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MET O 11 TECHNICAL NOTE NO 81

Monge - form surfaces, $z = f(x,y)$

PART 2 - Curvature, Directions, Conjugate Directions, Asymptotic Directions, Umbilics, Geodesics, Tensor Formalism

R. DIXON

1. Introduction

Part 3 will deal with the important topic of the form and use of operators on a Monge-form surface. This part, Part 2, deals with a collection of items most of which will probably not be required very often. They are gathered together in this Part 2 for the sake of completeness. The notation is the same as in Part 1.

Met O 11. (Forecasting Research Branch)
Meteorological Office
London Road
Bracknell, Berks, RG12 2SZ, UK

2. Curvature

At any point P on a surface there are two directions such that the normal at a consecutive point in either direction intersects the normal at P. These directions are the principal directions and they are at right angles to each other on the surface. Associated with these two principal directions are the two principal curvatures K_a & K_b.

The two principal curvatures K_a and K_b are given as the roots of the quadratic equation

$$(1+p^2+q^2)^2K^2 - \sqrt{1+p^2+q^2}\,[r(1+q^2) - 2pqs + t(1+p^2)]K - (rt - s^2) = 0 \quad (1)$$

The sum of the roots

$$J = K_a + K_b \quad (2)$$

is either called the first curvature or the mean curvature and from (1) it is given by

$$J = \frac{1}{(1+p^2+q^2)^{3/2}}\left[r(1+q^2) - 2pqs + t(1+p^2)\right] \quad (3)$$

The product of the roots

$$K = K_a K_b \quad (4)$$

is called the second curvature or the specific curvature or the Gaussian curvature, and from (1) it is given by

$$K = \frac{rt - s^2}{(1+p^2+q^2)^2} \quad (5)$$

The principal direction corresponding to K_a is given by either

$$\frac{dx}{dy} = -\frac{pqK_a\sqrt{1+p^2+q^2} - s}{(1+p^2)K_a\sqrt{1+p^2+q^2} - r} \quad (6)$$

or

$$\frac{dx}{dy} = -\frac{(1+q^2)K_a\sqrt{1+p^2+q^2} - t}{pqK_a\sqrt{1+p^2+q^2} - s} \quad (7)$$

The principal direction corresponding to K_b is given by

$$\frac{dx}{dy} = - \frac{pq\,K_b\sqrt{1+p^2+q^2} - s}{(1+p^2)K_b\sqrt{1+p^2+q^2} - r} \tag{8}$$

or

$$\frac{dx}{dy} = - \frac{(1+q^2)K_b\sqrt{1+p^2+q^2} - t}{pq\,K_b\sqrt{1+p^2+q^2} - s} \tag{9}$$

A line of curvature on a surface is a line such that the normals at consecutive points intersect. There are two orthogonal lines of curvature through each point on a surface. The differential equation of the lines of curvature is

$$\left[s(1+p^2) - rpq\right]dx^2 + \left[t(1+p^2) - r(1+q^2)\right]dx\,dy + \left[tpq - s(1+q^2)\right]dy^2 = 0 \tag{10}$$

In any general direction, determined by the ratio dx/dy, the curvature of the normal section is given by

$$K_N = \frac{1}{\sqrt{1+p^2+q^2}} \cdot \frac{r\,dx^2 + 2s\,dx\,dy + t\,dy^2}{(1+p^2)dx^2 + 2pq\,dx\,dy + (1+q^2)dy^2} \tag{11}$$

K_a and K_b are the maximum and minimum values taken by (11) as the direction given by dx/dy varies.

An umbilic is a point on the surface at which the curvature is the same for any normal section. The curvature does not vary with the direction of the normal section. At an umbilic the coefficients of (10) all vanish, giving the three conditions

$$s(1+p^2) = rpq \tag{12}$$

$$t(1+p^2) = r(1+q^2) \tag{13}$$

$$tpq = s(1+q^2) \tag{14}$$

3. Directions in general

At a point on the surface corresponding to the values (x, y), a direction will be determined by increments Δx and Δy in the parameters x and y. Some other direction will correspond to different increments δx and δy. For the first direction the small vector displacement is $\Delta \underline{r}$, and its length is Δs. For the second direction the small vector displacement is $\delta \underline{r}$ and its length is δs. Then since

$$\Delta \underline{r} = \frac{\partial \underline{r}}{\partial x} \Delta x + \frac{\partial \underline{r}}{\partial y} \Delta y \tag{15}$$

$$\delta \underline{r} = \frac{\partial \underline{r}}{\partial x} \delta x + \frac{\partial \underline{r}}{\partial y} \delta y \tag{16}$$

the angle θ between these two directions is given by

$$\Delta \underline{r} \cdot \delta \underline{r} = \left(\frac{\partial \underline{r}}{\partial x}\right)^2 \Delta x \, \delta x + \left(\frac{\partial \underline{r}}{\partial x} \cdot \frac{\partial \underline{r}}{\partial y}\right)(\Delta x \, \delta y + \Delta y \, \delta x) + \left(\frac{\partial \underline{r}}{\partial y}\right)^2 \Delta y \, \delta y \tag{17}$$

i.e.

$$\Delta s \, \delta s \cos\theta = E \Delta x \, \delta x + F(\Delta x \, \delta y + \Delta y \, \delta x) + G \Delta y \, \delta y \tag{18}$$

and, from (21), (22) and (23) of Part 1, for a Monge form surface this is

$$\Delta s \, \delta s \cos\theta = (1+p^2)\Delta x \, \delta x + pq(\Delta x \, \delta y + \Delta y \, \delta x) + (1+q^2)\Delta y \, \delta y \tag{19}$$

The angle θ is also given by

$$|\Delta \underline{r} \times \delta \underline{r}| = \Delta s \, \delta s \sin\theta = \left|\frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y}\right| (\Delta x \, \delta y - \Delta y \, \delta x) \tag{20}$$

i.e. from (29) of Part 1

$$\Delta s \, \delta s \sin\theta = \sqrt{1+p^2+q^2} \, (\Delta x \, \delta y - \Delta y \, \delta x) \tag{21}$$

and then (18) and (21) yield an expression which is devoid of the arclengths, namely

$$\tan\theta = \frac{\sqrt{1+p^2+q^2} \, (\Delta x \, \delta y - \Delta y \, \delta x)}{(1+p^2)\Delta x \, \delta x + pq(\Delta x \, \delta y + \Delta y \, \delta x) + (1+q^2)\Delta y \, \delta y} \tag{22}$$

An important special case is when one of the two directions is along one of the parametric curves as this case gives the angle between an arbitrary direction and the parametric curve. If the first direction above is along the parametric curve y = const. then $\Delta y = 0$ and (19) reduces to

$$\Delta s\,\delta s \cos\theta = (1+p^2)\Delta x\,\delta x + pq\,\Delta x\,\delta y \tag{23}$$

But, from (31) of Part 1, we have in this case

$$\Delta s^2 = (1+p^2)\Delta x^2 \tag{24}$$

and so

$$\Delta s = \sqrt{1+p^2}\,\Delta x \tag{25}$$

Substituting this in (23) and cancelling out Δx gives

$$\sqrt{1+p^2}\,\delta s \cos\theta = (1+p^2)\delta x + pq\,\delta y \tag{26}$$

as the required expression for the angle between the direction determined by $(\delta x, \delta y)$ and the parametric curve y = const. Similarly, from (21) we get

$$\sqrt{1+p^2}\,\delta s \sin\theta = \sqrt{1+p^2+q^2}\,\delta y \tag{27}$$

and thus

$$\tan\theta = \frac{\sqrt{1+p^2+q^2}\;\delta y}{(1+p^2)\delta x + pq\,\delta y} \tag{28}$$

and, of course, this could have been obtained directly from (22). The angle ψ between $(\delta x, \delta y)$ and the other parametric curve x = const. is found by putting $\Delta x = 0$ and a similar sequence of manipulations then leads to

$$\sqrt{1+q^2}\,\delta s\cos\psi = pq\,\delta x + (1+q^2)\,\delta y \tag{29}$$

and

$$\sqrt{1+q^2}\,\delta s\sin\psi = -\sqrt{1+p^2+q^2}\,\delta x \tag{30}$$

and

$$\tan\psi = -\frac{\sqrt{1+p^2+q^2}\,\delta x}{pq\,\delta x + (1+q^2)\,\delta y} \tag{31}$$

From (19) it follows that the two directions $(\Delta x, \Delta y)$ and $(\delta x, \delta y)$ will be at right-angles when

$$(1+p^2)\frac{\Delta x}{\Delta y}\frac{\delta x}{\delta y} + pq\left(\frac{\Delta x}{\Delta y} + \frac{\delta x}{\delta y}\right) + (1+q^2) = 0 \tag{32}$$

If a family of curves on the surface is given by the differential equation

$$P\,\delta x + Q\,\delta y = 0 \tag{33}$$

where P and Q are any functions of x and y, then

$$\frac{\delta x}{\delta y} = -\frac{Q}{P} \tag{34}$$

and substituting this into (32) yields

$$\left[pq - \frac{Q}{P}(1+p^2)\right]\Delta x + \left[(1+q^2) - pq\frac{Q}{P}\right]\Delta y = 0 \tag{35}$$

which is the differential equation of the set of curves which are orthogonal to the family (33).

Any equation of the form

$$P dx^2 + Q dx dy + R dy^2 = 0 \tag{36}$$

determines two directions on the surface, because it is a quadratic in dx/dy. If the two roots of (36) are denoted by $\Delta x/\Delta y$ and $\delta x/\delta y$ then

$$\frac{\Delta x}{\Delta y} + \frac{\delta x}{\delta y} = -\frac{Q}{P} \tag{37}$$

and

$$\frac{\Delta x}{\Delta y}\frac{\delta x}{\delta y} = \frac{R}{P} \tag{38}$$

Thus from (32) it is seen that (36) determines two orthogonal directions if

$$(1+p^2)R - pqQ + (1+q^2)P = 0 \tag{39}$$

The angle between the two directions determined by (36) may be obtained in terms of P, Q, and R by using (22). Dividing the top and bottom of (22) by $\Delta y\,\delta y$ gives

$$\tan\theta = \frac{\sqrt{1+p^2+q^2}\left(\frac{\Delta x}{\Delta y} - \frac{\delta x}{\delta y}\right)}{(1+p^2)\frac{\Delta x}{\Delta y}\frac{\delta x}{\delta y} + pq\left(\frac{\Delta x}{\Delta y} + \frac{\delta x}{\delta y}\right) + (1+q^2)} \tag{40}$$

and (37) and (38) may then be used to reduce this to

$$\tan\theta = \frac{\sqrt{1+p^2+q^2}\,(Q^2 - 4PR)}{(1+p^2)R - pqQ + (1+q^2)P} \tag{41}$$

Now the differential equation of the parametric curves themselves is simply

$$dx\,dy = 0 \tag{42}$$

which is (36) with P = R = 0 and Q = 1. In this case (41) reduces to

$$\tan\theta = \frac{\sqrt{1+p^2+q^2}}{pq} \tag{43}$$

which is identical to (46) of Part 1, where this expression for the angle between the parametric curves was found by a different route.

4. Conjugate Directions

If P and Q are adjacent points on the surface then in the limit as Q tends to coincide with P the conjugate directions at P are the direction given by PQ itself and the line of intersection of the tangent planes at P and Q. If, as before, $(\Delta x, \Delta y)$ and $(\delta x, \delta y)$ specify two different directions on a Monge surface
then they will be conjugate directions if

$$r\frac{\Delta x}{\Delta y}\frac{\delta x}{\delta y} + s\left(\frac{\Delta x}{\Delta y} + \frac{\delta x}{\delta y}\right) + t = 0 \tag{44}$$

Consequently the two directions represented by an equation of the form (36) will be conjugate directions if

$$rR - sQ + tP = 0 \tag{45}$$

Again, since the parametric curves have the differential equation (42), corresponding to P=R=0, Q=1 in (36) it follows that the parametric curves are conjugate only if s=o

If a family of curves on the surface is given by (33), then from (34) and (44) it follows that the differential equation of the conjugate family is

$$(sP - rQ)\Delta x + (tP - sQ)\Delta y = 0 \tag{46}$$

5. Asymptotic Directions

They are simply the directions which are self-conjugate and so in (44) we put $\Delta x/\Delta y$ equal to $\delta x/\delta y$ to obtain the differential equation of the asymptotic lines as

$$r\,dx^2 + 2s\,dx\,dy + t\,dy^2 = 0 \tag{47}$$

There are thus two asymptotic directions at each point of the surface and they are real if $s^2 - rt$ is positive.

From (42) and (47) it follows that the parametric curves will be asymptotic lines if r = t = o and s $\neq$ o.

6. Geodesics

Probably the most widely known definition of a geodesic on a surface is that it is the path of shortest distance between two given points. Mathematically this is not a particular useful definition. It is better to define a geodesic as a curve whose osculating plane at each point contains the normal to the surface. Equivalently a geodesic on a surface is a curve such that the principal normal to the curve coincides with the normal to the surface.

Unlike the other directions and lines discussed in the foregoing sections, the geodesics are not uniquely determined at a point on $z=f(x,y)$. Through any point there is an infinite number of geodesics, each geodesic being determined by the direction through the point.

From the preferred definition of a geodesic given above it follows that the space curvature of a geodesic on a Monge-form surface is given by

$$\kappa = \frac{1}{\sqrt{1+p^2+q^2}}\left(r x'^2 + 2s x'y' + t y'^2\right) \tag{48}$$

where the dashes indicate differentiation with respect to arc-length. This is so because, from the definition, the curvature of the geodesic will be the same as the normal curvature of the surface in that direction and so it is given by (11) which then reduces to (48) by virtue of (31) of Part 1.

The differential equations governing the geodesics are

$$(1+p^2+q^2)x'' + pr x'^2 + 2ps x'y' + pt y'^2 = 0 \tag{49}$$

$$(1+p^2+q^2)y'' + qr x'^2 + 2qs x'y' + qt y'^2 = 0 \tag{50}$$

These two equations are really one differential equation as they are also linked by (31) of Part 1 and they may be combined to give

$$(1+p^2+q^2)\frac{d^2y}{dx^2} = pt\left(\frac{dy}{dx}\right)^3 + (2ps-qt)\left(\frac{dy}{dx}\right)^2 + (pr-2qs)\frac{dy}{dx} - qr \tag{51}$$

Now y=const is one of the parametric curves. In this case (51) will be satisfied only if $qr=0$, i.e. the parametric curve y=const is a geodesic if $qr=0$. Similarly, the parametric curve x=const is a geodesic if $pt=0$.

7. Geodesic Curvature and the Torsion of a Geodesic

The geodesic curvature of a curve at a particular point is the curvature of the curve relative to the geodesic which touches it tangentially at that point. Denoting it by κ_g, for a Monge-form surface it is given by

$$\begin{aligned}\kappa_g = {} & x'\sqrt{1+p^2+q^2}\left(y'' + \frac{qr}{1+p^2+q^2}x'^2 + \frac{2qs}{1+p^2+q^2}x'y' + \frac{qt}{1+p^2+q^2}y'^2\right) \\ & - y'\sqrt{1+p^2+q^2}\left(x'' + \frac{pr}{1+p^2+q^2}x'^2 + \frac{2ps}{1+p^2+q^2}x'y' + \frac{pt}{1+p^2+q^2}y'^2\right)\end{aligned} \tag{52}$$

The curvature of a geodesic relative to itself will naturally be zero and it is seen that (52) does reduce to zero if (49) and (50) hold.

If θ is the angle between the geodesic and the parameter curve y=const. then the curvature of the geodesic relative to the parametric curve, in the tangent plane is given by

$$\frac{d\theta}{ds} = -\frac{q(r+s)}{(1+p^2)\sqrt{1+p^2+q^2}} \tag{53}$$

Then (52) and (53) can be added to give the curvature of the given curve relative to the parametric curve y=const.

The geodesic curvature of a curve of the family defined by (33) is given by

$$K_g = \frac{1}{\sqrt{1+p^2+q^2}}\frac{\partial}{\partial x}\left[\frac{pqQ-(1+q^2)P}{\sqrt{(1+p^2)Q^2-2pqPQ+(1+q^2)P^2}}\right] + \frac{1}{\sqrt{1+p^2+q^2}}\frac{\partial}{\partial y}\left[\frac{pqP-(1+p^2)Q}{\sqrt{(1+p^2)Q^2-2pqPQ+(1+q^2)P^2}}\right] \tag{54}$$

The torsion of a geodesic, regarded as a space curve, is the arc-rate of turning of its binormal and is given by

$$\tau = \frac{1}{1+p^2+q^2}\left\{\left[(1+p^2)s-pqr\right]x'^2+\left[(1+p^2)t-(1+q^2)r\right]x'y'+\left[pqt-(1+q^2)s\right]y'^2\right\} \tag{55}$$

8. Tensor Formalism

It is almost always possible to avoid the use of tensor formalism in these matters but since many texts make use of it this Part 2 is completed by listing certain basic quantities from z=f(x,y) in tensor notation. They are the covariant and contravariant components of the metric tensor g_{ij} and g^{ij}, the Christoffel symbols of the first kind $[ij,k]$ and the Christoffel symbols of the second kind Γ^i_{jk}, where, i, j, and k range over the values 1 and 2. The Monge-surface forms are covariant components of metric tensor

$$g_{11} = 1+p^2 \quad , \quad g_{12} = g_{21} = pq \quad , \quad g_{22} = 1+q^2 \tag{56}$$

contravariant components of metric tensor

$$g^{11} = \frac{1+q^2}{1+p^2+q^2} \quad , \quad g^{12} = g^{21} = -\frac{pq}{1+p^2+q^2} \quad , \quad g^{22} = \frac{1+p^2}{1+p^2+q^2} \tag{57}$$

Christoffel symbols of the first kind

$$[11,1] = pr \; , \; [11,2] = rq \; , \; [22,1] = pt \; , \; [22,2] = qt \tag{58}$$

$$[12,1] = [21,1] = ps \quad , \quad [12,2] = [21,2] = qs \tag{59}$$

Christoffel symbols of the second kind

$$\Gamma^1_{11} = \frac{pr}{1+p^2+q^2} \; , \; \Gamma^1_{12} = \Gamma^1_{21} = \frac{ps}{1+p^2+q^2} \; , \; \Gamma^1_{22} = \frac{pt}{1+p^2+q^2} \tag{60}$$

$$\Gamma_{11}^{2} = \frac{qr}{1+p^2+q^2} \; , \; \Gamma_{12}^{2} = \Gamma_{21}^{2} = \frac{qs}{1+p^2+q^2} \; , \; \Gamma_{22}^{2} = \frac{qt}{1+p^2+q^2} \qquad (61)$$

A study of the results presented in Parts 1 and 2 sheds quite a lot of light on the role played by these quantities in the theory of surfaces

R Dixon

R Dixon
Met O 11
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