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TURBULENCE & DIFFUSION NOTE NO. 204.

A New Gravity Wave Drag Parametrization Scheme for the  
Unified Model

by

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October 1990

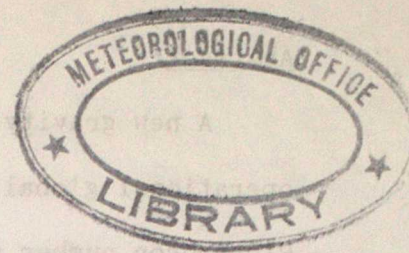
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## Abstract

A new gravity wave drag parametrization scheme is proposed for the operational global and Fine-Mesh models. The more obvious weaknesses of the Richardson number scheme described by Palmer, Shutts and Swinbank (1986) are remedied and several new aspects of stratified flow over mountains are represented. Improvements to the Richardson number scheme include:

- (1) a scheme for determining the occurrence of trapped lee waves which predicts their wavelength and amplitude profile.
- (2) an assumed functional form for the orographic variance spectrum function which facilitates a crude calculation of wave drag in the presence of anisotropy in the orography. It is also used in the estimation of lee wave momentum fluxes.
- (3) a treatment of the low Froude number regime (flow blocking) within (4)
- (4) a representation of high drag states (hydraulic jump response)

A gross simplifying assumption made by Palmer et al (1986) - that all the vertical momentum flux is carried by long, hydrostatic waves with no internal reflection - is considered to be the weakest aspect of that scheme. Trapped (or 'leaky') lee waves dominate the gravity wave response in strong wind synoptic situations and this, together with the hydraulic jump response for Froude number  $\leq 1$  implies more tropospheric drag and less



stratospheric drag than the Richardson number scheme.

## 1. Introduction

In the last five years gravity wave drag parametrizations schemes have been incorporated into most operational forecast models around the world. Their inclusion has corrected the greater part of a major systematic error that existed beforehand namely, excessive westerly flow bias. The quality of climate simulations depends heavily on the existence of gravity wave drag parametrization (it is easy to forget how serious an error existed prior to the implementation of the Richardson number scheme in the 11-level climate model). This uncomfortable fact suggests that the details of climate simulation may be sensitive to the form of gravity wave drag parametrization. *It is important therefore to ensure that this parametrization scheme has as much realism as possible.*

Compared to other parametrization schemes such as the deep convection scheme, the Richardson number scheme does not properly reflect the level of current understanding of the subject. Perhaps its most unsatisfactory feature is the lack of treatment of flow blocking effects. In practical terms, high sub-grid scale orographic variances have to be artificially suppressed to account for the fact that stratified flows tend to go around rather than over steep terrain slopes. Air may become trapped in valley or separate near peaks thereby reducing the effective sub-grid scale terrain height variation. The key non-dimensional flow parameter controlling this phenomenon is the Froude number  $F_r$  ( $= \frac{U}{Nh}$  where  $N$  is the buoyancy frequency,  $h$  is the height of the mountain or hill and  $U$  is the speed of the upstream flow) : flows contain blocked layers upstream if  $F_r \leq 1$ .

Palmer et al (1986) were not specific about the scale of gravity waves



which make the dominant contribution to the vertical momentum flux and made no allowance for trapping or partial internal reflection of wave energy. The surface drag was assumed to be dictated by the surface wind and static stability in accordance with linear theory (for an airstream with constant wind and buoyancy frequency). In practice, even the longest wavelength gravity waves experience partial internal reflection due to the height dependence of the Scorer parameter. Gravity waves forced by small scale orography (wavelengths  $< 20$  Km. ) are frequently trapped and take the form of long (  $> 50$  Km. ) lee wave trains. Studies of radiosonde data from stations in the British Isles have shown that in strong wind situations long, 'leaky' gravity waves are common with typical horizontal wavelengths in the range 15-25 Km. Numerical simulations with the Mesoscale model initialized with a single radiosonde ascent, and using a 3 Km. gridlength, have shown these long lee waves to be the dominant response and are accompanied by considerable surface drag. A very simple scheme for predicting lee wave drag profiles, suitable for operational implementation, will be discussed later.

Numerical simulations of flow over sufficiently high mountains frequently exhibit a hydraulic jump type of response with intense downslope winds beneath a deep well-mixed layer formed by wave breaking (Smith, 1985). Under these conditions the surface drag is considerably larger than estimates based on linear theory and the flow response is known as a 'high drag state'. Laboratory simulations by Rottman and Smith (1989) show the appearance of these high drag states in uniform flows with constant static stability for Froude numbers  $< 1$  . The distinctive character of the high drag state suggests that it should be accounted for in any parametrization scheme. A simple consistency model developed by Smith (1985) ( using Long's equation ) is used as a basis for determining the depth scale of wavebreaking associated with the



hydraulic jump layer.

Most current gravity wave drag parametrization schemes assume that the surface stress due to gravity wave form drag lies in the same direction as the surface wind. This is clearly not true in general as is exemplified by the case of a flow which is directed at an oblique angle to a two-dimensional mountain ridge : the drag always acts at right angles to the axis of the ridge. The scheme to be described here makes some simple assumptions about the Fourier transform spectrum of the sub-grid scale orographic height field which enable a crude representation of the anisotropy effect on the drag.

In Section 2 we discuss the various issues concerning the form of the orography. Section 3 outlines the computation of lee wave drag and a simple two-layer scheme for use in the unified model. The long wavelength, hydrostatic component of the wave field (allowing anisotropy in the orography ) will be covered in Section 4 and the treatment of high drag flow states will be outlined in Section 5. Finally, the component parts of the whole gravity wave drag parametrization scheme will be drawn together in Section 6.

## 2. The spectrum of the orographic variance

Following the mathematical development in Bretherton (1969), consider an isolated mountainous region within a rectangular region defined by  $0 \leq x \leq X$  and  $0 \leq y \leq Y$  with  $h(x,y)$ , the height of the orography, equal to zero outside of this area. Let  $\hat{h}(k,l)$  be the double Fourier transform of  $h(x,y)$  defined by:

$$\hat{h}(k,l) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) \exp -i(kx+ly) \, dx dy \quad (1)$$



with inverse given by:

$$h(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{h}(k,l) \cdot \exp i(kx+ly) \, dkdl \quad (2)$$

where  $\hat{h}(k,l) = \hat{h}^*(-k,-l)$  and  $*$  denotes the complex conjugate. Parseval's equality can then be used to get an expression for the mean variance since

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^2 \, dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{h}|^2 \, dkdl$$

or

$$\frac{1}{XY} \int_0^X \int_0^Y h^2 \, dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(k,l) \, dkdl \quad (3)$$

where  $A(k,l) = \frac{4\pi^2}{XY} |\hat{h}|^2$ .

It is clearly impractical to attempt to evaluate the functional form of  $A(k,l)$  for every grid box containing orography and even if this were possible, the computation of the trapped lee wave drag would not be well-posed (see Bretherton, 1969). The best we could hope for is some kind of ensemble average estimate of the drag over a range of orographic profiles characterised by the same values of a finite set of mean statistics (eg. the variance of the height about the mean). Bretherton suggests that  $\tilde{A}(K)$  defined by:



$$\tilde{A}(K) = \int_0^{2\pi} A(K, \phi) K d\phi \quad \text{where } k = K \cos \phi \text{ and } l = K \sin \phi,$$

may fit a power law of the form  $aK^b$  where  $-2 < b < -1$ . For the orography of North Wales he found  $b \cong -1.5$  by computing the mean of 90 one-dimensional spectra and assuming isotropy. Young and Pielke (1983) carried out a similar analysis for the mountains of West Colorado and found  $b \cong -1$  while Bannon and Yukas (1990) found  $b \cong -1.7$  for the Appalachian mountains.

In line with the above ideas, and in order to accommodate anisotropy in the representation of the orographic variance spectrum, we choose a spectrum function  $A(K, \phi)$  such that:

$$K \cdot A(K, \phi) = \left( \frac{K_0}{K} \right)^{1.5} \left[ a \cos^2 \phi + 2b \sin \phi \cos \phi + c \sin^2 \phi \right] \quad (4)$$

with  $b^2 < ac$ ,  $a > 0$  and  $c > 0$  to ensure positivity.

If the part of the wavenumber spectrum associated with the forcing of sub-grid scale gravity waves lies within the annular region defined by  $K_L < K < K_U$  then the corresponding contribution to the sub-grid scale variance  $\sigma$  is given by:

$$\sigma = \int_{K_L}^{K_U} \left( \frac{K_0}{K} \right)^{1.5} dK \cdot \int_0^{2\pi} \left( a \cos^2 \phi + 2b \sin \phi \cos \phi + c \sin^2 \phi \right) d\phi$$

which simplifies to:



$$\sigma = 2\pi K_0^{1.5} \left[ K_L^{-0.5} - K_U^{-0.5} \right] (a+c) \quad (5)$$

Now if 
$$\frac{1}{4\pi^2} \int_0^X \int_0^Y \left( \frac{\partial h}{\partial x} \right)^2 dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k^2 |\hat{h}|^2 dk dl$$

define its contribution within the wavenumber band  $K_L$  and  $K_U$  to be

$\sigma_{xx}$  so that, in polar form:

$$\sigma_{xx} = \int_{K_L}^{K_U} \int_0^{2\pi} K^2 \cos^2 \phi \cdot A(K, \phi) K dK d\phi$$

$$= \frac{\pi}{6} K_0^{1.5} \left[ K_U^{1.5} - K_L^{1.5} \right] (3a+c) \quad (6)$$

Similarly, we may define  $\sigma_{xy}$  and  $\sigma_{yy}$  by:

$$\sigma_{xy} = \int_{K_L}^{K_U} \int_0^{2\pi} K^2 \sin \phi \cos \phi \cdot A(K, \phi) K dK d\phi$$

$$\sigma_{yy} = \int_{K_L}^{K_U} \int_0^{2\pi} K^2 \sin^2 \phi \cdot A(K, \phi) K dK d\phi$$

which, on evaluating the integrals, give:



$$\sigma_{yy} = \frac{\pi}{6} K_0^{1.5} \left[ K_U^{1.5} - K_L^{1.5} \right] (a+3c) \quad (7)$$

and 
$$\sigma_{xy} = \frac{\pi}{3} K_0^{1.5} \left[ K_U^{1.5} - K_L^{1.5} \right] .b \quad (8).$$

Equations (6),(7) and (8) can be used to determine the constants a,b and c following the direct evaluation of  $\sigma_{xx}$ ,  $\sigma_{xy}$  and  $\sigma_{yy}$  from the gridded height field.

The parameters  $\sigma_{xx}$ ,  $\sigma_{xy}$  and  $\sigma_{yy}$  will be used in the computation of the hydrostatic gravity wave stress computation and the power law form of the spectrum function is used later to assign wavelength-dependent amplitudes to trapped lee waves.

### 3. Parametrization of the trapped lee wave contribution

#### (a) Theoretical development

Bretherton (1969) provides a thorough analysis of the computation of lee wave drag ( the 'line spectrum' in his terminology) and makes various cautionary remarks concerning the use of his surface drag formula in real situations. Even if nonlinear effects could be ignored, the spiky form of the spectrum function  $A(K,\phi)$ , and its convolution with the line spectrum response function, introduces indeterminacy into the calculation of the drag. Very small changes in the flow could cause dramatic changes in the lee wave field and drag. However, the fact that lee wave patterns are coherent for hundreds of kilometres parallel to complex orographic ridges suggests that this sensitivity is not real - nonlinearity may be increasing the



predictability of the phenomenon. In spite of these reservations, it is likely that linear theory captures the important physical factors influencing the size of the trapped lee wave drag and will be sufficiently accurate to warrant using in a parametrization scheme.

Firstly, define Fourier transforms for the vertical velocity  $w(x,y,z)$  according to:

$$w(x,y,z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{w}(k,l,z) \exp i(kx+ly) dkdl \quad (9)$$

where  $\hat{w}(k,l,z)$  satisfies the equation:

$$\frac{\partial^2 \hat{w}}{\partial z^2} + (l_s^2 - K^2) \hat{w} = 0 \quad (10)$$

and  $l_s(\phi,z)$  is the Scorer parameter given by:

$$l_s^2 = \frac{N^2}{U_n^2} - \frac{1}{U_n} \frac{\partial^2 U_n}{\partial z^2} \quad (11)$$

where  $U_n(\phi,z)$  is the component of the basic state wind in the direction of  $(k,l)$  (ie. at an angle  $\phi$  measured anticlockwise from the x axis). If  $\hat{u}(k,l,z)$  and  $\hat{v}(k,l,z)$  are the Fourier transforms of the x and y components of the horizontal wind vector ( $u(x,y,z)$  and  $v(x,y,z)$  respectively) then the incompressible form of the continuity equation



$$(11) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

implies that

$$iku + ilv + \frac{\partial \hat{w}}{\partial z} = 0 \quad (12).$$

Since the disturbance wind vector associated with each harmonic component of the transform is parallel to the wave vector,  $\hat{u}$  and  $\hat{v}$  are related according to  $\hat{u}l - \hat{v}k = 0$  and it can be shown that:

$$\hat{u} = \frac{i \cos \phi}{K} \cdot \frac{\partial \hat{w}}{\partial z} \quad (13).$$

Multiplying the transform expressions for  $u$  and  $w$  then integrating over the entire  $(x,y)$  plane can be shown to give:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uw \, dx dy = 4\pi^2 i \int_0^{\infty} \int_0^{2\pi} \frac{\cos \phi}{K} \frac{\partial \hat{w}}{\partial z} \hat{w}^* K \, dK d\phi$$

which is equivalent to:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uw \, dx dy = 4\pi^2 i \int_0^{\infty} \int_{-\pi/2}^{\pi/2} \left( \frac{\partial \hat{w}}{\partial z} \hat{w}^* - \frac{\partial \hat{w}^*}{\partial z} \hat{w} \right) \cos \phi \, dK d\phi \quad (14).$$

Using Bretherton's notation, we introduce a response function  $F(K, \phi)$  defined by:



$$F(K, \phi) = \left( \frac{\partial \hat{w}}{\partial z} \hat{w}^* - \frac{\partial \hat{w}^*}{\partial z} \hat{w} \right) / 2i \cdot |\hat{w}(K, \phi, 0)|^2 \quad (15)$$

where, in accordance with the lower boundary condition,

$$\hat{w}(K, \phi, 0) = iKU_0 \cos(\phi - \chi) \hat{h}(K, \phi)$$

and where  $U_0$  and  $\chi$  are the surface wind speed and direction respectively. The net momentum flux in the x direction averaged over the rectangular mountainous region is then given by:

$$\frac{1}{XY} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uw \, dx dy = -\frac{8\pi^2}{XY} U_0^2 \int_0^{\infty} \int_{-\pi/2}^{\pi/2} \hat{h}^2 \cdot F(K, \phi) \cos^2(\phi - \chi) \cdot \cos \phi \, K^2 \, dK d\phi$$

(16) .

Similarly, it can be shown that the corresponding mean y component of momentum flux is given by:

$$\frac{1}{XY} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} vw \, dx dy = -\frac{8\pi^2}{XY} U_0^2 \int_0^{\infty} \int_{-\pi/2}^{\pi/2} \hat{h}^2 \cdot F(K, \phi) \cos^2(\phi - \chi) \cdot \sin \phi \, K^2 \, dK d\phi$$

(17) .

Trapped lee waves occur for values of  $K$  and  $\phi$  which cause  $\hat{w}(K, \phi, 0)$  to become exceedingly small or zero : this leads to a pronounced spike in  $F(K, \phi)$ . Bretherton's derivation of this singular contribution to integrals



in eqs. (17) and (18) is too lengthy to reproduce here. He effectively shows that in the limit of no energy leakage out of the assumed trapped layer, the response function  $F(K, \phi)$  tends to:

$$\sum_{i=1}^I - \frac{\pi \left| \frac{\partial \hat{w}}{\partial z} \right|_{K=K_*^i, z=0}^2}{2K \int_0^\infty \left| \hat{w}(K_*^i, \phi, z) \right|^2 dz} \cdot \delta \left[ K - K_*^i(\phi) \right] \quad (18)$$

where  $K = K_*^i(\phi)$  for  $\phi_A^i < \phi < \phi_B^i$ ,  $i=1, I$  describes a set of arcs in the wavenumber plane along which trapped lee waves occur and  $\delta(\ )$  is the Dirac delta function. Note that the endpoints at  $\phi_A^i$  and  $\phi_B^i$  will not, in general, be precisely defined. Substituting (18) for  $F(K, \phi)$  in eqs. (16) and (17) gives the following combined expression for the mean vertical momentum flux due to trapped lee waves:

$$\frac{1}{XY} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{pmatrix} uw \\ vw \end{pmatrix} dx dy = \pi U_0^2 \sum_{i=1}^I \int_{\phi_A^i}^{\phi_B^i} \frac{K_*^i A(K_*^i, \phi) \left| \frac{\partial \hat{w}}{\partial z} \right|_{K=K_*^i, z=0}^2}{\int_0^\infty \left| \hat{w}(K_*^i, \phi, z) \right|^2 dz} \cos^2(\phi - \chi) \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} d\phi \quad (19)$$

using the definition of the spectrum function  $A(K, \phi)$  quoted earlier.

Eq. (19) should be compared to eq. (61) of Bretherton's paper: it appears that the factor of  $K_*^i$  appearing within the integral above is missing from Bretherton's analysis. This seems to be an error in his paper since eq. (19) is dimensionally correct.

In view of its rather fearsome nature, eq. (19) will require considerable simplification if it is to be used in a parametrization scheme. The first



thing we can do is remove the summation over all I resonance branches and select only the first internal wave mode ( ie. the one with no nodes ). This seems intuitively reasonable and is supported by observations such as those of Brown (1983). The next (rather brutal) step is to assume, in accordance with observations of lee waves, that the dominant contribution to the integral will come from wavevectors of similar orientation as the surface wind and remove the  $\phi$  dependence in the integrand of eq.(19) by setting  $\phi=\chi$ . The integrand will be assumed zero if  $\chi-\Delta\phi/2 < \phi < \chi+\Delta\phi/2$ . With these assumptions eq.(19) reduces to:

$$\frac{1}{XY} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{pmatrix} uw \\ vw \end{pmatrix} dx dy = \pi U_0^2 \frac{K_*(\chi) \cdot A(K_*, \chi) \left| \frac{\partial \hat{w}}{\partial z} \right|_{K=K_*(\chi), z=0}^2}{\int_0^{\infty} \hat{w}(K_*(\chi), \chi, z)^2 dz} \begin{pmatrix} \cos \chi \\ \sin \chi \end{pmatrix} \Delta\phi \quad (20)$$

(dropping the index i) where  $\Delta\phi$  is to be chosen arbitrarily and will ultimately form part of a 'tuning' constant. Before eq.(20) can be put to any practical use, it is necessary to solve an eigenvalue problem for  $\hat{w}$ . Since this would have to be solved for all operational model gridpoints with orography it must be extremely simple and economical on computer time.

#### (b) A variable depth two-layer model

A minimum requirement for the representation of trapped lee waves is to have two layers of different Scorer parameter ( or less realistically, a single layer bounded from above by a lid ). The difficulty with such a formulation is that the optimum position for the interface between the two layers is dependent on the vertical profile of wind and temperature. The approach taken here is to allow the thickness of the lowest layer to be an



unknown, at the expense of imposing a constraint on the phase difference of the lee wave across the layer.

Assume linear wave dynamics with the lower layer (1) extending between  $z=0$  and  $H$  and the upper layer (2) extending between  $z=H$  and  $2H$ . The Scorer parameter  $l_i(H)$  ( $i=1,2$ ) in each layer is assumed to be constant and will be taken as equal to the layer mean when applied to real profiles. It is, therefore, a function of  $H$ . The eigenvalue problem is :

$$\frac{\partial^2}{\partial z^2} \hat{w}_i + (l_i^2 - K_*^2) \hat{w}_i = 0 \quad i = 1, 2 \quad (22)$$

subject to  $\hat{w}_i(K_*, 0) = 0$  and with an evanescent solution in layer 2 (NB. it is convenient to revert to the usual two-dimensional notation here so that  $K_*$  should be understood to be the wavenumber component in the direction of the surface wind and  $l_i$  is evaluated taking the wind component in that direction). Lee wave modes can only exist if  $l_2^2 < K_*^2 < l_1^2$  and under these circumstances solutions will be of the form:

$$\hat{w}_1 = C \exp[-m_2 H] \cdot \sin m_1 z \quad (23)$$

$$\hat{w}_2 = C \sin(m_1 H) \cdot \exp[-m_2 z] \quad (24)$$

with  $m_1^2 = l_1^2 - K_*^2 > 0 \quad (25)$

and  $m_2^2 = K_*^2 - l_2^2 > 0, \quad (26)$

and where  $C$  is an arbitrary constant and continuity of vertical velocity has been assumed. This would not be true if the basic state wind was discontinuous



at the interface but since it is the Scorer parameter which is discontinuous, no inconsistency is implied. Continuity of disturbance pressure and basic state wind across the interface at  $z=H$ , together with the horizontal momentum equation, implies continuity of horizontal wind perturbation. Furthermore, the continuity equation

$$iK_* \hat{u} + \frac{\partial \hat{w}}{\partial z} = 0$$

requires that  $\frac{\partial \hat{w}}{\partial z}$  be continuous across the interface which forces  $m_1$  and  $m_2$  to be related by:

$$\tan(m_1 H) = - \frac{m_1}{m_2} \quad (27).$$

At this point, we introduce the phase difference constraint referred to earlier by choosing a fixed value for  $m_1 H$ . Since most observations of lee waves reveal a single amplitude maximum (eg. Brown, 1983),  $m_1 H$  should lie between  $\frac{\pi}{2}$  and  $\pi$ .

Defining  $\alpha$  such that  $\tan(m_1 H) = \alpha$ , then eqs.(25),(26) and (27) can be used to obtain the following expression for  $K_*$ :

$$K_*^2 = \frac{l_1^2 + \alpha^2 l_2^2}{1 + \alpha^2} \quad (28).$$

The lee wavelength ( $2\pi/K_*$ ) is therefore readily calculated if  $H$  is known. Using this expression for  $K_*$  in eq.(25) gives:



$$m_1^2 = \alpha^2 \left( l_1^2 - l_2^2 \right) \left( 1 + \alpha^2 \right)^{-1} \quad (29)$$

which can be expressed as an equation for H on using the definition of  $\alpha$ :

$$\frac{\alpha \left( l_1^2(H) - l_2^2(H) \right)^{1/2} H}{\left( 1 + \alpha^2 \right)^{1/2} \cdot \text{Tan}^{-1}(\alpha)} + 1 = 0 \quad (30)$$

To implement this simple model to find the wavelength and amplitude structure of lee waves, the function on the left hand side of eq.(30) is evaluated over increasing H (corresponding to progressively larger sets of operational forecast model levels being used to define the upper and lower layers) until it changes sign. H is then found by interpolation and the appropriate values of  $l_1(H)$  and  $l_2(H)$  are substituted into eq.(28). Eq.(27) and the chosen value of  $\alpha$  (implied by the phase difference constraint) give  $m_1$  and  $m_2$  which in turn determine the amplitude profile in eqs.(23) and (24).

We now have all the information required to evaluate the lee wave drag formula eq.(20). Substituting eqs.(23) and (24), performing the integral and rearranging leads ultimately to the following expression for the lee wave drag:

$$\frac{1}{XY} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{pmatrix} uw \\ vw \end{pmatrix} dx dy = \frac{\pi U_0^2 K_* A(K_*, \chi) \Delta \phi}{\bar{z}^3} \begin{pmatrix} \cos \chi \\ \sin \chi \end{pmatrix} \quad (31)$$



(cf. Smith, 1976) where

$$\bar{z} = \frac{H}{\gamma} \left( \frac{\gamma - 1}{2\alpha^2} \right)^{1/3} \quad (32)$$

and  $\gamma = \alpha^{-1} \tan^{-1}(\alpha)$ . In tests of this two-layer model with radiosonde data, good results were obtained when  $\alpha$  was set equal to  $\tan(0.6\pi)$  implying a lee wave amplitude peak at  $z = 5H/6$ . Differences between the lee wavelength (of the first internal mode) calculated with this model, and a solution of the wave equation using 100 levels, can be surmised from Table 1 together with the corresponding heights of the lee wave amplitude maxima.

TABLE 1.

Date	Sonde Station	2 - Layer model		100 level calculation	
		$l_x$ (Km)	$h_*$ (Km)	$l_x$ (Km)	$h_*$ (Km)
6/10/89	Caersws	17.7	3.9	21.8	4.1
12/10/89	Aberporth	3.5	1.2	4.6	1.0
18/10/89	"	6.3	1.8	5.6	1.8
19/10/89	"	9.7	3.1	9.8	2.3

The overall agreement suggests that the two-layer model will be adequate for the purposes of parametrization. The 'error' resulting from the use of the two-layer model is comparable with the expected discrepancy between linear theory and observations. Eq.(31) is probably the simplest expression arising from linear theory that could be used to estimate the lee wave drag. Nevertheless, it has to be admitted that we have no idea of its accuracy. Of



course in a parametrization scheme one would only expect it to be correct in some statistical average sense. It may turn out to be more expedient to substitute the whole trapped lee wave algorithm with a semi-empirical expression as do ECMWF.

(c) Distribution of stress with height

A general principle of wave/mean flow interaction theory requires that there be some wave transience or dissipation before the basic flow is forced by the waves. A permanent modification to the flow accompanies wave dissipation. Bretherton (1969) supposed that turbulence within thin layers might ultimately dissipate lee wave energy. Unfortunately, the cause of lee wave dissipation is still largely unknown. It may be that the waves can persist long enough for the synoptic flow to change sufficiently to allow them to radiate vertically. Under these circumstances the wave drag profile would resemble that associated with long hydrostatic waves but would be displaced downstream through horizontal ducting in the trapped lee wave phase. Since the horizontal group speed of lee waves is typically of the order of half the wind speed it takes about 5 to 10 hours to set up a 200 Km long lee wave train. On this time scale it is conceivable that boundary layer dissipation might damp the waves.

Bretherton showed that an infinite trapped lee wave train has a characteristic vertical momentum flux profile when integrated from far upstream to a point far downstream. For expediency of parametrization, it will be assumed that the forcing of the mean flow follows the vertical gradient of this profile. A brief resumé of Bretherton's formula for the momentum flux profile associated with lee waves will now be given and will then be



interpreted in terms of the two-layer model.

The wave equation for two-dimensional gravity wave may be written in terms of a perturbation streamfunction  $\psi$  giving:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + l^2(z) \cdot \psi = 0 \quad (33)$$

Now

$$\frac{\partial}{\partial z} \left[ \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial z} \right] = \frac{\partial^2 \psi}{\partial x \partial z} \cdot \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x} \left[ \frac{\partial^2 \psi}{\partial x^2} + l^2 \psi \right]$$

or  $-\frac{\partial}{\partial z} (uw) = \frac{1}{2} \frac{\partial}{\partial x} \left[ \left( \frac{\partial \psi}{\partial z} \right)^2 - \left( \frac{\partial \psi}{\partial x} \right)^2 - l^2 \psi^2 \right] \quad (34).$

Integrating eq.(34) with respect to  $z$  from a height  $z_*$  to  $\infty$ , and then again with respect to  $x$  from  $-\infty$  to  $x_*$  gives:

$$\int_{-\infty}^{x_*} uw|_{z=z_*} dx = \frac{1}{2} \int_{z_*}^{\infty} \left[ \left( \frac{\partial \psi}{\partial z} \right)^2 - \left( \frac{\partial \psi}{\partial x} \right)^2 - l^2 \psi^2 \right]_{x=x_*} dz \quad (35)$$

assuming that the perturbation streamfunction is zero at  $x = -\infty$ . Now if  $\psi_0(z)$  is the eigenfunction obtained from the vertical structure equation for the lee wave mode ( eq.(22) with  $\psi_0(0) = \psi_0(\infty) = 0$  ) and

$$\psi = \psi_0(z) \cos kx \quad (36)$$

then substituting eq.(36) into eq.(35) gives:



$$\int_{-\infty}^{x_*} uw|_{z=z_*} dx = - \frac{1}{2} \left[ \psi_0 \frac{d\psi_0}{dz} \Big|_{z=z_*} \cdot \cos^2 kx - k^2 \int_{z_*}^{\infty} \psi_0^2 dz \right] \quad (37).$$

If  $w = W_0(z) \sin kx$ , then  $\psi_0 = -W_0 / k$  and averaging eq.(37) over a wavelength yields:

$$\overline{uw} \Big|_{z=z_*} = - \frac{1}{4} \frac{W_0}{k^2} \frac{dW_0}{dz} \Big|_{z=z_*} - \frac{1}{2} \int_{z_*}^{\infty} W_0^2 dz \quad (38).$$

Note the interesting fact that the surface momentum flux is equal to twice the mean kinetic energy per unit area of the lee wave train. The reader is advised to consult Bretherton's paper for an interpretation of the oscillatory contribution to the momentum flux in eq.(37).

It is now a purely algebraic task to evaluate the momentum flux profile for the two-layer model by substituting eqs.(23) and (24) in eq.(38). From these it can be shown that:

$$C^2 = - \frac{4 \overline{uw} \Big|_{z=0} \exp(2m_2 H)}{H - \frac{\sin 2m_1 H}{2m_1} + \frac{\sin^2 m_1 H}{m_2}} \quad (39)$$

( where  $\overline{uw} \Big|_{z=0}$  is obtainable from eq.(31) )

$$\overline{uw} = - \frac{C^2}{4} e^{-2m_2 H} \left\{ H - z - \frac{\sin 2m_1 H}{2m_1} + \frac{l_1^2 \sin 2m_1 z}{2m_1 k^2} + \frac{\sin^2 m_1 H}{m_2} \right\}$$

for  $z < H$



(40)

and

$$\overline{uw} = -\frac{C^2}{4} \sin^2 m_1 H \exp(-2m_2 z) \frac{l_2^2}{k^2 m_2} \quad (41)$$

for  $z > H$ .

#### 4. Hydrostatic long wave drag with anisotropic orography

The original gravity wave drag parametrization scheme of Palmer et al (1986) assumed that most of the vertical momentum flux was carried by waves sufficiently long that  $k^2 \ll l_s^2$  and ignored the possibility of partial internal reflection caused by  $l_s$  variations with height. This makes the surface drag dependent on surface flow parameters only (ie. wind and buoyancy frequency) and greatly simplifies scheme. The surface drag was also assumed to be proportional to the sub-grid scale orographic height variance ignoring anisotropy associated with the organization of ridges and valleys in a preferred direction. A revised treatment of the drag due to these long waves which crudely incorporates these effects will now be presented.

Consider the case of an airstream with constant wind speed ( $U_0$ ), direction ( $\chi$ ) and buoyancy frequency ( $N$ ) flowing over irregular terrain characterised by the variance spectrum function  $A(K, \phi)$  given by eq.(4). The problem posed here is to obtain an expression for the wave drag due to wave vectors lying within the annulus defined by  $K_L < K < K_U$  in the wavenumber plane. The upper bound  $K_U$  could be viewed as the highest wavenumber lying outside the range in which trapped lee waves occur ( corresponding, say, to a



wavelength of  $\approx 25$  Km. ). The lower bound  $K_L$  represents the lowest unresolved wavenumber in a numerical forecast model. The component of total drag in the x direction is given, therefore, by the integral:

$$\frac{1}{XY} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uw \, dx dy = -2U_0^2 \int_{K_L}^{K_U} \int_{-\pi/2}^{\pi/2} KA(K, \phi) \cdot F(K, \phi) \cdot \cos^2(\phi - \chi) \cos \phi \, K dK d\phi \quad (42)$$

$$\text{with } KA(K, \phi) = \left( \frac{K_0}{K} \right)^{3/2} \cdot \left[ a \cos^2 \phi + 2b \sin \phi \cos \phi + c \sin^2 \phi \right].$$

Solutions to the wave equation ( eq.(10) ) under these conditions take the form:

$$\hat{w}(K, \phi, z) = \text{const.} \exp \left[ i K (x \cos \phi + y \sin \phi) + i l_s z \right]$$

$$\text{with } l_s = \frac{N}{U_0 \cos(\phi - \chi)}$$

Using eq.(15),  $F(K, \phi)$  is easily shown to be equal to  $l_s$ . With some algebraic manipulation, eq.(42) can be reduced to:

$$\frac{1}{XY} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uw \, dx dy = -\frac{\pi U_0 N K_0^{3/2}}{2} \left( K_U^{1/2} - K_L^{1/2} \right) (3a \cos \chi + 2b \sin \chi + c \cos \chi) \quad (43).$$

Similarly, the y component of the total vertical momentum flux is given by:



$$\frac{1}{XY} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} vw \, dx dy = -\frac{\pi U_0 N K_0^{3/2}}{2} \left( K_U^{1/2} - K_L^{1/2} \right) (a \sin \chi + 2b \cos \chi + 3c \sin \chi) \quad (44).$$

Now with the aid of eqs.(6),(7) and (8), the above equations may be written in the simpler forms:

$$\frac{1}{XY} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uw \, dx dy = \frac{U_0 N}{\hat{K}} \left( \sigma_{xx} \cos \chi + \sigma_{xy} \sin \chi \right) \quad (45)$$

$$\frac{1}{XY} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} vw \, dx dy = \frac{U_0 N}{\hat{K}} \left( \sigma_{xy} \cos \chi + \sigma_{yy} \sin \chi \right) \quad (46)$$

where 
$$\hat{K} = \frac{1}{3} \left\{ \frac{K_U^{3/2} - K_L^{3/2}}{K_U^{1/2} - K_L^{1/2}} \right\}.$$

Eqs.(45) and (46) are a simple generalization of the surface stress formulae used in the Palmer et al scheme with the orographic variance being replaced by combinations of  $\sigma_{xx}$ ,  $\sigma_{xy}$  and  $\sigma_{yy}$ .

The above drag formulae are for an airstream with uniform Scorer parameter - no partial internal reflection occurs. When a high degree of trapping exists then the analysis of Section 3 is applicable. For hydrostatic waves, trapping can only occur if the Scorer parameter approaches zero over a layer of considerable depth. Even in situations when lee waves are observed



there may well be a sizable contribution to the vertical momentum flux from the hydrostatic waves.

In order to allow for the effect of partial internal reflection, the following simple two-layer analysis provides a *transmission factor* which can be applied to the drag computed from eqs.(45) and (46). Solutions to eq.(22) with  $k = 0$  can be written in the form:

$$\hat{w}_1(\phi, z) = A_1 \exp[il_1(z-H)] + B_1 \exp[-il_1(z-H)] \quad (47)$$

$$\hat{w}_2(\phi, z) = A_2 \exp[il_2(z-H)] \quad (48)$$

where only the upward energy propagating mode has been selected for the upper layer. As before, continuity of  $\hat{w}$  and  $\frac{\partial \hat{w}}{\partial z}$  will be enforced at  $z=H$  but in this case  $\hat{w}_1(\phi, 0)$  is non-zero. A transmission factor ( $T$ ) can be defined by forming the ratio of the vertical momentum flux derived from eqs.(47) and (48) with the same quantity when the upper layer is absent (or has identical Scorer parameter to the lower layer) and is given by:

$$T = \frac{\hat{w}_2^* \frac{\partial \hat{w}_2}{\partial z}}{\hat{w}_2^* \frac{\partial \hat{w}_2}{\partial z} \Big|_{l_2=l_1}} = \frac{2l_1 l_2}{l_1^2 + l_2^2 + (l_1^2 - l_2^2) \cos 2l_1 H} \quad (49)$$

after some algebra.

## 5. Flow blocking and high drag states

One of the most unsatisfactory features of the Palmer et al



parametrization scheme is the need to limit the orographic variance to  $400 \text{ m}^2$  when it exceeds this value in the raw data. This was expedient at the time when interest centred on the correction of gross systematic errors in the operational forecast model : meanwhile other operational centres never implemented this blatant fix.

Whilst it was realised at the time that Froude number dependent terms were needed to circumvent the 'variance chop', some uncertainty surrounded the whole issue of wavebreaking just above the mountain tops under these conditions. It was felt that simply re-defining the orographic variance when the Froude number fell below unity ( to account for the reduction in effective terrain height due to flow blocking ) might miss out some important low-level wavebreaking effects. Since then, a number of papers have been published which have an important bearing on this problem. Numerical simulation and laboratory experiments have clearly shown that a distinctive hydraulic jump type of flow response sets in when the Froude number approaches unity from above (eg. Clark and Peltier, 1977; Peltier and Clark, 1979; Bacmeister and Pierrehumbert, 1988; Rottman and Smith, 1989 ). This has become known as the 'high drag state'. As the Froude number decreases below unity, flow blocking may accompany the hydraulic jump response. Considerable attention has been concentrated on the interpretation of Clark and Peltier's simulations. Without going into the details of the controversy regarding the mechanism by which the high drag state is achieved, suffice it to say that:

(i) wavebreaking is a key ingredient in setting up the hydraulic jump response

(ii) a consistency model put forward by Smith (1985) seems to



successfully relate the important physical parameters.

Not only is it possible to avoid the artificial limitation on the size of the orographic variance, Smith's model also gives a drag law which differs from that originating from linear theory ( or a nonlinear solution of the 'Long' type ) and predicts the depth over which the hydraulic jump state will occur. An expression for the non-dimensional depth of the hydraulic jump layer  $\hat{H}_0$  ( the upstream height of the dividing streamline in his terminology ) is obtained in terms of the non-dimensional mountain height  $\hat{h}_m$ , for an upstream flow with constant wind and buoyancy frequency ( Smith and Sun, 1987 ). This can be written, following Rottman and Smith (1989), as:

$$\hat{H}_0 = \hat{h}_m - \hat{\delta}_m + \cos^{-1}(\hat{h}_m / \hat{\delta}_m) + 2\pi n \quad 0 < \hat{h}_m < 0.985 \quad (50)$$

$$\hat{H}_0 = \frac{3\pi}{2} + \hat{d} + 2\pi n \quad \hat{h}_m > 0.985 \quad (51)$$

where  $n = 0, 1, 2, \dots$ ,

$$\hat{\delta}_m = - \frac{1}{2^{1/2}} \left[ \hat{h}_m^2 + \hat{h}_m (\hat{h}_m^2 + 4)^{1/2} \right]^{1/2} \quad (52)$$

$$\text{and the depth of the blocked layer } \hat{d} = \hat{h}_m - 0.985 \quad (53).$$

The vertical scale used in the non-dimensionalization is  $U_0 / N$  and so  $\hat{h}_m$  is equal to the inverse Froude number. The hydraulic jump state is only relevant when wavebreaking is possible which according to Miles and Huppert



(1969) sets in when  $\hat{h}_m > 0.85$ . The laboratory experiments of Rottman and Smith (1989) showed that eq.(51) works well when  $n=0$  : in dimensional terms this is given by:

$$H_0 = \left[ \frac{3\pi}{2} - 0.985 \right] \frac{U_0}{N} + h_m \quad (53)$$

provided that  $Nh_m / U_0 > 0.985$ .

Smith's theory also provides a simple expression for the drag (D) on an isolated two-dimensional obstacle in the high drag regime when  $\hat{h}_m < 0.985$ :

$$D = \rho \left( \hat{H}_0 - \frac{\pi}{2} \right)^3 \cdot \frac{U_0^3}{6N} \quad (54)$$

Unfortunately this formula is not strictly applicable when flow blocking sets in, and even if it was, its use in the parametrization problem would be unjustified since the terrain height is highly variable within a model grid square. It is interesting to note, however, that the  $U^3 / N$  factor in eq.(54) also appears in the expression for the saturation wave stress defined in Palmer et al (1986). In the context of the Richardson number wavebreaking scheme, it would be logical to define the surface stress to be equal to the saturation stress for cases where the magnitude of the stress computed from eqs.(45) and (46) exceeds the saturation value. When  $\rho$ ,  $U_0$  and  $N$  are constant the Richardson number scheme would then require the wave stress to be independent of height - in marked conflict with the hydraulic jump theory and results from numerical and laboratory experiments where the wave stress



decreases to a fraction of its surface value at  $z=H_0$ . Bearing in mind the above remarks, some semi-empirical formulae will now be proposed which cover the hydrostatic regime and imply eqs.(45) and (46) at large Froude number. Let the components of the surface stress vector  $(\tau_{sx}, \tau_{sy})$  be given by:

$$\left. \begin{aligned} \tau_{sx} &= \rho \hat{K}^{-1} \left( U_0^3 / N \right) \left\{ \frac{\sigma_{xx}}{\sigma} \cos \chi + \frac{\sigma_{xy}}{\sigma} \sin \chi \right\} \hat{h}_1 \hat{h}_2 \\ \tau_{sy} &= \rho \hat{K}^{-1} \left( U_0^3 / N \right) \left\{ \frac{\sigma_{xy}}{\sigma} \cos \chi + \frac{\sigma_{yy}}{\sigma} \sin \chi \right\} \hat{h}_1 \hat{h}_2 \end{aligned} \right\} \quad (55)$$

where

$$\hat{h}_1 = \begin{cases} \frac{N \sigma^{1/2}}{\alpha U_0} & \text{if } < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$\hat{h}_2 = \begin{cases} \frac{N \sigma^{1/2}}{\beta U_0} & \text{if } < 1 \\ 1 & \text{otherwise} \end{cases}$$

and  $\alpha$  and  $\beta$  are constants to be chosen. The first thing to notice about the above expressions is that they reduce to eqs.(45) and (46) if  $\sigma^{1/2} < \frac{\alpha U_0}{N}$  and the drag is proportional to the wind speed. If  $\frac{\alpha U_0}{N} < \sigma^{1/2} < \frac{\beta U_0}{N}$  then the drag follows a square law in  $U_0$  and for  $\sigma^{1/2} > \frac{\beta U_0}{N}$  a cubic law is followed consistent with Smith's formula. Alternatively, for the same basic flow, the drag is proportional to the variance for  $\sigma^{1/2} < \frac{\alpha U_0}{N}$ ; the standard deviation in the intermediate regime, and is independent of the orographic heights when  $\sigma^{1/2} > \frac{\beta U_0}{N}$ . This formulation builds in the idea that the



surface stress asymptotes to a constant value for sufficiently large orographic variance. Preliminary experimentation with the constants  $\alpha$  and  $\beta$  suggests that  $\alpha = 0.3$  and  $\beta = 1$  may be appropriate. Since  $U_0/N$  is typically of the order of 1 Km, this choice of  $\alpha$  and  $\beta$  suggests that the linear drag law is valid for orographic height standard deviations of up to 300 m : constancy of the drag occurs for the extreme cases where  $\sigma^{1/2} > 1$  Km.

The next important step is to decide on a simple wave stress saturation law like that of Palmer et al (1986). Eqs.(55) could equally well be interpreted as an expression for the wave stress components  $\tau_x$  and  $\tau_y$  at any height provided that  $\sigma$ ,  $\sigma_{xx}$ ,  $\sigma_{xy}$  and  $\sigma_{yy}$  are imagined to relate to the height of an isentropic surface. We shall define saturation of the wave stress to occur when  $\sigma^{1/2} = \frac{\alpha U_0}{N}$  so that

$$\tau_x^* = \rho \hat{K}^{-1} \left( U_0^3 / N \right) \left\{ \frac{\sigma_{xx}}{\sigma} \cos \chi + \frac{\sigma_{xy}}{\sigma} \sin \chi \right\} \left( \frac{\alpha}{\beta} \right) \quad (56)$$

$$\tau_y^* = \rho \hat{K}^{-1} \left( U_0^3 / N \right) \left\{ \frac{\sigma_{xy}}{\sigma} \cos \chi + \frac{\sigma_{yy}}{\sigma} \sin \chi \right\} \left( \frac{\alpha}{\beta} \right)$$

where  $\tau_x^*$  and  $\tau_y^*$  are the x and y components of the saturation stress. For a single, plane harmonic gravity wave, convective instability (and therefore wave saturation) is identified with the point at which the vertical displacement amplitude equals  $\frac{U_0}{N}$ . Since we are attempting to represent the effect of an ensemble of waves, the corresponding condition on  $\sigma^{1/2}$  should reflect the difference between the rms vertical displacement and the largest vertical displacements in the isentropic surface. As regards the



vertical distribution of the wave stress with height, it will be assumed, in the absence of any observational guidance, that the stress falls linearly with height in the layer below  $H_0$  to a fraction (  $1/3$  ? ) of its surface value at  $z = H_0$ .

#### 6. Proposed form for the new parametrization scheme

Before discussing a possible flow diagram through which the parametrization scheme could be implemented, a few minor topics need to be addressed. Firstly, the vertical distribution of wave stress associated with the long, hydrostatic waves will be assumed to be governed by the saturation hypothesis as in Palmer et al (1986) though with the saturation stress given by eqs.(56). The wave stress (  $\tau$  ) at any level is assumed to be equal to that of the level immediately below : if  $\tau$  is found to be greater than  $\tau_*$  at any level then it is set equal to  $\tau_*$ . When a critical level is encountered (  $U$  passes through zero ) all of the wave stress will be assumed to be absorbed.

The second point of concern relates to the meaning of 'surface wind' and static stability used in the surface stress formulae and Froude number. Since the wind and potential temperature profiles are highly variable in the boundary layer, there is a high level of indeterminacy in the choice of suitable values appropriate to the excitation of gravity waves. It would seem reasonable to assume that the relevant surface wind and static stability are dependent on the height of the sub-grid scale orography. For instance, one could choose the surface wind to be the model wind interpolated to a height (above the mean orography) equal to  $\sigma^{1/2}$ . Similarly, one could base the static stability on the difference in potential temperature between that at a height  $\sigma^{1/2}$  above the surface and the lowest model level.



The overall structure of the proposed parametrization scheme is represented by the flow diagram in Fig. 1.

point in the layer below  $H_0$  to a fraction  $(1/3)^2$  of its surface value at  $z$

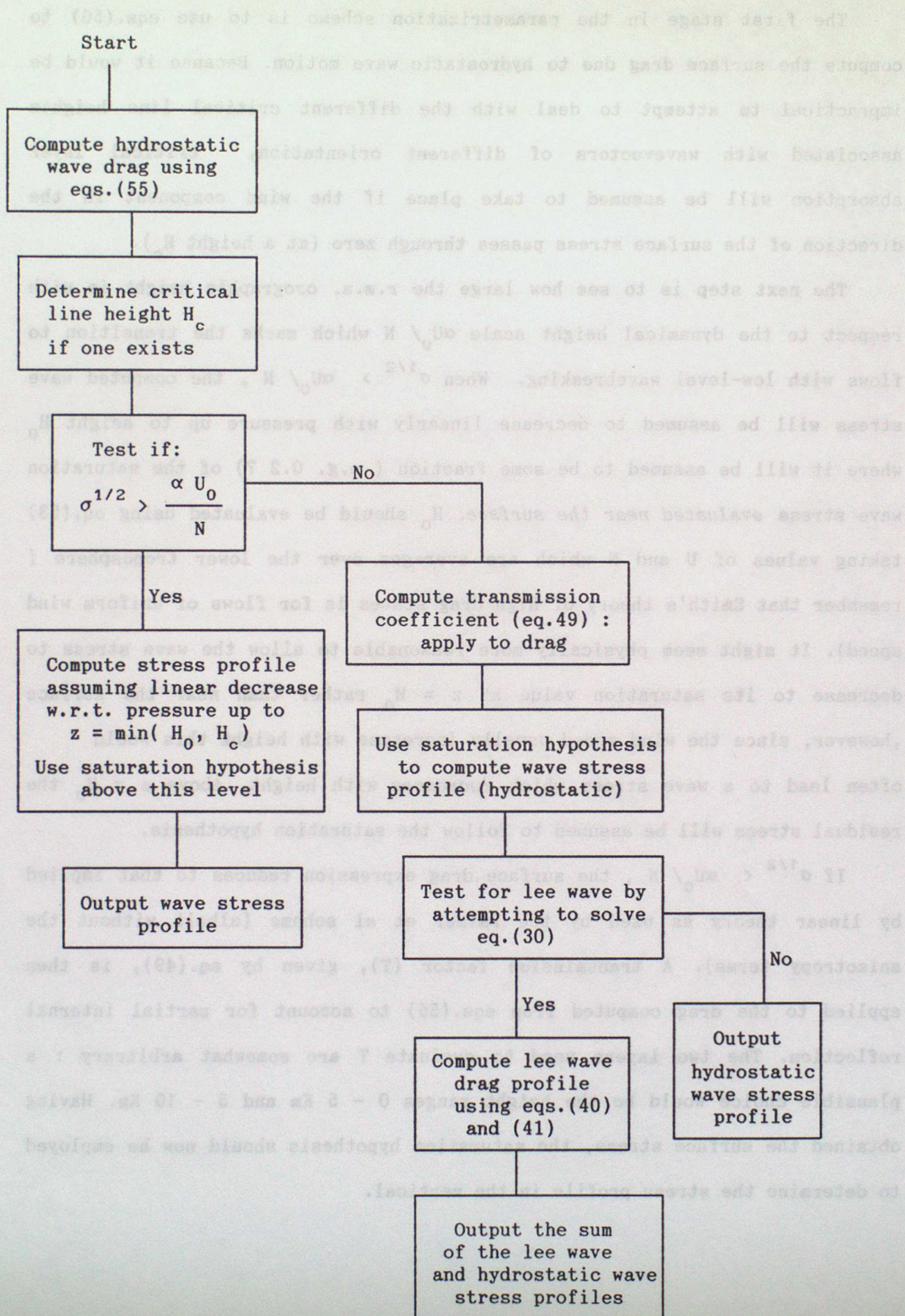
$$= H_0.$$

#### 6. Proposed form for the new parametrization scheme

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The second point of concern relates to the meaning of 'surface wind' and static stability used in the surface stress formula and Prandtl number. Since the wind and potential temperature profiles are highly variable in the boundary layer, there is a high level of indeterminacy in the choice of suitable values appropriate to the extraction of gravity waves. It would seem reasonable to assume that the relevant surface wind and static stability are dependent on the height of the sub-grid scale orography. For instance, one could choose the surface wind to be the model wind interpolated to a height (above the mean orography) equal to  $0.15 H_0$ . Similarly, one could base the static stability on the difference in potential temperature between that at a height  $0.15 H_0$  above the surface and the lowest model level.







The first stage in the parametrization scheme is to use eqs.(50) to compute the surface drag due to hydrostatic wave motion. Because it would be impractical to attempt to deal with the different critical line heights associated with wavevectors of different orientation, critical layer absorption will be assumed to take place if the wind component in the direction of the surface stress passes through zero (at a height  $H_c$ ).

The next step is to see how large the r.m.s. orographic height is with respect to the dynamical height scale  $\alpha U_0 / N$  which marks the transition to flows with low-level wavebreaking. When  $\sigma^{1/2} > \alpha U_0 / N$ , the computed wave stress will be assumed to decrease linearly with pressure up to height  $H_0$  where it will be assumed to be some fraction ( e.g. 0.2 ?) of the saturation wave stress *evaluated near the surface*.  $H_0$  should be evaluated using eq.(53) taking values of  $U$  and  $N$  which are averages over the lower troposphere ( remember that Smith's theory of high drag states is for flows of uniform wind speed). It might seem physically more reasonable to allow the wave stress to decrease to its saturation value at  $z = H_0$  rather than near the surface ,however, since the wind speed usually increases with height this would often lead to a wave stress which *increases* with height. Above  $z = H_0$  the residual stress will be assumed to follow the saturation hypothesis.

If  $\sigma^{1/2} < \alpha U_0 / N$ , the surface drag expression reduces to that implied by linear theory as used by the Palmer et al scheme (albeit without the anisotropy terms). A transmission factor ( $T$ ), given by eq.(49), is then applied to the drag computed from eqs.(55) to account for partial internal reflection. The two layers used to evaluate  $T$  are somewhat arbitrary : a plausible choice would be the height ranges 0 - 5 Km and 5 - 10 Km. Having obtained the surface stress, the saturation hypothesis should now be employed to determine the stress profile in the vertical.



Next a test for trapped lee waves is carried out. This involves computing the function on the left-hand side of eq.(30) for various  $H$ . Successively increasing values of  $H$  (corresponding to the height of model levels) are used to compute mean Scorer parameters in layers 1 and 2. Layer 1 could include all levels up to and including that at height  $H$  : layer 2 should be defined to be roughly the same thickness as layer 1. Since the energy in trapped lee waves is usually confined to the troposphere, the search for lee waves should stop when  $H$  gets to a height of about 6 Km. At this point layer 2 may include part of the stratosphere and raise the value of  $l_2$ . There is probably no need to include all model levels which fall in the boundary layer when defining  $l_1$ . If the function in eq.(30) passes through zero or falls within a chosen distance from zero (tests on radiosonde data show 0.2 to be an appropriate value) then it will be assumed that some form of resonant lee wave can exist. Under these circumstances the lee wave component of the surface drag is given by eqs.(31) and (32), and the height distribution of  $\overline{\rho u w}$  by eqs.(40) and (41) ( note that the incompressible analysis of that section is trivially adapted to the compressible case without change to the momentum flux expression : the error involved is negligibly small )

It would be unwise to be any more specific about the implementation of the scheme at this stage since this will almost certainly change during the fine tuning stage when the model's response can be examined. It is not clear how economical the scheme will turn out to be, particularly the lee wave calculation. The computational overheads are certainly going to be much greater than currently implemented schemes. It should be possible to make economies by using a fixed lee wave drag profile scaled against  $H$  instead of repeatedly evaluating eqs.(40) and (41). Additional storage is required in the



new scheme for the orographic datasets needed to describe the anisotropy of the orographic height variance spectrum ( ie.  $\sigma_{xx}$  ,  $\sigma_{yy}$  and  $\sigma_{xy}$  ) .

One would hope that with the stronger scientific basis to the new scheme, some modest improvements in model climatology could be seen. Even if these are not immediately forthcoming, the greater realism of the drag parametrization ensures that future improvements in the parametrization of other physical processes, or in aspects of the model's formulation, will stand a better chance of realizing their potential benefits. A degraded form of the gravity wave drag parametrization scheme might happen to be associated with better skill scores with the current state of the model, yet block future progress by covering up deficiencies in other parametrization schemes. In other words, one must resist the temptation to look for semi-empirical optimization of systematic errors and have faith in scientifically-reputable formulations.

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