme. Let $\underline{X}_n$ be a vector representing the state of the model atmosphere after n time steps of integration. The components of $\underline{X}_n$ are the entire set of grid point variables after n time steps. The advection stage of the integration cycle may be schematically represented as follows

$$\hat{\underline{X}}_n = (\underline{\underline{I}} + \underline{\underline{A}})\underline{X}_n$$

where $\underline{\underline{A}}$ is a matrix operator representing the two-step Lax-Wendroff integration through a time step δt.

The adjustment stage of the integration cycle consists of a sequence of m operators, each representing an explicit integration of the adjustment terms through a time step $\delta t/m$. The effects of friction, topography, and diabatic heating are included in these shorter time steps and so the state of the model atmosphere after $n+1$ integration cycles is given by

$$\underline{X}_{n+1} = (\underline{\underline{I}} + \underline{\underline{B}})^m \hat{\underline{X}}_n$$

where $\underline{\underline{B}}$ is a matrix operator representing an adjustment step. Notice that, since a splitting technique is used, operators representing different parts of the equations are multiplied together to make up the complete integration cycle which is therefore represented schematically by

$$\underline{X}_{n+1} = (\underline{\underline{I}} + \underline{\underline{B}})^m (\underline{\underline{I}} + \underline{\underline{A}})\underline{X}_n$$

The advantage of splitting is that the requirement for linear computational stability reduces to a series of criteria which guarantee that each operator in the product is non-amplifying. In the present case a pair of criteria emerge, one being a condition on the eigenvalues of $\underline{\underline{A}}$ and the other a condition on the eigenvalues of $\underline{\underline{B}}$. The computational economy of the scheme relies on the fact that the relatively expensive advection terms are computed with a time step δt limited only by the wind speed, whilst the relatively inexpensive adjustment terms are computed more frequently, using a time step $\delta t/m$ which satisfies the restrictive criterion imposed by fast moving gravity waves. It is also important for economy that the effects of diabatic heating are calculated only once per integration cycle. However, experience has shown that it is essential for stability that the changes due to diabatic heating, once calculated, be divided uniformly among the m short adjustment steps. Changes due to topography and friction have been similarly treated in the present scheme.

3. The integration scheme for the adjustment terms

In the pressure coordinate system used by the 10 level model the terms describing gravity-inertia oscillations are the following.

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = fv - g\dfrac{\partial h}{\partial x} \quad ; \quad \dfrac{\partial v}{\partial t} = -fu - g\dfrac{\partial h}{\partial y} \; ; \\[2ex] \dfrac{\partial h'}{\partial t} = -\omega\left(\dfrac{\partial h'}{\partial p} + \dfrac{h'}{\gamma p}\right) ; \quad \dfrac{\partial h}{\partial t} = -\omega\dfrac{\partial h}{\partial p} \ \text{at } 1000 \text{ mb} \; ; \\[2ex] \text{and} \quad \dfrac{\partial \omega}{\partial p} = -\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) \end{array}\right\} \quad (1)$$

Here u and v are horizontal wind components, ω is the vertical velocity in pressure co-ordinates, h is the height of an isobaric surface and h' the thickness of a 100mb layer between two isobaric surfaces. f, g, and γ have their conventional significance.

The following integration scheme has been used for these terms.

$$u_{i+1} = u_i + \delta t' \left[fv_i - g\left(\frac{\partial h}{\partial x}\right)_{i+1} \right]$$

$$v_{i+1} = v_i + \delta t' \left[-fu_i - g\left(\frac{\partial h}{\partial y}\right)_{i+1} \right]$$

$$h'_{i+1} = h'_i + \delta t' \left[-\omega_i \left(\frac{\partial h'}{\partial p} + \frac{h'}{\gamma p}\right)_i \right]$$

$$h_{i+1} = h_i + \delta t' \left[-\omega_i \left(\frac{\partial h}{\partial p}\right)_i \right] \quad \text{at } 1000 \text{ mb}.$$

Here $\delta t'$ represents $\delta t/m$ and i is an index for the m short adjustment steps. Notice that the new values h_{i+1} must be calculated first, and that these values are then used to calculate u_{i+1} and v_{i+1}. The scheme is based on that described in one space dimension by Ames (1969), and it allows longer time steps than other explicity integration schemes for equations (1). It is, however, of only first order accuracy in time.

4. Note on finite difference grids

The split explicit integration scheme was first tested on the Eliassen staggered grid (Figure 1(a)) as used in the original 10 level model (Benwell et al 1971). A single adjustment step was used in each integration cycle ($m=1$). Linear analysis indicated that the scheme should be stable with time steps more than double those previously used in the 10 level model, and this was confirmed by trial integrations. The scheme was, however, unsuccessful because the forecasts were contaminated by a spatial computational mode (Gerrity and McPherson 1970). This mode is a result of the evolution of distinct solutions on two independent lattices of grid points. Special numerical procedures adopted at the lateral boundaries of the computational area reduced the impact of the computational mode but nontheless the forecasts were unacceptable.

The scheme was therefore developed for the alternative staggered grid shown in Figure 1(b). The spatial computational mode does not occur on this grid since there are no possibilities of lattice separation. With a grid separation δx the time step of the adjustment stage must satisfy the criterion $\delta t' \leq \delta x/\sqrt{2}c$, where c is the speed of the external gravity wave, for linear computational stability. With the 100km grid mesh of the 10 level model a time step of 12 minutes for the entire integration cycle, with four 3 minute adjustment steps, allows for wind speeds up to 100 m/sec.

5. Assessment of the economical explicit scheme.

Figure 2 illustrates a 36 hour forecast computed using the split explicit integration scheme, which may be compared with a semi-implicit integration from the same initial data illustrated by Burridge (1974). The two schemes give essentially identical results.

The computing time taken by the forecasts with the two schemes is very similar, with a small advantage to the semi-implicit scheme. In some forecasts (not the one illustrated here) slight roughnesses develop near the lateral boundaries with the explicit scheme. These roughnesses disappear if the time step for the complete cycle is reduced to 10 minutes. This would, of course, slightly enhance the advantage of the semi-implicit scheme.

The economical explicit scheme shares with the semi-implicit scheme the possible disadvantages of splitting, in terms of time truncation errors, and the possible disadvantages of using first order accurate schemes for the adjustment terms.

The economical explicit scheme might have advantages over the semi-implicit scheme in other co-ordinate systems (e.g. sigma coordinates) where the isolation of a linear description of gravity waves for implicit treatment is more difficult than in pressure coordinates.

References

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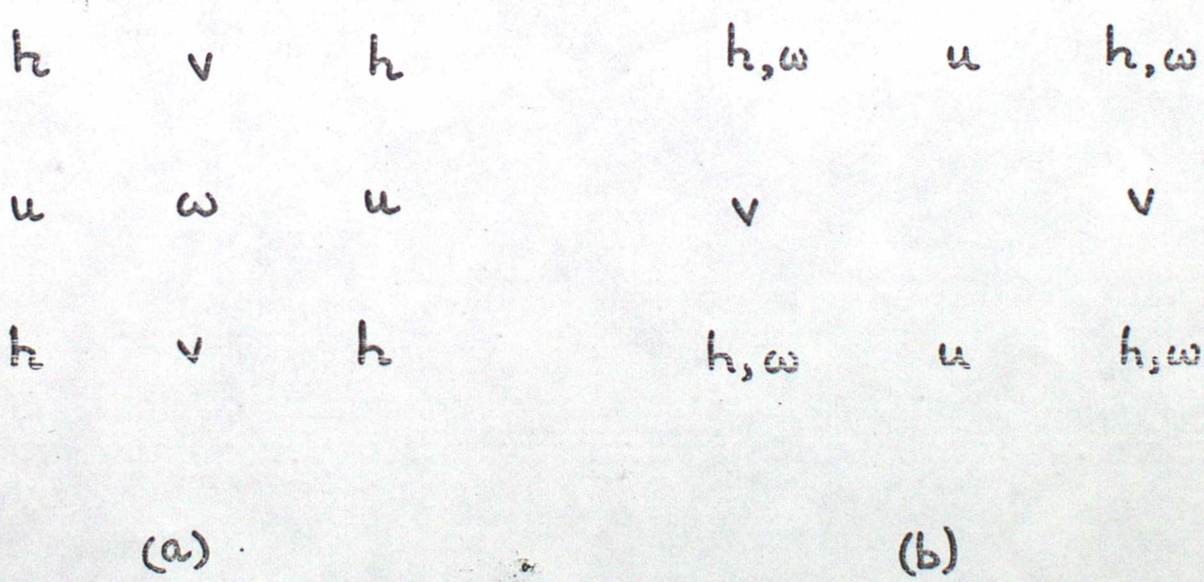

Figure 1. Staggered grid arrangements used with the split explicit integration scheme.

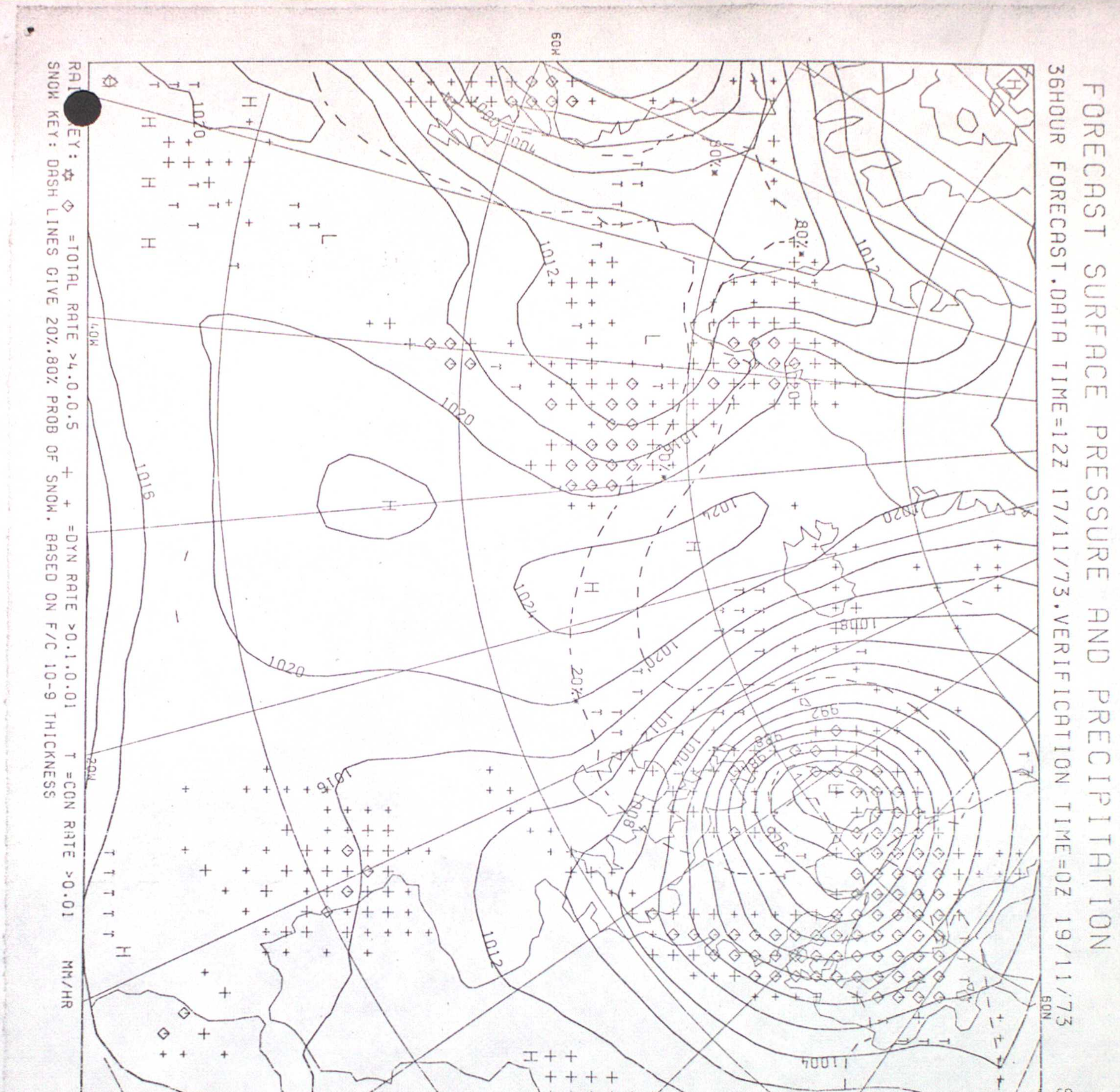

Figure 2 A 36 hour forecast computed with the economical explicit integration scheme. This may be compared with results presented by Burridge (1974).