A Review of some recent Theories of the Boundary Layer of the Atmosphere

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Introduction

First of all it is appropriate to say a few words about the construction of the paper. Before the review really starts an attempt is made to establish a satisfactory notation. As was soon apparent when first reading through many of these papers there has grown up a great deal of confusing notation so that one letter may mean quite different things in different places and almost every useful combination of the variables is somewhere defined by a letter, often quite unnecessarily. The table of notation here suggested is an attempt to fit in with the most common usage, especially here in the 'West', but taking into account the very substantial contribution made in this field by Russian scientists.

The reviewed papers are tabulated in the main table which summarises the main facts of each. The papers are subdivided according to the general method employed and the basic assumptions of each method are clearly displayed. Then for each paper a note is made of the particular assumptions made, the method of solution and the main conclusions reached.

To supplement this rather concise summary the main text gives a commentary on some of the various methods, and at the end of the paper there are several appendices which enlarge upon various parts of the papers where these are considered either very important or where some elucidation of the original paper seems desirable.

One of the main aims of the study of the boundary layer is to optimise the procedures used in numerical forecasting and general circulation models to allow for the effect of the Earth's surface on atmospheric motions. With the vorticity-models it is important to know the induced vertical velocity at the top of the boundary layer and where this top is. With primitive equations models, which are becoming more widely used, the need for a detailed boundary layer theory is less apparent although this may be an illusion. In these latter models the lowest level at which calculations are made is often well within the boundary layer and the height of the boundary layer is held fixed in the sense that the shearing stress is already assumed zero at the second level up. The shearing stress at the surface is estimated from empirical relationships, taking into account the general nature of the stability, which relate the shearing stress at the ground to the wind speed at the lowest level in a way rather similar to that discussed in the section on similarity theory methods. The vertical velocity is obtained directly from the divergence equation, without further dependence on the theories that are reviewed below.

Clearly it will be important at some stage to decide more precisely in what way the forecasting models should deal with the surface and to concentrate our study of the boundary layer accordingly.

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At some later time when the capacity of computers increases several fold primitive equation models will no doubt be developed with two or more levels within the boundary layer so that the energy balance near the surface may be significantly improved in its representation. When this happens these theories of the boundary layer will become of more practical interest, but until that time the models will presumably continue to represent the effects of the surface in a very empirical way.

A very good review called the "Dynamics of the Atmospheric Boundary Layer: a Review" by Zilitinkevitch, Laikhtman and Monin (1967) should be consulted for further reading.

Notation

One of the most confusing things in this subject is the notation. In some papers a great number of letters are defined, sometimes unnecessarily and sometimes without regard to customary usage that has been established in earlier works. Furthermore Russian and 'Western' usage seems to have diverged rather, for example as in the symbol to represent heat flux where Russians have tended to use q and where in the West H has been more commonly used.

It would be clearly beneficial if the notation could be standardised and the table below offers a suggested scheme which attempts to cover the needs of those who work in the field of atmospheric turbulence including boundary layer theory and diffusion. It is virtually impossible at this stage to remove all ambiguity but by and large one letter is allowed to represent more than one quantity only where the context of its use would clearly point to which meaning was involved.

TABLE OF NOTATION

a	(i) Deacon's power-law formula for the wind shear $\frac{du}{dz} = az^{-\beta}$ (ii) In the theory of approximate similarity to relate K_H, K_b (the diffusivities for heat and turbulent energy b) and to b and l : $K_H = a_\theta b^{\frac{1}{2}} l, K_b = a_b b^{\frac{1}{2}} l, \epsilon l = ab^{3/2}$.
A	(i) $\sigma_w = Au_*$. In neutral conditions A is about 1.05. (ii) Used in the similarity theory methods in the relationship between u_*/u_g and Ro : $A(\mu) = \ln \left[u_*/f z_0 \right] - k u_g/u_*$.
b	turbulent energy $= \frac{1}{2} (\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$.
B	Used in the similarity theory methods in the relationship between u_*/v_g and Ro : $B(\mu) = - k v_g/u_*$.
C_D	drag coefficient.
C	as for A and B, C is involved in the relationship for $T_*/\delta\theta$.
C_p	heat capacity at constant pressure.
D	(i) as for A and B, D is involved in the relationship for $q_*/\delta q$. (ii) rate of deposition of diffusing material on a surface. (iii) D(r) = structure function (in the description of turbulence).
e	exponential function.
E	vertical water vapour flux.

/E(x)

$E(n)$	three-dimensional energy spectrum.
f	(i) the coriolis parameter. (ii) reduced frequency $^{nz}/_{\bar{u}}$ in spectrum analysis.
$F(n)$	one-dimensional longitudinal energy spectrum.
F	used fairly generally as a functional relationship symbol.
g	acceleration due to gravity.
G	full geostrophic wind speed.
$G(n)$	one-dimensional lateral energy spectrum.
h	$h = ku_*/f$; or more generally the height of the boundary layer.
H	vertical heat flux.
i	intensity of turbulence.
k	(i) von Karman's constant whose value is about 0.41. (ii) sometimes used instead of κ for wavenumber.
K	diffusivity for momentum.
K_H	diffusivity for heat.
l	length scale of turbulence. Sometimes used with suffix.
L	the Monin-Obukhov length $L = Tu^3{}_*/kg\overline{\theta' w'}$.
$\mathcal{L}$	latent heat.
m	a constant which relates the roughness length z_o to the surface shearing stress over the sea : $gz_o = mu^2{}_*$. Its value has not been finally decided, but lies somewhere 0.01 - 0.04.
n	frequency (in spectrum analysis).
p	pressure.
q	specific humidity.
q_*	humidity scale $E \equiv u_* q_*$.
Q	source strength.
r	distance of separation between two points.
R	$R(t)$, $R(x)$ etc : correlation function.
Ro	The Rossby number $= G/fz_o$.
s	averaging time.
S	temperature stratification parameter $= \frac{g}{T} \; \frac{\delta\theta}{fG}$
t	time
t_E etc	time-scales : Eulerian, Lagrangian etc.
T	(i) temperature (ii) time of particle travel.
T_*	temperature scale $H = u_* T_*$.
u	the velocity in the x-direction. $\bar{u}$ is the mean wind speed.
u_*	often used as the full friction velocity at the ground (in which case $v_* = o$) but can sometimes be used as a function of z in conjunction with v_*.
u_g	the component of G along the x-axis (usually chosen as the direction of the surface wind).

v	the velocity in the y-direction.
v_*	(see u_*) : the friction velocity in the y-direction.
v_g	the component of G along the y-axis.
w	vertical velocity.
w_e	the entrainment velocity at the top of the boundary layer.
x	direction, usually in the direction of the mean wind.
X	distance travelled downwind in time T.
y	lateral direction.
Y	lateral distance travelled in time T.
z	height above ground.
z_o	surface roughness.
$\overline{Z}$	height of the centre of gravity of a plume.
α	(i) angle between the surface wind and the isobars. (ii) sometimes used in the approximate expansion of ϕ_m : $\phi_m = 1 + \alpha\zeta$
β	(i) Ratio of the Lagrangian time-scale to the Eulerian time-scale of turbulence (ii) Deacon's power law formula for the wind shear $\frac{du}{dz} = az^{-\beta}$.
γ	surface tension.
Γ	adiabatic lapse rate.
ϵ	rate of energy dissipation.
ζ	(i) the relative vorticity. (ii) non-dimensionalised height = $^z/L$.
η	absolute vorticity ($\zeta + f$).
θ	(i) potential temperature. (ii) lateral angular spread in diffusion from a point source.
κ	(i) wavenumber. (ii) thermal diffusivity.
λ	wavelength = $^{\bar{u}}/n$.
Λ	used in the boundary layer to describe the variation of length-scale with z : $l = kz\Lambda(\frac{z}{L})$.
μ	stability parameter = $^h/L$.
ν	kinematic viscosity.
ξ	time.
ρ	density.
σ	standard deviation of the turbulent velocity fluctuations.
$\sigma_u, \sigma_v, \sigma_w$	standard deviations of the components u, v and w.
σ_p	standard deviation of the plume cross-section; a function of x.
τ	(i) sampling time. (ii) shearing stress.

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ϕ	latitude. May also be used for any angle.
ϕ_m, ϕ_H	non-dimensionalised velocity and temperature gradients e.g. $\phi_m = \frac{kz}{u_*}\frac{du}{dz}$.
χ	concentration of diffusing material.
ψ	an angle.
Ψ	Used in semi-empirical length-scale theories of the boundary layer to define l. There is no agreed definition of Ψ but is used to give l as follows: $l = -k\,\Psi/\frac{d\Psi}{dz}$.
ω	occasionally used for frequency.
Ω	angular velocity of the Earth.

General survey of the papers

The main Table in this section summarises the main details of all the papers that have been considered. In the first column the general methods employed are indicated; the second column gives their general nature and the assumptions involved.

The first method involves solving the equations of momentum using an empirical form for the eddy diffusivity K(z). This is a very simple and direct approach which has the virtue that a wind profile is obtained which is not very sensitive to K(z) and which agrees reasonably well with experimentally determined profiles provided the form of K(z) is roughly correct. On the other hand it gives us little understanding of the relationship between the wind profile and the diffusivity and the momentum flux, and therefore fails to answer some of the more complex questions of the boundary layer. Little more need be said about this method.

The second method (B) is rather more subtle about its empiricism. It has as its roots a suggestion by von Kármán (1930) that in the constant stress region, the length-scale must be determined by the velocity profile and, ignoring the turning of the wind and any buoyancy effects, he proposed that

$$l = -k\,\frac{du/dz}{d^2u/dz^2} \qquad (1)$$

This simple law gives the correct solution when $u = \frac{u_*}{k}\ln\frac{z}{z_o}$ near the ground, for then $l = kz$.

Outside the constant stress region, the turning of the wind can no longer be ignored and the three papers in (B) explore how allowance may be made for this in neutral conditions (or when some simple allowance for stability, such as that in Ruzin's second paper, suffices). The general attitude has been to replace du/dz by some function Ψ and to define l as:

$$l = -k\,\frac{\Psi}{\frac{d\Psi}{dz}} \qquad (2)$$

Clearly this is dimensionally correct but nevertheless is an assumption. The majority of papers in this method (B), and in (C) where the effects of buoyancy are considered, put the dynamical part of Ψ as the wind shear:

/ Ψ =

$$\Psi = \left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]^{\frac{1}{2}} \qquad (3)$$

But the wind shear is a vector and it does not seem entirely natural to place in the denominator of (2) the derivative of the modulus of a vector, but rather one would expect to see the modulus of the derivative of the vector. One purely practical issue supports this : the derivative of a modulus can very easily become zero, but the modulus of the derivative of a vector very rarely does because it requires the two component derivatives to pass through zero simultaneously.

Accepting the generalised form of the numerator, the latter form of the denominator would give:

$$l = k\left[\frac{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2}{\left(\frac{\partial^2 u}{\partial z^2}\right)^2 + \left(\frac{\partial^2 v}{\partial z^2}\right)^2}\right]^{\frac{1}{2}} \qquad (4)$$

which is still a natural extension (1). As the Table shows, Ruzin et al.(1963) did suggest this form but as far as is known he never attempted to solve for the wind profile.

Certainly the form of l given by (2) and (3) did not yield very good solutions as can be noted from the Table (Blackadar B(a)). For example the height of the gradient wind level appears to be about four times higher than usually observed. Appendix I gives an analysis which supports (4). It supposes that an element of air is displaced due to turbulence along the vertical. Because of the variation of the mean wind shear with height the trajectory of the element bends back towards its original position in a way exactly analogous to Rossby waves in the horizontal. The amplitude of the element's displacement can be expressed in terms of its velocity and the vertical vectorial gradient of the mean wind velocity shear. Averaging over all likely displacements, a form for l (associated with the mean amplitude) is found which is identical to (4).

In method (C) the problem of non-neutral conditions is considered. Monin, as early as 1950, suggested using the turbulent energy equation (although he then carried the solution out only for zero heat flux). This equation is given in column 2 of the Table. Since the first term in this is reminiscent of Ψ in (3) the natural way of extending Ψ to include the effects of buoyancy appeared to be to include one or more of the other terms on the left hand side of the energy equation; namely, the work done against buoyancy term and the energy diffusion term. Zilitinkevitch and Laikhtman (1965) included the buoyancy term but ignored the turning of the wind and thereby stood a reasonable chance of avoiding the difficulties discussed above for Method (B). Their analysis is both analytic and elegant and is given, with some clarification, in Appendix 2. The results are reasonably encouraging especially in neutral and stable conditions where the theoretical wind profile seems to fit the data very well. On the unstable side the values of u(z) seem rather on the large side and this, it is claimed, is a results of omitting the energy diffusion term. The authors attempt to correct for this omission and are thereby able to bring the theoretical profile into much better agreement with the data. It should be noted that the authors have assumed both the shearing stress and the heat flux constant and therefore the results are applicable only to the surface layer. However the paper provides a valuable check and application of equation (4) of the Appendix which should be of value if the technique can be extended to the whole boundary layer.

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The last paper in this Method (C), that by Bobyleva, Zilitinkevitch and Laikhtman, carry through in a remarkable way the numerical solution of the full set of equations with the various constants empirically specified. This is an outstanding piece of work and it seems a great pity that the final results are rather unconvincing, especially with respect to the variation of $K(z)$ with height in neutral conditions. As in Monin's(1950) analytic solution, the numerical results show

(i) $K \propto z$ up to about 1 km
(ii) K has its maximum at about 3 km
(iii) K does not start to fall rapidly until $z \simeq 10$ km.

These heights all seem much too great; the observational evidence suggests that the K - maxima lie clustered around the 200-500 m level (see for example Zilitinkevitch et al (1967). In view of earlier results, it seems possible that a major source of error lies in accepting equation (2) above as it stands.

The fourth method, (D), abandons this semi-empirical form of the turbulent energy equation and returns to the basic equations of motion. From these equations it is straightforward algebra to derive the Friedmann-Keller equations which express the time rates of change of single-point second-order products of the velocity and temperature fluctuations (see the Table). These equations are then simplified in a variety of ways until the number of unknowns equals the number of equations so that a solution is, in theory at least, possible.

Appendix 3 describes in much more detail the analysis for D(b) and thereby shows the general method and the sort of deductions that may be made. The method is certainly very complex and quite a large number of assumptions have to be made. No complete solution of the equations has yet been found, although that is not to say that the method has been without its rewards. For example, as Appendix 3 shows, it is possible to confirm the empirical connection between the velocity shear and the shearing stress. Other equally interesting results are deduced.

The fifth method, E, the similarity theory method, is discussed in more detail in the next section and in Appendices 4 and 5.

Finally in Appendix 6 we come to a scheme for inserting the effects of the boundary layer, where appropriate, into large-scale numerical models. The scheme is based on that in a paper by Charnock and Ellison (1967). It is appropriate to remind ourselves of the comments made at the conclusion of the Introduction in considering this scheme. The applicability of the scheme to the atmosphere depends on Charnock and Ellison's observation that in about 50 per cent of all situations over the oceans the boundary layer consists of an unstable layer extending from the surface to the base of a deep stable layer, in which the intensity of turbulence is quite small, and which is affected by the boundary layer only through entrainment and by vertical changes in the height of the interface.

Similarity Theory Methods

Brought down to the very simplest terms these methods depend on the possibility of being able to express the unknown variables in non-dimensional form, there being suitable argument for saying only one length-scale and one time-scale (or velocity-scale) and one temperature-scale can be relevant in doing this. The non-dimensionalised forms are then postulated to be universal in character, which will always hold as long as the scales remain the only relevant ones.

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Four contributions are listed in the Table. For the second of these there is only the briefest of summaries to go on, so no further comment will be made. The first is by Kazansky and Monin (1961), and is given in some detail in Appendix 4. They solve the equations of momentum having assumed the relevant length-scale is $h = Ku_*/f$ and velocity-scale is u_*. One further assumption is made, namely that the shearing stress changes significantly from its ground-level value within a layer in which the eddy diffusivity K can be said to vary linearly with height. In the mathematics this means that Φ can be put equal to 1 in equation (2) Appendix 4, the complex momentum equation, which can then be solved, subject to the lower boundary conditions, in ascending powers of $\zeta = Z/h$. Since h is large, the expansion is valid only for small ζ because of the assumptions, and higher powers than the first in ζ are neglected.

Two important deductions follow, namely that u_*/G and α can both be expressed in terms of the Rossby number G/fz_o. Only the first of these relations was actually given by Kazansky and Monin in their original paper but as the appendix shows, the second relation easily follows.

In the third paper by Gill (1967) quite a different technique is used. Gill recognises that different length-scales are relevant in different parts of the layer; thus for small z the surface roughness z_o is all important whereas for large z, the boundary layer thickness h becomes the important scale. He then makes one further assumption, namely that there exists some intermediate region where the two universal functions (the one expressed in terms of z/z_o, the other in terms of z/h) are equally valid. This strong constraint determines the form of the universal functions within this region. The log-law for the wind profile follows, as well as the two relationships given by Kazansky and Monin's method. One may reasonably have some doubts about the validity of Gill's technique since it is questionable whether either universal function can be assumed valid in a region in which <u>both</u> length-scales are relevant.

The fourth paper is by Blackadar (1967). His technique bears some resemblance to Gill's but, to the present author, appears to avoid the pitfalls just discussed. Since Blackadar's description of the method in his paper is rather concise, a somewhat more expanded version is offered in Appendix 5.

Blackadar points out that a non-dimensional combination of the basic parameters can be constructed, namely fz_o/u_* (or equivalently z_o/h). This is a very small quantity, something of $O(10^{-5})$. Since it must enter the two universal forms, suggested by Gill, for small z and large z, we may expand both of them in ascending powers of fz_o/u_*.

Having allowed for this extra parameter the region of validity of each expansion is now much greater than in Gill's case and it is reasonable to suppose that there exists some intermediate layer where both expansions are valid.

Neglecting powers of fz_o/u_* of the first and higher orders, and equating the resulting forms for u/u_* we obtain a relationship which is assumed valid over a range of z and for all Rossby numbers.

Relations identical to those of Gill then follow. It should be noted that two unknown constants of integration, A and B, are involved. Various people have attempted to determine these from experimental data but no very definite conclusion has been reached as can be seen from this Table:

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	Charnock and Ellison	Gill	Zilitinkevitch and Chalikov
A	2	1.2 - 2.2	4
B	2 - 3	4.5	8

TABLE Very approximate values of the two constants A and B.

As can be seen from Figures 11 and 12 in Monin and Zilitinkevitch (1967) A and B vary quite rapidly with stability and this may be the reason for the discrepancies in the Table which are for near neutral conditions.

Conclusions

The position within the constant stress region is largely well established and theory and experimental data are in good agreement. The real interest of these new theories must therefore lie in how well they represent conditions within the rest of the boundary layer.

The position may be summarised as follows:-

1. The form of the wind profile can be predicted with reasonable accuracy but without implying precise forms for K, the length-scale or the shearing stress. In fact many of the solutions for K predict an unrealistically high level for the position of its maximum.

2. There is now good theoretical support for a unique relation between the geostrophic drag coefficient and the surface Rossby number in neutral conditions. Experimental evidence generally supports this conclusion, although the exact values of the two universal constants A and B are still rather uncertain. Theoretical treatments for non-neutral conditions have been made by Blackadar (1967) along very similar lines, and by Bobileva et al (1965) in their solution of the momentum and turbulent energy equations. Monin and Zilitinkevitch (1967) have shown that it is quite possible to fit satisfactory empirical curves $A = A(\mu)$ and $B = B(\mu)$ through the available data points over a wide range of their stability parameter μ. How well the theoretical curves and the empirical curves match is not at present known.

3. Application of the so-called Friedmann-Keller equations, even when grossly simplified, has been very difficult and the results not very rewarding.

4. The Gill-Blackadar theory indicates that in neutral conditions the log-profile should extend well outside the constant stress layer.

5. In non-neutral conditions theoretical results are in quite good agreement with the data obtained within the surface layer, especially in stable conditions (see Zilitinkevitch and Laikhtman, 1965). But above the surface layer very much more experimental confirmation is required. Further clarification is also needed on the relation between the temperature scale T_* and the average lapse rate in the boundary, so that empirical and theoretical relations for the heat input may be objectively examined.

/6.

6. Methods B and C which extend von Karman's relation between the length-scale and the wind profile offer further scope for investigation. In the planned programme of work at Cardington the following should become available:

(i) wind profiles,
(ii) measurements of stress at two (or more) levels,
(iii) estimates of energy dissipation ϵ.

We may follow Pasquill((1967) : J. of Atmos. Environ.) to obtain the length-scale:

$$l \propto \sigma^3/\epsilon$$

where σ is the standard deviation of the vertical velocity fluctuations. (This relation is directly analagous to that given in Method C).

These values of l might then be investigated in terms of the local wind profile in the light of possible forms for the generalised von Karman-relationship to decide which, if any, is satisfactory.

/References

References

Blackadar, A.K., 1961 "The Vertical Distribution of Wind and Turbulent Exchange in a Neutral Atmosphere", published 1962, J.Geophys.Res.67,8,p.3095.

1967 "External Parameters of the wind flow in the barotropic boundary layer of the atmosphere". Report of the GARP Study Conference.

Bobileva, I.M., Zilitinkevitch, S.S and Laikhtman, D.L., 1965 "Turbulent exchange in the thermally-stratified planetary boundary layer of the atmosphere". Acedemy of Sciences of the USSR., Moscow, International Colloq. of fine-scale structure of the Atmosphere.

Charnock, H., and Ellison, T.H., 1967 "The boundary layer in relation to large-scale motions of the atmosphere and ocean". Report of the GARP Study Conference.

Deacon, E.L. 1949 "Vertical diffusion in the lowest layers of the atmosphere". Quart. J.R.Met.Soc., 75,89.

Gill, A.E. 1967 "The turbulent Ekman Layer". Unpublished manuscript, Department of Applied Mathematics and Theoretical Physics, University of Cambridge.

Kazansky, A.B., and Monin, A.S. 1961 "The dynamic interaction between the atmosphere and the surface of the Earth". Izv. ANSSSR, Ser. Geofiz,5,p.786.

Lettau, H.H. 1962 "Theoretical wind spirals in the boundary layer of a barotropic atmosphere". Beitr. Phys. Atmosph., 35,Nos.3-4, p.195.

Monin, A.S. 1950 "Dynamic turbulence in the atmosphere". Izvestia ANSSSR, Ser.Geofiz and Geograph., 14, 3, p.232.

1965(a) "On the symmetry properties of turbulence in the surface layer of air". Izv. ANSSSR, Ser.Atm. & Oceanic Phys., 1, 1, p.45.

1965(b) "Structure of an atmospheric boundary layer". Ibid, 1, 3, p.258.

1965(c) "On the atmospheric boundary layer with inhomogeneous temperature". Ibid, 1, 5, p. 490.

Monin, A.S., and Zilitinkevitch, S.S., 1967 "Planetary boundary layer and large-scale atmospheric dynamics". Report of the GARP Study Conference.

/Monin, A.S.

Monin, A.S., and Yaglom, A.M. 1965 "Statistical hydromechanics". Translation 1966, US Dept. of Commerce, Clearinghouse for Federal scientific and Technical Information, Joint Publications Research Service, Washington. TT : 66-34191.

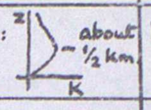
Ruzin, M.L. 1963 "The vertical profile of the coefficient of Turbulence in the atmospheric boundary layer". Trans. (Trudy) All-Union Scientific Meteorological Conference, 7, Gidrometeoizdat.

Ruzin, M.L., Boldyreva, N.A. and Savel'eva, T.A. 1963 "Some results of calculation of the coefficient of turbulent exchange in the atmospheric boundary layer". Trans (Trudy) Len. Hydrometeorol. Inst., No. 15.

Von Kármán, T. 1930 Nachr. Ges. Wiss. Göttingen, Math-physik. Klasse; or see O.G. Sutton's Micrometeorology", 1953, McGraw-Hill Publishing Co. Ltd. p.81.

Zilitinkevitch, S.S., and Laikhtman, D.L., 1965 "Turbulent Conditions in the near-surface layer of the atmosphere". Izv., Ser.Atmos. and Oceanic Phys., 1, 2, p.150.

Zilitinkevitch, S.S., and Laikhtman, D.L., and Monin, A.S. 1967 "Dynamics of the Atmospheric boundary layer : A review". Ibid. 3, 3, p.297.

Method	General Assumptions of Method	Author & Year	Particular Assumptions	General Method of Solution	Conclusions.
Empirical length-scale A.	Solve the equations of momentum either in conjunction with (i) An empirical form for $K(z)$ or (ii) $K(z) = l^2(z)\left[\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right]^{1/2}$ plus an empirical form for $l(z)$	(i) numerous	the better models give $K(z)$ behaving like this: 	Very simple direct solution	Wind profile not very sensitive to form of K. Conversely $K(z)$ not easily determined from wind profile.
		(ii) Blackadar 1961 Lettau 1962	neutral stability empirical $l(z) = kz\left(1+\frac{kfz}{0.00027G}\right)^{-1}$ similar.	numerical solution	Spiral obtained with log-form in lowest 20 metres. Reasonably good fit for surface wind angle and u_*/G as a function of R_o.
Semi-empirical length-scale B.	(i) neutral stability (ii) relate l to wind profile in a generalised way to that suggested by von Karman: $l = -k\frac{du}{dz}\Big/\frac{d^2u}{dz^2}$. (iii) $K(z)$ related to l as in A(ii).	(a) Blackadar 1961	$l = -k\,\Psi\Big/\frac{d\Psi}{dz}$ where $\Psi = \left[\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right]^{1/2}$ = wind shear	Converted non-linear mom. eqns to two linear diff. eqns plus two linear integral eqns. Solved numerically.	Not very good fit to observed data. Surface wind angle half that observed. $\frac{u_*}{G}$ too big. Height of b.l. too great by factor of about 4.
		(b) Ruzin 1963	similar to Blackadar except that Ruzin attempted to generalise equation for l by inserting a stability factor $\bar{m}$.	–	–
		(c) Ruzin et al. 1963 (quoted by Monin)	$l = k\bar{m}\left[\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right]^{1/2}\Big/\left[\left(\frac{d^2u}{dz^2}\right)^2+\left(\frac{d^2v}{dz^2}\right)^2\right]^{1/2}$ $\bar{m}$ is a complex stability factor.	No information available but suspect that no numerical solution has been obtained.	–
Energy-equation method. C.	(i) Turbulent energy equation is: $K_m\left[\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right] - K_H\frac{g}{\theta}\frac{d\theta}{dz} + \frac{d}{dz}K_b\frac{db^2}{dz} = \varepsilon$ (b^2 = turbulent energy) production by wind shear; work done against buoyancy; diffusion of energy; energy dissipation. (ii) $l = -k\,\Psi\Big/\frac{d\Psi}{dz}$ where Ψ is some part of the left-hand side of the energy equation. (iii) hypothesis of approximate similarity: $K_m = \sqrt{b}\,l$, $K_H = a_\theta K_m$, $K_b = a_b K_m$, $\varepsilon = a\,b^{3/2}/l$ (iv) usually the heat flux H is made constant.	(a) Monin 1950	$a_\theta = 0$, $a = 1$. For (ii) Monin had $l = kz$ Two solutions corresponding to $a_b = 0$ and $a_b = 1/4$. Since $a_\theta = 0$ Monin is considering NEUTRAL conditions	numerical evaluation after putting equations into non-dimensional form.	(i) effect of diffusion term negligible in neutral case. (ii) u_*/G too big by almost a factor of 2. (iii) Surface wind angle too small.
		(b) Ruzin 1965	$a_b = 0$. Assumed either H or $\frac{d\theta}{dz}$ constant in layer.	Attempted to derive analytic first approximations to the solution.	–
		(c) Zilitinkevich & Laikhtman 1965	(i) No v-component of wind. Within surface layer (ii) $\Psi = \left(\frac{du}{dz}\right)^2 - a_\theta\frac{g}{\theta_0}\frac{d\theta}{dz}$ {N.B. no square root. θ_0 is an average θ in layer. (iii) $a_b = 0$, although a correction is determined. (no energy diffusion)	Very cunning analytical solution. Wind profile etc given in dimensionless coordinates for all stabilities.	Reasonably good fit of wind profile to data for neutral & stable cases. In unstable conditions fit is improved if correction made for diffusion term.
		(d) Bobyleva, Zilitinkevich & Laikhtman 1965	(i) Full momentum and energy equations. (ii) $\Psi = \sqrt{\varepsilon/K_m}$ (iii) a_θ and a_b and a specified empirically. (iv) H = const (note: eqn (6) has a −ve sign omitted) (v) In stable conditions $b \to 0$ and in unstable cond. $b \sim z^{1/3}$ as $z \to \infty$	New variables are introduced $\eta = K\frac{du}{dz}$ and $\sigma = K\frac{dv}{dz}$. The momentum eqns are differentiated & they and the energy eqn expressed in terms of η & σ. Eqns solved numerically for 5 values of μ (stability)	Solutions vary strongly with μ. Data on u_*/G against R_o fit theory curve for μ about +10 (stable), while data on wind angle α fit theory curve for μ about 5. K_m increases up to max. at around 3Km in neutral case, at ∞ in unstable case and at one or two hundred metres in stable case. Maximum (neutral) seems much too high. Low-level jets predicted in stable cases at 100-200 m.
Semi-empirical theory based on the FRIEDMANN-KELLER equations D.	(i) The momentum equations are used to derive, in conjunction with the temperature equation: $\frac{\partial T}{\partial t} + (\underline{V}.\underline{\nabla})T = \chi\frac{\partial^2 T}{\partial z^2}$, the Friedmann-Keller equations which express the time rates of changes of the following: $\overline{u'^2}, \overline{v'^2}, \overline{w'^2}, \overline{T'^2}, \overline{u'v'}, \overline{u'w'}, \overline{v'w'}, \overline{u'T'}, \overline{v'T'}$ and $\overline{w'T'}$ (ii) These equations are simplified in the following ways: 1. time rates of change are put zero. 2. third order moments in the primed quantities are put zero thereby neglecting effect of eddy interaction. 3. effect of coriolis force on velocity fluctuations assumed indirect only. 4. some small terms neglected. 5. terms involving p' are handled by using semi-empirical relations in terms of velocity products similar to those used by Davydov (based on Poisson eqn relating pressure & velocity $\nabla^2 p = -\rho\frac{\partial^2(u_i u_j)}{\partial x_i \partial x_j}$ and analogy with molecular collision theory). 6. Other terms with symmetry properties expressed in simple tensor form.	(a) Monin 1965 (a)	Thermally stratified but restricted to surface layer by assuming mean pressure field makes no contribution. Assume parameters & coefficients in eqns can be expressed in terms of b & l.	Although no complete solution is obtained, various interesting results are deduced, e.g. $(\sigma_u^2 - \sigma_v^2)\sigma_w^2 = 2u_*^4$	If φ_m could be specified, all other variables could be expressed in terms of $\zeta = z/L$.
		(b) Monin 1965 (b)	Neutral stability but deals with whole layer. (T' terms omitted)	Verifies from these simplified equations various results ..... No wind profiles emerge.	Finds $\tau_{xz} \propto l\sigma_w^2\frac{d\bar{u}}{dz}$ etc. And relations between σ_u^2 etc and the $\frac{d\bar{u}}{dz}$ and $\frac{d\bar{v}}{dz}$.
		(c) Monin 1965 (c)	Thermally stratified and deals with whole layer. Assumes that $l(z)$ can be specified empirically. Heat flux assumed constant.	No complete solutions are sought ... only the effect of stability on b, α etc comparing them with neutral values.	In theory it should be possible to solve the equations given $l(z)$ but the problems are very formidable.

[redacted]

Method	General Assumptions of Method	Author & Year	Particular Assumptions	General Method of Solution	Conclusions
Similarity Theory Method E	It is assumed that all the profile variables may be expressed in non-dimensional form in terms of either the "internal" or "external" parameters and that these forms are equivalent since the "internal" parameters can themselves be expressed in terms of the "external" parameters. The internal parameters are: u_*, α, T_*, H and μ. The external parameters are: f, G, z_0 and $\delta\theta = \theta_{00} - \theta_0$.	Kazansky & Monin 1961	Assumes K varies linearly with height over a depth for which the shearing stress has altered significantly from its surface value. In the non-neutral case the turbulent energy equation of C is assumed valid.	Solves the two momentum eq^{ns} in complex form for small z/h retaining terms up to the first power in z/h.	Obtains the geostrophic drag coefficient $\frac{u_*}{G}$ and the angle α between the surface wind and the isobars in terms of R_0 and two parameters A & B which depend only on μ.
		Monin & Zilitinkevich 1967	Includes the effect of thermal stratification and the vertical flux of water vapour E, by using the heat balance equation, specific humidity balance eq^n and the turbulent energy equation.	Method only very briefly summarised in this paper.	Relations between T_* and q_* (the humidity scale) and R_0 and further parameters depending on μ are given.
		Gill 1967	Assumes away from surface ($\underline{V} - \underline{G}$) depends only on f, u_*, z but near to the surface $\underline{V}$ depends only on u_*, z_0, z. Curiously assumes these regions overlap. Neutral stability.	Puts velocities into non-dimensional form in each region and then studies outcome of assumption of overlap.	Obtains log-type profile which agrees with conventional log profile in surface layer. Also obtains same relationships as Kazansky & Monin (1961).
		Blackadar 1967	Assumes the principles of 'asymptotic similarity' whereby having expressed the wind in non-dimensional form as Gill did, and noting that $fz_0/u_* = O(10^{-5})$ it is possible to expand the profiles in powers of this quantity in which the coefficients are functions of z/z_0 provided $z < h$ or functions of fz/u_* provided $z > z_0$. The regions where these asymptotic expansions are valid are assumed to overlap since fz_0/u_* is so small.	By equating the leading terms of these expansions in the overlap region, the same relationships that Gill obtained follow.	Since the wind profiles are strictly only valid in this overlap region nothing can be directly deduced about the profile when z is of the same order as z_0. According to this theory then, if the log profile holds down to z_0 it must be because it provides the best extrapolation from the overlap region.
Insertion of Boundary Layer into large-scale models. F	The boundary layer should have a definite top, height h, above which there is no turbulence and the vertical velocity w remains virtually constant. h is a function of horizontal position & time. w at top of b.l. can be expressed in terms of the surface drag and the absolute vorticity. The temperature profile maintains a universal similarity-theory form. Radiation may be neglected. Entrainment of air through the upper surface and heat flux from the ground (or sea) are the only means by which heat enters the system.	Charnock & Ellison 1967.	Assumes u not a function of z in h-equation (or can be replaced by a mean u). No cross-isobaric advection of h. Other assumptions are: The heat flux is given. The surface drag coefficient can be specified. The profiles of θ & q are similar and are determined by either $\delta\theta$ (or δq). θ at the lower surface is constant (for the sea).	Equations for the heat content of the layer, the temperature at the top & the bottom and the height h are solved with respect to time from some initial state	No computations have yet been carried out so it is difficult to forsee whether the scheme is practical or not. The aim is to predict h and w, and if possible the statistical quantity of cloud etc.

APPENDIX 1

The relation between the length-scale and the mean velocity profile

The length-scale of turbulence, at any level z, reflects the probable vertical displacements that elements of air are likely to undergo at that level as a result of the turbulence.

The following simple model sets out to estimate the amplitude of these vertical fluctuations.

Suppose in the neighbourhood of z, the mean wind $\underline{u}$ has a vertical shear $\underline{f}(z)$ (which is analogous to the coriolis parameter f which is the horizontal shear over the surface of the Earth when there is no relative motion). Define $\beta = \left|\frac{d\underline{f}(z)}{dz}\right|$ and let us suppose that β is virtually a constant for individual element motions.

Then $\left|\underline{f}(z_1) - \underline{f}(z_2)\right| \simeq \beta(z_1 - z_2)$ provided $z_1 - z_2$ is not too large.
Consider an element of air which is displaced from a level z_1. If it starts with no vorticity relative to its environment, its total vorticity f_1 is approximately conserved so that when it reaches z_2:

$$K_s V + f_2 = f_1$$

where K_s is the curvature of its trajectory, V is the approximately constant turbulent velocity along it, and f_2 is the new environmental vorticity or shear at z_2.

$$\text{Hence} \quad K_s V = f_1 - f_2 = -\beta z$$

$$\text{where} \quad z = z_2 - z_1 .$$

Treating β/V as a constant, and using the definition of K_s:

$$\frac{d^2z}{dx^2} = -\frac{\beta z}{V}\left[1 + \left(\frac{dz}{dx}\right)^2\right]^{3/2}$$

where x is the distance of downwind travel relative to axes moving with the velocity $\underline{u}$.
Platzman has given the solution of this equation. He put $\frac{dz}{dx} = \tan\psi$ so that ψ is the inclination of the trajectory to the x-axis, and has an initial value ψ_i. The equation then simplifies to

$$\frac{d}{d\psi}z^2 = -\frac{2V}{\beta}\sin\psi$$

with a solution: $$z^2 = \frac{2V}{\beta}(\cos\psi - \cos\psi_1).$$

The amplitude of the trajectory, which is a measure of the length-scale l when suitably averaged, is given by

$$A = \left[\frac{2V}{\beta}(1 - \cos\psi_i)\right]^{1/2}$$

β is by definition: $\left[u''^2 + v''^2\right]^{1/2}$

V is a typical vertical velocity of the turbulence, proportional to σ_w and

APPENDIX 1

in neutral conditions $\sigma_w \sim B' u_*$ where $B' = O(1)$.
That is, $V \simeq B\sqrt{\tau/\rho}$.

The length-scale in the vertical is therefore:

$$l = \lambda \left[\frac{2\sqrt{\tau/\rho}}{\left(u''^2 + v''^2\right)^{1/2}} \right]^{1/2}$$

where λ is some constant of proportionality of $O(1)$, which may be found by requiring that $l = kz$ in the surface-layer where the log wind profile holds.

Hence $$kz = l = \lambda \left[\frac{2u_*}{u_*/kz^2} \right]^{1/2} = \lambda \sqrt{2k}\, z$$

i.e. $$\lambda = \sqrt{\frac{k}{2}}.$$

Thus $$l = \frac{k^{1/2} \, (\tau/\rho)^{1/4}}{\left(u''^2 + v''^2\right)^{1/4}}$$

Now $\frac{\tau}{\rho} = \sqrt{\frac{\tau}{\rho}}\, l \frac{\partial}{\partial z} \underline{u}$, which implies $\frac{\tau}{\rho} = l^2 \left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right]$

Eliminating $\frac{\tau}{\rho}$ gives: $$l = k \left[\frac{u'^2 + v'^2}{u''^2 + v''^2} \right]^{1/2}.$$

Zilitinkevich and Laikhtman (1965).

The motion is assumed uni-directional with no turning of the wind with height. The shearing stress and the heat flux are taken to be constant, thus u_* and T_* are constant. The vertical diffusion of turbulent energy is assumed zero (although subsequently correction can be made for this omission).

Both the diffusivity K and the rate of energy dissipation ε are expressed in terms of the basic internal parameters b (b^2 = turbulent energy) and l (the length-scale):

$$K = b^{1/2} l$$

$$\varepsilon = a\, b^{3/2}/l$$

Extending von Karman's relation for l to allow for the variation of potential temperature with height, the authors postulate that

$$l = -\bar{\kappa}\, \frac{\left(\frac{du}{dz}\right)^2 - a_\theta \frac{g}{\theta} \frac{d\theta}{dz}}{2 \frac{du}{dz} \frac{d^2u}{dz^2} - a_\theta \frac{g}{\theta} \frac{d^2\theta}{dz^2}}$$

where $\bar{\kappa}$ is related to von Karman's constant k, and a_θ is defined by the ratio of the heat diffusivity to the momentum diffusivity.

The variables are now made dimensionless by using L, u_* and T_* and other constants are absorbed in the process so that the following equations are derived:

$$K \frac{du}{dz} = 1 , \quad K \frac{d\theta}{dz} = 1 \qquad \ldots\ldots\ldots (1)$$

$$K \left[\left(\frac{du}{dz}\right)^2 - \frac{d\theta}{dz} \right] - \varepsilon = 0 \qquad \ldots\ldots (2)$$

$$K = b^{1/2} l , \quad \varepsilon = \frac{b^{3/2}}{l} \qquad \ldots\ldots (3)$$

$$l = - \left[\left(\frac{du}{dz}\right)^2 - \frac{d\theta}{dz} \right] \Big/ \left[\frac{du}{dz} \frac{d^2u}{dz^2} - \frac{1}{2} \frac{d^2\theta}{dz^2} \right] \qquad \ldots\ldots (4)$$

$$K \to 0 \text{ as } z \to 0 \qquad \ldots\ldots\ldots (5)$$

From (1) $\frac{du}{dz} = \frac{d\theta}{dz}$. Write $x = \frac{du}{dz}$ and $y = \sqrt{b}$. The equations then become:

$$\begin{cases} Kx = 1 \quad \text{(from (1))} \;; & x = \varepsilon + 1 \quad \text{(from (2))} \\ K = yl \quad \text{(from (3))} \;; & y^3 = \varepsilon l \quad \text{(also from (3))} \\ \frac{dx}{dz} = - \frac{2x(x-1)}{l(2x-1)} & \text{(from (4))} \end{cases}$$

Hence $K(\varepsilon + 1) = 1$

i.e. $yl\left(\frac{y^3}{l} + 1\right) = 1$

$$y^4 + yl = 1$$

or $$l = \frac{1-y^4}{y} \quad \ldots\ldots\ldots \quad (6)$$

Also since $K = yl$ it follows that

$$K = 1 - y^4 \quad \ldots\ldots\ldots \quad (7)$$

and $$\varepsilon = y^4/(1-y^4) \quad \ldots\ldots\ldots \quad (8)$$

Thus $$x = \varepsilon + 1 = \frac{1}{1-y^4} \quad \ldots\ldots \quad (9)$$

Substituting into the lengh-scale equation

$$\frac{dx}{dz} = -\frac{\frac{2}{1-y^4}\left(\frac{1}{1-y^4}-1\right)}{\frac{1-y^4}{y}\left(\frac{2}{1-y^4}-1\right)} = -\frac{1}{(1-y^4)^2}\cdot\frac{2y^5}{1+y^4} \quad \ldots\ldots \quad (10)$$

Hence from (9) : $$\frac{dx}{dz} = \frac{dx}{dy}\cdot\frac{dy}{dz} = \frac{1}{(1-y^4)^2}\cdot 4y^3\frac{dy}{dz}$$

Comparing this with (10) gives

$$2\frac{dy}{dz} = -\frac{y^2}{1+y^4} \quad \ldots\ldots\ldots \quad (11)$$

which is readily solved subject to the following boundary condition:

$K \to 0$ as $z \to 0$ implies that $y \to 1$ as $z \to 0$.

$$z = \frac{2}{y} - \frac{2}{3}y^3 - \frac{4}{3} \quad \ldots\ldots\ldots \quad (12)$$

Now $$\frac{du}{dy} = \frac{du}{dz}\Big/\frac{dy}{dz} = -\frac{2}{1-y^4}\cdot\frac{1+y^4}{y^2} \quad \text{from (9) and (11)}$$

$$= -\frac{2}{y^2} - \frac{2}{1-y^2} + \frac{2}{1+y^2}$$

Therefore $$\underline{u = \frac{2}{y} + 2\tan^{-1}y - \ln\frac{1+y}{1-y}} \quad \ldots\ldots\ldots \quad (13)$$

where y is related to the non-dimensionalised height by (12), and the equation is valid within the constant stress and heat flux layer.

It follows from (12) and (13) that :

(i) in neutral conditions, as $L \to \infty$, $z \to 0$: $u = \ln z + \text{const}$

(the ordinary non-dimensionalised log-profile)

(ii) in free convection conditions, as $L \to -0$, $z \to -\infty$: $y \to -\left(\frac{3z}{2}\right)^{1/3}$:

$$u + \text{const} = \theta + \text{const} = -\left(\frac{16}{3z}\right)^{1/3}$$

$$K = -\left(\frac{3z}{2}\right)^{4/3}, \quad b = \left(\frac{3z}{2}\right)^{2/3}, \quad l = \frac{3z}{2}, \quad \varepsilon = -1$$

(iii) in strong inversions, as $L \to +0$, $z \to \infty$, $y \to 2/z$:

$$u + \text{const} = \theta + \text{const} = z$$

$$K = 1, \quad b = 0, \quad l = z/2, \quad \varepsilon = 0.$$

The universal wind profile (13) fits the data very well for neutral and stable conditions, but in unstable conditions account has to be made of energy diffusion.

Monin (1965): "Structure of an atmospheric boundary layer".

The analysis will be given in an abbreviated form but in sufficient detail to give a feeling of the general technique employed in these semi-empirical theories based on the Friedmann Keller equations. Neutral stability is assumed in this analysis.

The equations of motion are:

$$\left.\begin{aligned} \frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} &= -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + (f_k u_j - f_j u_k) - g\lambda_i + \nu \nabla^2 u_i \\ \frac{\partial u_\alpha}{\partial x_\alpha} &= 0 \end{aligned}\right\} \quad \ldots (1)$$

where λ_i are the components of $(0,0,1)$.

To form the Friedmann Keller equations the momentum equations are multiplied by u_k' etc to form diagnostic equations for the second order velocity products, i.e. equations which start:

$\frac{\partial}{\partial t}\overline{u_i'^2}$, $\frac{\partial}{\partial t}\overline{u_i' u_j'}$, etc. (note that the mean values of all the variables have been subtracted).

Now certain simplifying assumptions are made. Some of them can be applied equally to equations (1) as to the Friedmann Keller equations. They are:

(i) the time rates of change of the averaged equations are put zero.

(ii) terms involving ν are ignored when gradients of means are involved.

(iii) terms in f are neglected because the effect of the coriolis force is most pronounced on the mean wind profile and only significantly affects the turbulent components implicitly through the mean profile.

(iv) all third order moments of the velocity fluctuations are neglected thereby removing all inter-eddy effects. Each eddy therefore is assumed to interact only with the mean wind.

(v) some other small terms are dropped.

The simplified F-K equations then become:

$$\tau_{xz}\frac{\partial \bar{u}}{\partial z} - \nu\overline{(\nabla u')^2} + \frac{1}{\rho}\overline{p'\frac{\partial u'}{\partial x}} = 0$$

$$\tau_{yz}\frac{\partial \bar{v}}{\partial z} - \nu\overline{(\nabla v')^2} + \frac{1}{\rho}\overline{p'\frac{\partial v'}{\partial y}} = 0$$

$$- \nu\overline{(\nabla w')^2} + \frac{1}{\rho}\overline{p'\frac{\partial w'}{\partial y}} = 0$$

$$-\sigma_w^2\frac{\partial \bar{u}}{\partial z} + \frac{1}{\rho}\overline{p'\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)} = 0$$

$$-\sigma_w^2\frac{\partial \bar{v}}{\partial z} + \frac{1}{\rho}\overline{p'\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right)} = 0$$

and $$\tau_{xz}\frac{\partial \bar{v}}{\partial z} + \tau_{yz}\frac{\partial \bar{u}}{\partial z} + \frac{1}{\rho}\overline{p'\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)} = 0$$

Now assuming local isotropy and that we can adopt the hypothesis of similarity by which the turbulence parameter can be expressed in terms of the turbulent energy b^2 and the length-scale l, we ~~can~~ can put the turbulent energy dissipation:

$$\nu\,\overline{\nabla u_i' \nabla u_j'} = \frac{c_1}{3}\frac{b^3}{l}\delta_{ij}$$

$$\text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Similarly we adopt a semi-empirical formula for the pressure terms, suggested by Davydov (1959), based on

(i) an analogy with the theory of molecular collisions in Boltzmann's kinetic theory in gases.

(ii) Poisson's equation which relates pressure fluctuations to the velocity fluctuation field.

which can be expressed as follows:

$$\frac{1}{\rho}\overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} = -c_2\frac{b}{l}\left(\overline{u_i'u_j'} - \frac{b^2}{3}\delta_{ij}\right) - c_3\frac{b^3}{l}\left(\lambda_i\lambda_j - \frac{1}{3}\delta_{ij}\right)$$

The F-K equations then become:

$$\tau_{xz}\frac{\partial \bar{u}}{\partial z} - \frac{2c_1 - c_3}{6}\frac{b^3}{l} - \frac{c_2}{2}\frac{b}{l}\left(\sigma_u^2 - \frac{b^2}{3}\right) = 0$$

$$\tau_{yz}\frac{\partial \bar{v}}{\partial z} - \frac{2c_1 - c_3}{6}\frac{b^3}{l} - \frac{c_2}{2}\frac{b}{l}\left(\sigma_v^2 - \frac{b^2}{3}\right) = 0$$

$$-\frac{c_1 + c_3}{3}\frac{b^3}{l} - \frac{c_2}{2}\frac{b}{l}\left(\sigma_w^2 - \frac{b^2}{3}\right) = 0$$

$$\tau_{xz} = \frac{l\sigma_w^2}{c_2 b}\frac{\partial \bar{u}}{\partial z}, \quad \tau_{yz} = \frac{l\sigma_w^2}{c_2 b}\frac{\partial \bar{v}}{\partial z}, \quad \tau_{xy} = -\frac{2l^2\sigma_w^2}{c_2^2 b^2}\frac{\partial \bar{u}}{\partial z}\frac{\partial \bar{v}}{\partial z}$$

which give basis to the general hypothesis that the shearing stress is proportional to the velocity gradient.

On this basis the diffusivity K is seen to be $l\sigma_w^2/c_2 b$.

Eliminating the τ and b it is a matter of simple algebra to show that:

$$\sigma_u^2 = Al^2\left(\frac{\partial \bar{u}}{\partial z}\right)^2 + Bl^2\left(\frac{\partial \bar{v}}{\partial z}\right)^2$$

$$\sigma_v^2 = Bl^2\left(\frac{\partial \bar{u}}{\partial z}\right)^2 + Al^2\left(\frac{\partial \bar{v}}{\partial z}\right)^2$$

$$\sigma_w^2 = Cl^2\left[\left(\frac{\partial \bar{u}}{\partial z}\right)^2 + \left(\frac{\partial \bar{v}}{\partial z}\right)^2\right]$$

where A, B and C are constants expressible in terms of c_1, c_2 and c_3. Near the ground $\frac{\partial \bar{v}}{\partial z} \simeq 0$ and $l = kz$. Thus $A = \sigma_u^2/u_*^2$ and $B = \sigma_v^2/u_*^2$, $C = \sigma_w^2/u_*^2$.

APPENDIX 3

Experimental data then suggests that:

$A \simeq 5.8$, $B \simeq 2.9$ and $C \simeq 0.64$ (but see below*)

It follows that $c_1 \simeq \frac{1}{29}$, $c_2 \simeq \frac{1}{5}$ and $c_3 \simeq -\frac{1}{13}$

We may also deduce the following expressions:

$K = l^2 s$ and $\varepsilon = l^2 s^3$

where s is the shear: $s = \left[\left(\frac{\partial \bar{u}}{\partial z}\right)^2 + \left(\frac{\partial \bar{v}}{\partial z}\right)^2 \right]^{1/2}$.

One final interesting deduction may be quoted which is valid in the surface layer of the atmosphere, even when conditions are unstable and when there is a horizontal heat flux. It is:

$$(\sigma_u^2 - \sigma_v^2)\,\sigma_w^2 \simeq 2\,u_*^4 .$$

The values of A, B and C almost conform to this relationship.

* However more recent estimates by King (Q.J.R.Met.S. April 1965), which have since been confirmed, put $A = 9.6$, $B = 4.8$ and $C = 1.7$ which do not fit the relationship anywhere near as well.

APPENDIX 4 [redacted]

Kazansky & Monin (1961) [see Monin & Yaglom p.318]

The equations of motion are:

$$\left.\begin{aligned} \frac{\partial}{\partial z} K \frac{\partial u}{\partial z} + fv &= \frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial}{\partial z} K \frac{\partial v}{\partial z} - fu &= \frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots (1)$$

Differentiate both these equations with respect to z, multiply the second by i and add. Calling $K\left(\frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z}\right) = F_1$ and assuming $\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) = \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right) = 0$ we obtain:

$$\frac{\partial^2}{\partial z^2} F_1 - i F_1 \frac{f}{K} = 0$$

Define the 'height' of the boundary layer $h = \frac{ku_*}{f} = O(10^4)$.

On dimensional grounds, then, $F_1 = u_*^2 F\left(\frac{z}{h}\right)$ where $F(0) = 1$ by definition

and $K = ku_* z\, \Phi\left(\frac{z}{h}\right)$ where for small $\frac{z}{h}$: $\Phi \sim 1$.

Writing $\zeta = \frac{z}{h}$ and remembering that here u_* is the ground value of the friction velocity and is therefore constant.

$$\frac{u_*^2}{h^2}\frac{\partial^2 F}{\partial \zeta^2} - i u_*^2 F \cdot \frac{f}{ku_* h \zeta \Phi(\zeta)} = 0$$

i.e.

$$\zeta \frac{\partial^2 F}{\partial \zeta^2} = \frac{iF}{\Phi} \quad \ldots\ldots\ldots\ldots (2)$$

Furthermore from equations (1), multiplying the first by i and subtracting:

$$i\frac{\partial F_1}{\partial z} + f\left[u + iv\right] = \frac{1}{\rho}\left[i\frac{\partial p}{\partial x} - \frac{\partial p}{\partial y}\right] = \frac{i}{\rho}\left[\frac{\partial p}{\partial x} + i\frac{\partial p}{\partial y}\right] = fGe^{i\alpha}$$

where α = angle between the surface wind and the isobars. The final identity follows since $\frac{1}{\rho}\frac{\partial p}{\partial x} = fG\sin\alpha$, $\frac{1}{\rho}\frac{\partial p}{\partial y} = -fG\cos\alpha$.

Hence $u + iv = Ge^{i\alpha} - i\frac{u_*}{k}F'(\zeta)$ remembering the definition of F.

At $z = z_0$, $u + iv = 0$

$$\therefore F'(\zeta_0) = -\frac{k}{u_*} i G e^{i\alpha} = \frac{k}{u_*} G \exp\left[i\left(\frac{3\pi}{2} + \alpha\right)\right]$$

and

$$\frac{F'(\zeta_0)}{\zeta_0} = \frac{kG}{u_*}\cdot\frac{ku_*}{fz_0}\exp\left[i\left(\frac{3\pi}{2}+\alpha\right)\right] = k^2 \mathcal{R}_o e^{i\left(\frac{3\pi}{2}+\alpha\right)} \quad \ldots\ldots (3)$$

Now h is large so we may expand $F(\zeta)$ in an asymptotic form for $z \ll h$. Provided we may put Φ equal to 1 in the region of validity of the expansion a solution of equation (2) is:

$$F(\zeta) = 1 + \left(\gamma + i\ln\left[\zeta/\zeta_*\right]\right)\zeta + O(\zeta^2)$$

where ζ_* is the value of ζ at some level. This constant is omitted in Monin and Yaglom p.319. The solution is obtained by writing $F = a + ib$.

On inserting in (2) and equating the real and imaginary parts we obtain

$\zeta a'' = -b$, $\zeta b'' = a$ (where the primes denote differentiation w.r.t. ζ)

Thus $\zeta(\zeta a'')'' = -a$ and $\zeta(\zeta b'')'' = -b$

with boundary conditions $a(0) = 1$, $b(0) = 0$.

[redacted]

APPENDIX 4

γ is a constant of integration just as ζ_* is.

Differentiating:

$$F'(\zeta) = \gamma + i \ln \frac{\zeta}{\zeta_*} + i + \ldots$$

Thus $$F'(\zeta_0) \approx F'(\zeta_1) + i \ln \frac{\zeta_0}{\zeta_1} \qquad \ldots\ldots \text{④}$$

where $\zeta_1 = \frac{z_1}{h}$ and z_1 is the height to which K remains approximately linear, i.e. $\Phi \simeq 1$.

From ③ $$k^2 R_0 = \frac{|F'(\zeta_0)|}{\zeta_0}$$

Substituting in for $F'(\zeta_0)$ from ④ and using the definitions of h and ζ_0 we obtain:

$$\frac{kG}{u_*} = \left| F'(\zeta_1) - i \ln \zeta_1 k + i \ln \frac{z_0 f}{u_*} \right|$$

Writing $F'(\zeta_1) - i \ln \zeta_1 k = -B + iA$ where B and A are postulated to be constants in the neutral case but are allowed to vary with μ when the layer is thermally stratified. From ④ it is clear that ζ_1 on the l.h.s. of this eqn. could be replaced by ζ provided $\zeta_0 \le \zeta \le \zeta_1$.

Then $$\frac{kG}{u_*} = \left| -B + i\left(A + \ln \frac{G}{u_* R_0}\right) \right| \qquad \text{since } \frac{z_0 f}{u_*} \equiv \frac{G}{u_* R_0}. \quad R_0 = \frac{G}{z_0 f}$$

i.e. $$\left(\frac{kG}{u_*}\right)^2 = B^2 + \left(A - \ln R_0 + \ln \frac{G}{u_*}\right)^2 \qquad \ldots\ldots\ldots \text{⑤}$$

i.e. $$\ln R_0 = A + \ln \frac{G}{u_*} + \left[\frac{k^2 G^2}{u_*^2} - B^2\right]^{1/2} \qquad \ldots\ldots\ldots \text{⑥}$$

When the turbulent energy equation is used the following values for A and B are found for neutral stability.

$$A = 2.49 \qquad B = +1.69$$

Now from ③ $$\frac{\text{Imag.}\,(F'(\zeta_0))}{\text{Real}\,(F'(\zeta_0))} = \tan\left(\frac{3\pi}{2} + \alpha\right) = -\cot\alpha.$$

i.e. $$\tan\alpha \approx \frac{-\gamma}{1 + \ln \zeta_0/\zeta_*}$$ using the expansion at the top of the page.

But from the definition of B and A, we have $\gamma = B$ & $1 - \ln\zeta_* = A + \ln k$

Hence $$\tan\alpha = + \frac{B}{A + \ln k\zeta_0}$$

and $$\sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{+B}{\sqrt{[A + \ln k\zeta_0]^2 + B^2}}$$

Since $k\zeta_0 \equiv G/u_* R_0$ the denominator is identical to the r.h.s. of ⑤ and hence

$$\sin\alpha = + \frac{B u_*}{kG} \qquad \ldots\ldots\ldots\ldots \text{⑦}$$

N.B. Kazansky & Monin did not derive ⑦ in their (1961) paper but clearly we have seen that they could have done without any further extension of the theory.

Boundary Layer Profile by Blackadar (1967).

The parameters affecting the profile are z_0, u_*, G, and f. These are not independent since u_* is determined by z_0, G and f (or, over the sea, both u_* and z_0 are mutually interrelated to G and f).

We assume on dimensional grounds that

$$\frac{G}{u_*} = f\left(\frac{fz_0}{u_*}\right) \qquad \text{where typically} \quad \frac{fz_0}{u_*} = O(10^{-5}).$$

Because of this relation we do not need to include G explicitly in the general wind profile relation:

$$\frac{u}{u_*} = F\left(\frac{z}{z_0}, \frac{fz_0}{u_*}\right)$$

Since fz_0/u_* is so small we may make an asymptotic expansion of F in terms of it, valid provided z/z_0 does not become comparable with $(fz_0/u_*)^{-1}$; i.e. the expansion is valid provided $z \ll h$.

The expansion is:

$$\frac{u}{u_*} = f_{10}\left(\frac{z}{z_0}\right) + f_{11}\left(\frac{z}{z_0}\right)\frac{fz_0}{u_*} + f_{12}\left(\frac{z}{z_0}\right)\left(\frac{fz_0}{u_*}\right)^2 + \cdots\cdots$$

$$\simeq f_{10}\left(\frac{z}{z_0}\right) \qquad \text{over a large range of } z.$$

For large z (near $z = h$) this particular form of the expansion is not very suitable because as we have noted $\frac{z}{z_0} = O\left(\frac{1}{\frac{fz_0}{u_*}}\right)$.

It is better then to rearrange the independent variable in F to be:

$\frac{z}{z_0} \times \frac{fz_0}{u_*} = \frac{fz}{u_*}$ and $\frac{fz_0}{u_*}$. At these greater heights

$\frac{fz}{u_*} = O(1)$. We also know that $u \to u_g$. Since $\frac{u_g}{u_*}$ is a function

of fz_0/u_* we can "pull" u_g/u_* out of F to comply with this condition on u:-

$$\frac{u - u_g}{u_*} = F'\left(\frac{fz}{u_*}, \frac{fz_0}{u_*}\right)$$

Again we may expand in terms of fz_0/u_*:

$$\frac{u - u_g}{u_*} = f_{20}\left(\frac{fz}{u_*}\right) + f_{21}\left(\frac{fz}{u_*}\right)\frac{fz_0}{u_*} + \cdots\cdots$$

$$\simeq f_{20}\left(\frac{fz}{u_*}\right) \qquad \text{over a large range of } z.$$

If fz_0/u_* is as small as we believe it to be the range of z for which these asymptotic expansions are valid are sufficiently large that they overlap.

There $f_{10}\left(\frac{z}{z_0}\right) \equiv \frac{u_g}{u_*} + f_{20}\left(\frac{fz}{u_*}\right)$ over the region of overlap.

Write $\frac{z}{z_0} = \zeta$, $\frac{fz_0}{u_*} = \varepsilon_0 = \varepsilon_0(R_0)$, $\frac{fz}{u_*} = \zeta\varepsilon_0$

Hence $f_{10}(\zeta) = \frac{u_g}{u_*} + f_{20}(\zeta\varepsilon_0)$

Differentiate with respect to ζ, and secondly with respect to R_0 :

$f_{10}'(\zeta) = \varepsilon_0 f_{20}'(\zeta\varepsilon_0)$ since $\frac{u_g}{u_*}$ does not depend on ζ.

$0 = \frac{d}{dR_0}\left(\frac{u_g}{u_*}\right) + \zeta f_{20}'(\zeta\varepsilon_0)\frac{d\varepsilon_0}{dR_0}$ since ζ is independent of R_0.

Eliminating f_{20}' : $\zeta f_{10}'(\zeta) = -\left(\frac{d}{dR_0}\ln\varepsilon_0\right)^{-1}\frac{d}{dR_0}\left(\frac{u_g}{u_*}\right) = \text{constant}$

since the lefthand side is a function of ζ only whilst the righthand side is a function of R_0 only.

$\therefore\ \zeta f_{10}'(\zeta) = \frac{1}{k}$, say.

$\therefore\ f_{10}(\zeta) = \frac{\ln\zeta}{k}$ i.e. $\underline{\frac{u}{u_*} = \frac{1}{k}\ln\frac{z}{z_0}}$

Also $\underline{\frac{u_g}{u_*} = \frac{1}{k}\left(\ln\frac{u_*}{fz_0} - A\right)}$

For the y components $f_{10}\left(\frac{z}{z_0}\right)$ is zero by choice of axes. Hence $v/u_* \simeq 0$ and $\frac{d}{dR_0}\left(\frac{v_g}{u_*}\right) = 0$

Hence $\underline{\frac{v_g}{u_*} = \text{constant} = -\frac{B}{k}}$, say.

One interesting thing to note about this analysis is that the log-profile for u is strictly valid only in the overlap region where $z \gg z_0$, and in so far as it is valid for smaller z, in practise, this appears to be fortuitous.

Three constants of integration are left to be determined : k, A and B. k is now well established as 0.41 but, as the Table in the main text shows, A and B are still only roughly known.

<u>A scheme for inserting the Boundary Layer into a Large-scale Model based on that suggested by Charnock & Ellison (1967).</u>

The aim is to forecast the changes and magnitude of the depth of the boundary layer and the vertical velocity induced at the top.

It is assumed that a definable interface exists between the boundary layer and the free atmosphere above. The interface is represented by $h(\underline{x}, t)$. Air from the free atmosphere may be entrained into the boundary layer with a velocity w_e and the air at the interface may itself have a vertical velocity w (which must persist up through the free atmosphere).

Velocity shear within the boundary layer is neglected compared to the mean wind as far as the advection of the interface is concerned. The Lagrangian rate of change of h is therefore determined solely by w and w_e:

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = w + w_e \qquad \cdots\cdots\cdots (1)$$

From conditions in the free atmosphere, and allowing for the thermal wind, calculate the geostrophic wind at the ground that would exist if there were no friction.

From this the friction velocity u_* (or surface stress τ_0) at the ground can be estimated from empirical relationships such as that displayed in Figure 8, Monin (1967) which relates u_*/G to Ro. The surface geostrophic wind will also give the relative vorticity ζ, and this information is now sufficient to solve for w.

The exact form of the w-equation is perhaps open to question. Charnock & Ellison give the following:

(i) The momentum equation, when $\underline{G}$ is the frictionless geostrophic velocity, is

$$\frac{\partial \underline{\tau}}{\partial z} = f\,\underline{k} \wedge (\underline{V} - \underline{G})$$

where $\underline{k} = (0,0,1)$ and $\frac{d}{dt}(\underline{V} - \underline{G})$ has been neglected.

Hence $\underline{k} \wedge \frac{\partial}{\partial z}\left(\frac{\underline{\tau}}{f}\right) = \underline{G} - \underline{V}$

since $\underline{k} \cdot (\underline{V} - \underline{G}) = 0$

Therefore $\frac{\partial}{\partial z}\left[\underline{k} \wedge \frac{\underline{\tau}}{f}\right] = \underline{G} - \underline{V} = -\underline{V}'$, the ageostrophic wind velocity.

$$\text{i.e.} \quad -\underline{k} \wedge \frac{\underline{\tau}_0}{f} = -\int_0^h \underline{V}'\,dz \qquad \cdots\cdots\cdots (2)$$

Now $\frac{\partial w}{\partial z} = -\underline{\nabla} \cdot \underline{V} = -\underline{\nabla} \cdot \underline{V}' - \underline{\nabla} \cdot \underline{G} \equiv \frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial z}$, say

Using (2) $w_1 = -\underline{\nabla} \cdot \int_0^h \underline{V}'\,dz = -\underline{\nabla} \cdot \left(\underline{k} \wedge \frac{\underline{\tau}_0}{f}\right) = \underline{k} \cdot \underline{\nabla} \wedge \frac{\underline{\tau}_0}{f} \equiv \mathrm{curl}_H \frac{\underline{\tau}_0}{f}$

For the frictionless component we can turn to the simple vorticity equation:

$$\frac{\partial \eta}{\partial t} + \underline{\nabla} \cdot \eta \underline{G} = 0$$

i.e. $$\frac{d\eta}{dt} + \eta\, \underline{\nabla} \cdot \underline{G} = 0$$

$$\therefore \quad \underline{\nabla} \cdot \underline{G} = -\frac{1}{\eta}\frac{d\eta}{dt} = -\frac{d}{dt} \ln \eta$$

$$\therefore \quad w_2 = -\int_0^h \underline{\nabla} \cdot \underline{G}\, dz = h \frac{d}{dt} \ln \eta$$

Combining w_1 and w_2: $$w = \mathrm{curl}_H \frac{\underline{\underline{\tau}}_0}{f} + h \frac{d}{dt} \ln \eta \qquad \text{..........} \quad (3).$$

An alternative form to this equation is:

(ii) The full vorticity equation is $\frac{d}{dt}\eta + \eta \underline{\nabla} \cdot \underline{V} = \frac{\partial}{\partial z}(\underline{\nabla} \wedge \underline{\tau})$

Now $$\underline{\nabla} \cdot \underline{V} + \frac{\partial w}{\partial z} = 0$$

$$\therefore \quad \int_0^h \frac{d\eta}{dt} dz - \int_0^h \eta \frac{\partial w}{\partial z} dz = \underbrace{(\underline{\nabla} \wedge \underline{\tau})_h}_{=0} - (\underline{\nabla} \wedge \underline{\tau})_0$$

$$\therefore \quad h \frac{d\bar{\eta}}{dt} - \bar{\eta}\, w(h) = -\underline{\nabla} \wedge \underline{\tau}_0$$

where we have a defined an average η in the boundary layer.

An approximation is made here, namely that the two terms yield the same $\bar{\eta}$.

Hence $$w(h) = h \frac{d}{dt} \ln \bar{\eta} + \frac{1}{\bar{\eta}} \underline{\nabla} \wedge \underline{\tau}_0 \qquad \text{.............} \quad (4).$$

As one can see the difference between equations (3) and (4) lies in the curl term. Overall I believe (4) to be the better equation for our purposes.

The next step is to estimate the heat flux. Over the sea this is comparatively easy since the temperature difference $\theta_h - \theta_0$ can be readily estimated, since the sea temperature is comparatively conservative quantity. Over land the problem is that much harder.

But given $\theta_h - \theta_0$, and assuming a universal form for the temperature profile the heat flux follows. (see for example Charnock and Ellison, equation (3)). Two other parameters are also obtained: the Monin-Obukhov length L, and the average potential temperature $\bar{\theta}$ in the layer.

Two important equations now have to be obtained, both dealing with temperature changes in the boundary layer.

1. Define $$I(t) = \int_0^{h(t)} \theta(t,z)\, dz$$

then $$\frac{d}{dt} I(t) = \int_0^{h(t)} \frac{d\theta}{dt} dz + \frac{dh}{dt} \theta(t,h).$$

Now $\frac{dI}{dt} \equiv \frac{d}{dt}(\bar{\theta}h) = \lim_{\Delta t \to 0}\left[\frac{(h+[w+w_e]\Delta t)(\bar{\theta}+\Delta\bar{\theta}) - h\bar{\theta}}{\Delta t}\right]$

where $\Delta\bar{\theta} = \frac{w_e}{h}\left[\theta(t,h) - \bar{\theta}\right]\Delta t$ since ignoring the heat flux at this stage (this is easily added in afterwards) and considering the heat content of a given column of air:

$$(h + w_e\Delta t)\,\bar{\theta}(t+\Delta t) = h\bar{\theta}(t) + w_e\,\Delta t\,\theta(t,h).$$

Hence $\frac{d}{dt}(\bar{\theta}h) = \bar{\theta}\,(w+w_e) + \frac{w_e}{h}\left(\theta(t,h) - \bar{\theta}\right)h = w\bar{\theta} + w_e\,\theta(t,h).$

Therefore
$$\int_0^{h(t)} \frac{d\theta}{dt}\,dz = w\bar{\theta} + w_e\,\theta(t,h) - (w+w_e)\,\theta(t,h)$$
$$= w\left[\bar{\theta} - \theta(t,h)\right] \qquad \ldots\ldots\ldots\ldots (5).$$

Adding the effect of the heat flux we obtain:
$$\int_0^{h(t)} \frac{d\theta}{dt}\,dz = w\left[\bar{\theta} - \theta(t,h)\right] + \overline{w'\theta'_s} \qquad \ldots\ldots\ldots\ldots (6)$$

The first term is the integrated local temperature change, the second term is the effect of the vertical motion acting on the lapse rate, and the third term is the effect due to the surface heat flux.

2. The second equation we need is:
$$\frac{d}{dt}\,\theta(h) = w_e\left(\frac{\partial\theta}{\partial z}\right)_h \qquad \ldots\ldots\ldots\ldots (7)$$

which follows since the temperature at the top of the boundary layer must equal that at the bottom of the free atmosphere. Only entrainment eating into this temperature profile affects $d\theta(h)/dt$. There is no contribution from w since w lifts the air in the free atmosphere as well as in the boundary layer.

To return to the forecast scheme, the next step is to solve equation (6) for $\int_0^h \frac{d\theta}{dt}\,dz$. Since the temperature profile has a universal form, this immediately implies a change in $\theta(h)$ for a given $\overline{\theta' w'}$. We may now solve equation (7) for w_e and determine the change in h using equation (1).

The cycle may be repeated as often as required.