

## BASIC HYDRODYNAMIC AND THERMODYNAMIC EQUATIONS (Revision)

### MOMENTUM

$$\frac{D\mathbf{V}}{Dt} = -2\mathbf{\Omega} \wedge \mathbf{V} - \frac{1}{\rho} \nabla p + \mathbf{g} + \mathbf{F}$$

In component form, for frictionless motion ( $\mathbf{F}=0$ ) and neglecting terms due to the earth's curvature, we have :-

$$\frac{Du}{Dt} - fv = \frac{1}{\rho} \frac{\partial p}{\partial x} \quad (1)$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (2)$$

(noting that  $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$ )

The momentum equation for vertical motion is usually reduced, via the hydrostatic approximation, to a diagnostic equation :-

$$\frac{\partial p}{\partial z} = -\rho g \quad (3)$$

### CONTINUITY

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

With the assumption of incompressibility ( $\frac{D\rho}{Dt}=0$ ) this reduces to

$$\nabla_h \cdot \mathbf{V} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

### THERMODYNAMICS

$$\left. \begin{aligned} dQ &= C_v dT + p d\alpha \\ \text{or } dQ &= C_p dT - \alpha dp \end{aligned} \right\} \begin{aligned} &\text{1st Law of Thermodynamics} \\ &(C_p = C_v + R) \end{aligned}$$

for adiabatic processes  $dQ=0$ ; with this assumption, and  $\theta = T \left( \frac{1000}{p} \right)^{R/C_p}$ ,  $\frac{D}{Dt}$  of the latter equation leads to

$$\frac{D\theta}{Dt} = 0 \quad (5)$$

### EQUATION OF STATE (perfect dry gas)

$$p\alpha = RT \quad (6)$$



Boussinesq approximation . Variations in density are ignored except in the gravitational force term in the vertical momentum equation . In the horizontal momentum equations ,  $\rho$  is assumed constant ( some convenient "standard" or "reference" value ) . This approximation also involves the incompressibility assumption in the full ( z-coordinate ) continuity equation .

The set of equations (1) - (5) , with the frictionless , hydrostatic , adiabatic and Boussinesq approximations is often expressed in the form :-

$$\frac{Du}{Dt} - f v + \frac{\partial \phi}{\partial x} = 0 \quad (1a)$$

$$\frac{Dv}{Dt} + f u + \frac{\partial \phi}{\partial y} = 0 \quad (2a)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (3a)$$

$$\frac{D\theta}{Dt} = 0 \quad (4a)$$

$$\frac{\partial \phi}{\partial z} = \left( \frac{g}{\theta_0} \right) \theta \quad (5a)$$

where  $\phi$  is geopotential  $(= \int_0^z g dz)$ , and  $z = \left( \frac{R\theta_0}{gK} \right) \left( 1 - \left( \frac{p}{p_0} \right)^K \right)$  .

$K = R/c_p$  and  $p_0$  and  $\theta_0$  are standard or reference values of pressure and potential temperature .

The vertical coordinate  $z$  here is little different from physical height in the lower troposphere ( it is exactly the same if  $\frac{\partial \theta}{\partial z} = 0$  )

A further alternative format for these basic equations , using pressure ,  $p$  , as the vertical coordinate , is given later .



Geostrophic approximation

$$\frac{D\mathbf{V}}{Dt} = 0$$

leading to  $|\mathbf{V}_g| = \frac{1}{\rho f} \nabla_h p = \frac{1}{f} \nabla_p \phi$ , the geostrophic wind equation

The Rossby number,  $Ro$ , is often used as a measure of the relative magnitudes of the acceleration and Coriolis terms in the momentum equations.

$$Ro = \frac{u^2/L}{f u} = \frac{u}{f L}$$

( $L$  is a length scale). The smallness of  $Ro$  is a measure of the validity of the geostrophic approximation.

Thermal wind

$$v_g = \frac{1}{f} \frac{\partial \phi}{\partial x} \quad (7)$$

$$u_g = -\frac{1}{f} \frac{\partial \phi}{\partial y} \quad (8)$$

The hydrostatic equation (3) can be expressed as  $\partial \phi / \partial p = -RT/p$ ; using this relationship and differentiating (7) and (8) w.r.t.  $p$

$$p \frac{\partial v_g}{\partial p} = \frac{\partial v_g}{\partial (\ln p)} = -\frac{R}{f} \left( \frac{\partial T}{\partial x} \right)_p \quad (9)$$

$$p \frac{\partial u_g}{\partial p} = \frac{\partial u_g}{\partial (\ln p)} = \frac{R}{f} \left( \frac{\partial T}{\partial y} \right)_p \quad (10)$$

In vector form this is  $\frac{\partial \mathbf{V}_g}{\partial (\ln p)} = -\frac{R}{f} \mathbf{k} \wedge (\nabla T)_p$ , the thermal wind equation

The thermal wind for a layer  $p_0$  to  $p_1$  is given by

$$\mathbf{V}_T = \mathbf{V}_g(p_1) - \mathbf{V}_g(p_0) = -\frac{R}{f} \int_{p_0}^{p_1} (\mathbf{k} \wedge \nabla T) d \ln p.$$

Assuming  $\bar{T}$  is the mean temperature for the layer  $p_0$  to  $p_1$ , the  $x, y$  components of  $\mathbf{V}_T$  can be expressed as

$$u_T = -\frac{R}{f} \left( \frac{\partial \bar{T}}{\partial y} \right) \ln \left( \frac{p_0}{p_1} \right) \quad (11)$$

$$v_T = \frac{R}{f} \left( \frac{\partial \bar{T}}{\partial x} \right) \ln \left( \frac{p_0}{p_1} \right) \quad (12)$$



## Barotropic atmosphere

In a purely barotropic atmosphere, density depends only on the pressure, so that isobaric surfaces are also surfaces of constant density. For an ideal gas the isobaric surfaces will also be isothermal if the atmosphere is barotropic. So  $(\nabla T)_p = 0$  and

$$\frac{\partial \tilde{V}_g}{\partial(\ln p)} = 0.$$

## Baroclinic atmosphere

In a baroclinic atmosphere, surfaces of constant pressure and constant density (isopycnics) intersect, and the geostrophic wind varies with height, the wind shear being related to the horizontal temperature gradient by the thermal wind equation:

$$\frac{\partial \tilde{V}_g}{\partial(\ln p)} = -\frac{R}{f} (\underline{k} \wedge \nabla T)_p$$

Deformation. In hydrodynamics, this is a measure of the change of shape of a fluid element produced by space variations of fluid velocity. In meteorology, the term is used mainly in respect of the kinematical development of frontogenesis and frontolysis. Flow associated with a Col, for example, represents a deformation field of motion.

$$B_1 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

stretching deformation

$$B_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

shearing deformation

$$\text{Total deformation } B = (B_1^2 + B_2^2)^{1/2}.$$

Vorticity. A measure of the rotation or spin of fluid elements

$$\text{Vertical component of relative vorticity } \zeta = \underline{k} \cdot \nabla \wedge \underline{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

$$\text{Often expressed in the form } \zeta = \underbrace{V/r}_{\text{curvature}} - \underbrace{\partial V / \partial n}_{\text{shear}}$$

$r$  = radius of trajectory,  $\underline{n}$  is directed to the left of the flow.

$$\text{Absolute vorticity } \zeta_a = \zeta + f.$$



### VORTICITY EQUATION. (using $z$ as vertical coordinate)

Using the momentum equations (1) and (2)

$$\frac{\partial}{\partial y} \left\{ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - f v \right\} = \frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\frac{\partial}{\partial x} \left\{ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + f u \right\} = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$

Subtracting these two equations, and given that  $f = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ , some algebraic manipulation gives:

$$\frac{D}{Dt} (f + f) = - \underbrace{(f + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)}_A - \underbrace{\left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right)}_B + \frac{1}{\rho^2} \underbrace{\left( \frac{\partial \rho}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y} \frac{\partial p}{\partial x} \right)}_C \quad (13)$$

or, vectorially:

$$\frac{D}{Dt} (f + f) = - (f + f) \nabla_h \cdot \underline{V} + \underline{k} \cdot \left( \frac{\partial \underline{V}}{\partial z} \wedge \nabla w \right) - \underline{k} \cdot (\nabla \alpha \wedge \nabla p)$$

In equation (13): term A is the divergence term.

B is the tilting or twisting term.

C is the solenoidal term.

For most synoptic-scale motions the tilting, solenoidal and vertical advection of relative vorticity ( $w \partial f / \partial z$ , part of  $\frac{D}{Dt} (f + f)$ ) terms are normally assumed to be negligible in comparison with the other terms, and (13) reduces to:

$$\frac{D}{Dt} (f + f) = - f \nabla_h \cdot \underline{V} \quad (14)$$

or, via the continuity equation:

$$\frac{D}{Dt} (f + f) = f \frac{\partial w}{\partial z} \quad (14a)$$

Here  $\frac{D}{Dt} h \equiv u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}$ .

This approximate form of the vorticity equation does not remain valid close to atmospheric fronts (horizontal scale  $\lesssim 100$  km). On these scales the vertical advection, tilting and solenoidal terms may all be significant.



## BASIC EQUATIONS

(isobaric coordinates in the vertical)

### MOMENTUM

$$\frac{Du}{Dt} - fv = -\frac{\partial \phi}{\partial x} \quad (15)$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial \phi}{\partial y} \quad (16)$$

where  $\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_p + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p + \omega\frac{\partial}{\partial p}$ ,  $\omega = \frac{Dp}{Dt}$

The hydrostatic equation is expressed as :

$$\frac{\partial \phi}{\partial p} = -\alpha = -\frac{RT}{p} \quad (17)$$

### CONTINUITY

$$\nabla_p \cdot \underline{V} + \frac{\partial \omega}{\partial p} = 0 \quad (18)$$

### THERMODYNAMICS

$$c_p \frac{D(\ln \theta)}{Dt} = \frac{DS}{Dt} \quad (19)$$

Here, diabatic heating is allowed for in  $DS/Dt$ , the rate of change of entropy.

(for adiabatic processes  $DS/Dt = 0$  and, thus,  $D\theta/Dt = 0$ )

### EQUATION OF STATE

In terms of potential temperature, this is :

$$\theta = \frac{p\alpha}{R} \left(\frac{1000}{p}\right)^k \quad (20)$$

These six equations, (15) - (20), form a closed set which completely determines the relationships among the dependent variables  $u, v, \omega, \phi, \alpha$  and  $\theta$  (provided  $DS/Dt$  is specified)



## VORTICITY EQUATION (p-coordinates)

In the isobaric coordinate system, the vorticity equation is :

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f) \nabla_p \cdot \underline{V} + \left( \frac{\partial u}{\partial p} \frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p} \frac{\partial \omega}{\partial x} \right)$$

or, expanding  $\frac{D}{Dt}$ ,

$$\frac{\partial \zeta}{\partial t} = -\underline{V} \cdot \nabla(\zeta + f) - \omega \frac{\partial \zeta}{\partial p} - (\zeta + f) \nabla_p \cdot \underline{V} + \left( \frac{\partial u}{\partial p} \frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p} \frac{\partial \omega}{\partial x} \right) \quad (21)$$

(here  $\nabla \equiv \nabla_p$ )

The various terms here are, reading from left to right :

1. local change of relative vorticity
2. horizontal advection of absolute vorticity
3. vertical advection of relative vorticity
4. divergence term
5. tilting or twisting term.

The solenoidal term which appeared in the z-coordinate version, equation (13), is automatically excluded when the horizontal derivatives are evaluated on isobaric surfaces.

For mid-latitude synoptic-scale motions, we can simplify (21) by :-

- a. neglecting the vertical advection and tilting terms
- b. neglecting  $\zeta$  compared to  $f$  in the divergence term
- c. approximating the horizontal velocity by the geostrophic wind in the advection term
- d. replacing the relative vorticity by its geostrophic value.

We can also assume that the Coriolis parameter has a constant value  $f_0$  except where it appears differentiated in the advection term, in which case  $\partial f / \partial y (= \beta)$  is assumed constant; this is usually referred to as the beta-plane approximation.

Applying the above approximations, we obtain the quasi-geostrophic vorticity equation :

$$\frac{\partial \zeta_g}{\partial t} = -\underline{V}_g \cdot \nabla(\zeta + f) - f_0 \nabla_p \cdot \underline{V} \quad (22)$$

where  $\zeta_g = f_0^{-1} \nabla^2 \phi$ ,  $\underline{V}_g = f_0^{-1} \underline{k} \wedge \nabla \phi$ . Note that  $\underline{V}$  is not replaced by its geostrophic value in the divergence term.



Alternatively, with  $\nabla \cdot \underline{V}$  eliminated using the continuity equation:

$$\frac{\partial S_g}{\partial t} = - \underline{V}_g \cdot \nabla (S + f) + f_0 \frac{\partial \omega}{\partial p} \quad (22a)$$

Since  $S_g$  and  $\underline{V}_g$  are both defined in terms of  $\phi$ , this equation can be used to diagnose the  $\omega$  field (for synoptic-scale motions) provided that  $\phi$  and  $\partial\phi/\partial t$  are known.

### THERMODYNAMIC EQUATION in terms of $\phi$

Eliminate  $\theta$  from the thermodynamic equation using the equation of state as in (20), i.e.

$$\theta = \frac{p\alpha}{R} \left( \frac{1000}{p} \right)^k$$

$$\ln \theta = \ln \alpha - \left( \frac{R}{c_p} - 1 \right) \ln p + \text{constant}$$

differentiation on an isobaric surface gives

$$\left( \frac{\partial \ln \theta}{\partial x} \right)_p = \left( \frac{\partial \ln \alpha}{\partial x} \right)_p, \text{ etc}$$

Expanding the total derivative  $D/Dt (\ln \theta)$  in (19) and substituting  $\ln \alpha$  in place of  $\ln \theta$  in the partial derivatives evaluated at constant  $p$ , we get

$$\frac{\partial \ln \alpha}{\partial t} + u \frac{\partial \ln \alpha}{\partial x} + v \frac{\partial \ln \alpha}{\partial y} + \omega \frac{\partial \ln \alpha}{\partial p} = \frac{1}{c_p} \frac{DS}{Dt}$$

$$\text{So } \frac{\partial \alpha}{\partial t} + u \frac{\partial \alpha}{\partial x} + v \frac{\partial \alpha}{\partial y} - \sigma \omega = \frac{\alpha}{c_p} \frac{DS}{Dt} \quad (23)$$

where  $\sigma = -\frac{\alpha}{\theta} \frac{\partial \theta}{\partial p}$ , the static stability

For synoptic scale systems, to a first approximation, the horizontal velocity components may be replaced by their geostrophic values, and if we also assume  $DS/Dt$  is negligible, the thermodynamic equation can be reduced to the approximate form in equation (24)



$$\underbrace{\frac{\partial \alpha}{\partial t}}_A = - \underbrace{V_g \cdot \nabla \alpha}_B + \underbrace{\sigma \omega}_C \quad (24)$$

Both  $\alpha$  and  $\sigma$  can be expressed in terms of  $\phi$  or  $\partial\phi/\partial p$ , so that (24) involves only two dependent variables,  $\phi$  and  $\omega$ .

Term A is effectively the local rate of change of temperature on an isobaric surface.

Term B is proportional to the horizontal advection of temperature by the geostrophic wind on an isobaric surface.

Term C is normally called the adiabatic cooling (heating) term; it represents the adiabatic temperature changes resulting from rising and expansion (sinking and compression) of air parcels.

The corresponding thickness change  $(\frac{\partial h'}{\partial t})$  equation, which is often quoted, is :

$$\frac{\partial h'}{\partial t} = -R \int_{p_0}^{p_1} \left[ -V_g \cdot \nabla T + (\Gamma_a - \Gamma) \omega \right] d \ln p \quad (25)$$

where  $\Gamma = \frac{\partial T}{\partial p}$ ,  $\Gamma_a =$  adiabatic temperature lapse rate.

(the diabatic heating term is also ignored in this equation)



## GEOPOTENTIAL TENDENCY EQUATION

Defining the geopotential tendency  $\chi \equiv \frac{\partial \phi}{\partial t}$ , the thermodynamic equation (24) may be written

$$\frac{\partial \chi}{\partial p} = \underline{V}_g \cdot \nabla \alpha - \sigma \omega \quad (26)$$

and the quasi-geostrophic vorticity equation (22a) may be written (using the relationship  $\zeta_g = f_0^{-1} \nabla^2 \phi$ ):

$$\nabla^2 \chi = -f_0 \underline{V}_g \cdot \nabla (\zeta_g + f) + f_0^2 \frac{\partial \omega}{\partial p} \quad (27)$$

Now  $\frac{f_0^2}{\sigma} \frac{\partial}{\partial p}$  (26) + (27) yields

$$\left( \nabla^2 + \frac{f_0^2}{\sigma} \frac{\partial^2}{\partial p^2} \right) \chi = \underbrace{-f_0 \underline{V}_g \cdot \nabla (\zeta_g + f)}_B + \underbrace{\frac{f_0^2}{\sigma} \frac{\partial}{\partial p} (\underline{V}_g \cdot \nabla \alpha)}_C \quad (28)$$

(assuming  $\sigma$  to be independent of pressure)

This is the geopotential tendency equation

Term A involves only second derivatives in space of the  $\chi$  field; for wave-like disturbances, this term can easily be shown to be proportional to  $-\chi$  (Holton (1972)).

Term B is proportional to the advection of absolute vorticity by the geostrophic wind. For physical interpretation the term is usually divided into two parts

$$\underline{V}_g \cdot \nabla (\zeta_g + f) \equiv \underline{V}_g \cdot \nabla \zeta_g + \sigma_g \frac{\partial f}{\partial y}$$

These two parts represent the geostrophic advections of relative vorticity and planetary vorticity. For typical wave-like perturbations in a basic zonal flow these two terms tend to oppose each other. The vorticity advection term cannot change the amplitude of wave-like disturbances, but only acts to propagate them horizontally. The mechanism for amplification or decay of mid-latitude systems is contained in term C, the differential temperature advection term. In a developing baroclinic wave this tends to be a maximum (in magnitude) at the trough and ridge lines.



## OMEGA EQUATION

A diagnostic equation for the vertical motion ( $\omega$ ) field can be obtained by eliminating  $\chi$  between (26) and (27).

Taking the derivative w.r.t.  $p$  of (27) and subtracting the horizontal Laplacian of (26),  $\chi$  is eliminated and we obtain the so-called omega equation

$$\underbrace{\left(\nabla^2 + \frac{f_0^2}{\sigma} \frac{\partial^2}{\partial p^2}\right)}_A \omega = \underbrace{\frac{f_0}{\sigma} \frac{\partial}{\partial p} \left[ \vec{V}_g \cdot \nabla (S_g + \beta) \right]}_B + \underbrace{\frac{1}{\sigma} \nabla^2 \left[ \vec{V}_g \cdot \nabla \alpha \right]}_C \quad (29)$$

It involves only derivatives in space and is therefore a diagnostic equation for the  $\omega$  field in terms of the instantaneous  $\phi$  field (remembering that  $S_g = f_0^{-1} \nabla^2 \phi$ ,  $\alpha = -\partial \phi / \partial p$ , and  $\sigma$  can be expressed in terms of  $\phi$ )

The omega equation, unlike the continuity equation, gives a measure of  $\omega$  which does not depend on accurate observations of the horizontal wind field. In fact, direct wind observations are not required at all. This method is also superior to the vorticity equation method of obtaining the  $\omega$  field since no knowledge of the vorticity tendency is required; only observations of  $\phi$  at a single time are required to determine the  $\omega$  field using equation (29).

As with the geopotential tendency equation the terms in the omega equation are each subject to straightforward physical interpretation. Term A for instance is, for a wave-like perturbation, simply proportional to  $-\omega$ ; term B is the differential vorticity advection, and term C, the horizontal Laplacian of the thermal advection, is proportional to the negative of the thermal advection. Terms B and C are often referred to as the "forcing" terms for vertical motion.

Holton (1972) gives an account of the physical significance of the terms B and C individually, but, as Hoskins et al. (1978) have pointed out, with the omega equation written in this form there can be substantial cancellation between the two terms, and the simple addition of a speed of  $U$  to the motion of the whole system alters the extent of this cancellation, and thus the relative



magnitudes and phases of the terms (but does not affect the resultant). So the effect of each term considered in isolation can be misleading in attempting to diagnose the magnitude, and even the sign, of the vertical velocity.

Hoskins et al. (1978) equivalent form of equation (29), using  $w$  (rather than  $\omega$ ) for the vertical motion within the 2-coordinate system of equations listed earlier ((1a) - (5a)), is :-

$$N^2 \nabla^2 w + f_0^2 \frac{\partial^2 w}{\partial z^2} = f_0 \frac{\partial}{\partial z} (\underline{V}_g \cdot \nabla \underline{S}_g) - \frac{g}{\theta_0} \nabla^2 (\underline{V}_g \cdot \nabla \theta) \quad (29a)$$

where  $N = \left( \frac{g}{\theta_0} \cdot \frac{\partial \theta}{\partial z} \right)^{1/2}$ , the Brunt-Vaisala frequency.

$\theta_0$  is a standard value of potential temperature and  $\theta(z)$  is a standard distribution of potential temperature in the vertical.

In Sutcliffe's development theory (see, for example, Handbook of Weather Forecasting, Ch. 12) an expression is derived, subject to a series of approximations, for the difference between the horizontal divergence at two levels and criteria for the vertical motion obtained. In fact Sutcliffe's development equation can be regarded as an alternative form of the omega equation (with added assumptions). Sutcliffe's work was presented in a rather simple form in order that its practical applications could be easily carried out and understood; it does in fact give a good idea of the large-scale system movement and development, but the details near active frontal zones are missed.

### References

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