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CALCULATION OF NIGHT MINIMUM TEMPERATURES

By R. FROST, B.A.

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CALCULATION OF NIGHT MINIMUM TEMPERATURES

By R. FROST, B.A.

In 1932 D. Brunt[1*] calculated the grass-minimum temperature on the assumption that the temperature changes at the surface of the earth depend only upon the readiness with which the loss of heat from the surface is compensated by the conduction of heat upwards from the lower layers of the earth.

The assumption that the earth is the sole reservoir from which the heat is extracted is not obvious *a priori* and indeed appears to be incorrect. An inspection of any two consecutive upper air ascents made in the evening and the following morning respectively shows that a considerable quantity of heat has been extracted from the lower layers of the air during the night.

E. Gold[2] subsequently shewed that day-maximum temperatures could be calculated on the assumption that most of the insolation reaching the earth's surface was used in warming the lowest layers of the atmosphere, the rest being lost by radiation back to space or utilised in evaporating surface water.

The success of Gold's method suggests that the main physical factors have been taken into account. In this paper therefore, it will be assumed that the temperature changes at the surface of the earth depend mainly upon the readiness with which the loss of heat from the surface is compensated by the transfer of heat downwards from the air and by the heat released by the deposition of dew.

The eddy equation of diffusion of heat was shown by G. I. Taylor[3] to be

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(K_z \frac{\partial \theta}{\partial z}\right) \qquad \text{(1)}$$

where θ is the potential temperature and K_z is the coefficient of eddy diffusion at a height z above the surface of the ground.

If the net loss of heat by radiation from the ground is R_N, then

$$R_N = \left[K_z \varrho c_p \frac{\partial \theta}{\partial z}\right]_{z=0}$$

where ϱ and c_p are respectively the density and specific heat of the air. This assumes that the loss of heat from the surface outwards is equal to the flow of heat from the air to the surface. Strictly we should write

$$R_N - \Delta - \varepsilon = \left[K_z \varrho c_p \frac{\partial \theta}{\partial z}\right]_{z=0}$$

where Δ is the heat released by the deposition of dew and ε is the heat supplied from the earth. In the first instance we shall neglect both Δ and ε, and for a first approximation we shall take R_N to be a constant. The neglect of Δ and ε clearly gives an upper limit to the fall of temperature.

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* The index figures refer to the bibliography on p. 6.

Writing $$K_z \frac{\partial \theta}{\partial z} = S,$$

we have
$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial t}\left(K_z \frac{\partial \theta}{\partial z}\right)$$
$$= K_z \frac{\partial}{\partial z} \frac{\partial \theta}{\partial t}$$
$$= K_z \frac{\partial^2}{\partial z^2}\left(K_z \frac{\partial \theta}{\partial z}\right)$$
$$= K_z \frac{\partial^2 S}{\partial z^2}. \quad .. \quad (2)$$

Under inversion conditions Sverdrup[4] has shewn that in the lowest layers of the atmosphere K_z is represented by a power law

$$K_z = az^{1-m}$$

where m is a positive fraction and a is a constant.

Thus we require a solution of

$$\frac{\partial S}{\partial t} = az^{1-m} \frac{\partial^2 S}{\partial z^2} \quad .. \quad (3)$$

with the boundary condition that $S = \dfrac{R_N}{\varrho c_p}$ at $z = 0$.

An inspection of the dimensions of equation (3) suggests that it should be possible to find a solution $S = f(\xi)$ where $\xi = z^{m+1}/[(m+1)^2\, at]$.

Denoting differentiations with respect to ξ by dashed letters we obtain

$$-f'(\xi)\frac{\xi}{t} = az^{1-m}\frac{\partial}{\partial z}\left[f'(\xi)\frac{z^m}{(m+1)\,at}\right]$$
$$= \frac{m}{m+1}\frac{f'(\xi)}{t} + \frac{f''(\xi)\,.\,\xi}{t}$$

Therefore $$f''(\xi)\,.\,\xi + f'(\xi)\left[\xi + \frac{m}{m+1}\right] = 0 \quad .. \quad (4)$$

whence $$S = f(\xi) = \frac{R_N}{\varrho c_p \Gamma\left(\frac{1}{m+1}\right)} \int_\xi^\infty \xi^{-m/(m+1)} e^{-\xi}\, d\xi \quad .. \quad (5)$$

But $$S = K_z \frac{\partial \theta}{\partial z} = az^{1-m}\frac{\partial \theta}{\partial z}$$

therefore $$\frac{\partial \theta}{\partial z} = \frac{R_N z^{m-1}}{a\varrho c_p \Gamma\left(\frac{1}{m+1}\right)} \int_\xi^\infty \xi^{-m/(m+1)} e^{-\xi}\, d\xi$$

and hence integrating by parts we obtain

$$\theta = \theta_0 + \frac{R_N}{am\varrho c_p \Gamma\left(\frac{1}{m+1}\right)}\left[z^m \int_\xi^\infty \xi^{-m/(m+1)} e^{-\xi}\, d\xi - e^{-\xi}\{(m+1)^2 at\}^{m/(m+1)}\right] \quad (6)$$

where $$\xi = \frac{z^{m+1}}{(m+1)^2 at}$$

In this equation R_N is measured in gram calories per square centimetre per second and t in seconds.

Before applying this result to any practical problem it is advisable to consider the conditions under which it has been derived. Reference to equation (5) shews that, at $z = 0$, S is a constant at all times except at $t = 0$ when it is zero. Thus the solution represented by (6) corresponds to the case when the outward radiation is initially zero at a time $t = 0$ and then jumps to the value R_N. This corresponds approximately to the conditions at sunset, or even better to the case when the sky having been overcast with low cloud suddenly clears at sunset.

Values of K_z under inversion conditions at various heights have been computed by Cowling and White[5] from June temperature observations at Leafield, and are given in Table I together with values of K_z calculated from the power law $K_z = 40z^{2/3}$. It can be seen that the calculated values are in close agreement with the observed values.

TABLE I

Height in metres	1·2	12·4	30·5	57·4	87·7
K_z (observed)	10^3	$4 \cdot 5 \times 10^3$	$8 \cdot 5 \times 10^3$	12×10^3	20×10^3
K_z (calculated)	10^3	$4 \cdot 5 \times 10^3$	$8 \cdot 5 \times 10^3$	$13 \cdot 2 \times 10^3$	$17 \cdot 5 \times 10^3$

Considering (6) we find that the fall of temperature at the ground at a time t after sunset with $a = 40$ and $m = \frac{1}{3}$ is, from equation (6),

$$\theta_0 - \theta = \frac{R_N \,.\, (\frac{16}{9})^{\frac{1}{4}} \,.\, (40)^{\frac{1}{4}} \,.\, t^{\frac{1}{4}}}{40 \,.\, \frac{1}{3} \,.\, \frac{1}{800} \,.\, 0 \cdot 24 \,.\, \frac{4}{3} \,.\, 0 \cdot 92}$$

since $\xi = 0$ for $z = 0$, $m = \frac{1}{3}$, $\varrho = \frac{1}{800}$ gm./cm.^{3}, $\Gamma(\frac{3}{4}) = \frac{4}{3}\Gamma(\frac{7}{4})$, $\Gamma(\frac{7}{4}) = 0 \cdot 92$,

i.e. $$\theta_0 - \theta = R_N \,.\, 5 \cdot 9 \,.\, 10^2 \,.\, t^{\frac{1}{4}}. \qquad .. \quad (7)$$

Now the mean value for R_N for June at Benson according to Brunt is $2 \cdot 1 \times 10^{-3}$ gm. cal./sq. cm./sec.,* and hence the fall of temperature at the ground in the 7 hours between sunset and sunrise is

$$2 \cdot 1 \,.\, 10^{-3} \,.\, 5 \cdot 9 \,.\, 10^2 \,.\, 12 \cdot 6 = 15 \cdot 6°\text{C}. \qquad .. \quad (8)$$

Similarly it may be calculated from equation (6) that the fall of temperature at 4 ft. above the ground is 13·0°C.

N. K. Johnson[6] gives a mean temperature curve for clear days and nights in June. The mean fall of temperature at 4 ft. from sunset to sunrise, shewn by this curve, is 9°C., which is lower than the theoretical value given above as we should expect.

The decrease in the fall of temperature caused by the deposition of dew can be calculated without much difficulty. G. Yamamoto[7], who measured the rate of deposition of dew from the air at Hukuoka in Japan with an apparatus which automatically recorded the weight of dew formed on the upper surface of a glass pan placed flush with the surface of a grass lawn, found that the rate of deposition of dew was comparatively slow at first, rising to a maximum some two or three hours later and then decreasing again slowly until sunrise. The variations in the rate of deposition were however small, and for a first approximation the rate of deposition of dew may be taken as a constant.

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* A very good example has been noted from the upper ascents at Bircham Newton on the night of June 21 and morning of June 22, 1942. The area on the tephigram enclosed by the two upper air ascent curves and the surface pressure is equal to 0·78 squares, which gives an average flow of heat of $2 \cdot 1 \,.\, 10^{-3}$ gm. cal./sq. cm./sec. throughout the night, which is equal to the net loss of heat per second quoted by Brunt.

Dew was formed on 126 nights in the year with a mean nightly dewfall of 0·086 mm. and a maximum nightly dewfall of 0·18 mm. These rates are clearly underestimates of the actual dewfall on the grass lawn, for the amount of dew varies in accordance with the sum of the surface areas of the blades of grass which are exposed to the night sky and not as the surface area which is covered by the grass; the actual dewfall is probably about three times the measured dewfall.

No measurements of the rate of deposition of dew in this country are available, but Shaw quotes the following estimates for the depth of water deposited annually by dew: 1·0–1·5 in. in the neighbourhood of London and 1·6 in. in Worcestershire. If the number of dew nights in England is comparable with the number of dew nights in Japan, this would give an average nightly deposition of dew of 0·2–0·3 mm.

Now equation (1) can equally be used for the diffusion of water vapour provided the potential temperature is replaced by x the humidity mixing ratio and the solution of the equation which obeys the boundary conditions

$$x = x_0 \text{ when } t = 0$$

$$\lim_{z \to 0} K_z \varrho \frac{\partial x}{\partial z} = \delta$$

where δ is the rate of deposition in centimetres and is assumed constant, is clearly

$$x = x_0 + \frac{\delta}{am\varrho\Gamma\left(\frac{1}{m+1}\right)} \left[z^m \int_{\xi}^{\infty} \xi^{-m/(m+1)} e^{-\xi} d\xi - e^{-\xi} \{(m+1)^2 at\}^{m/(m+1)} \right] \qquad (9)$$

Thus the fall in humidity mixing ratio at the ground at a time t after sunset is

$$x_0 - x = \frac{\delta \,.\, (\frac{16}{9})^{1/4} \,.\, (40)^{1/4} \,.\, t^{1/4}}{40 \,.\, \frac{1}{3} \,.\, \frac{1}{800} \,.\, \frac{4}{3} \,.\, 0{\cdot}92} \qquad .. \quad (10)$$

and therefore the fall of the humidity mixing ratio at the surface in 7 hours between sunset and sunrise is

$$x_0 - x = \delta \,.\, 1{\cdot}8 \,.\, 10^3 \qquad .. \quad (11)$$

whilst from (9) it may be calculated that the fall in humidity mixing ratio in 7 hours at a height of 4 ft. above the ground is

$$x_0 - x = \delta \,.\, 1{\cdot}5 \,.\, 10^3 \qquad .. \quad (12)$$

The average fall of the humidity mixing ratio at a height of 4 ft. above the ground for all radiation nights at Boscombe Down in June and July, 1942, between the hours of 2200 and 0400 G.M.T. was 1·2 grams of water per kilogram of air. This period is not quite the same as in (12), but the error introduced by this would be small as the decrease in humidity mixing ratio varies as $t^{1/4}$.

Thus $1{\cdot}2 \,.\, 10^{-3} = \delta \,.\, 1{\cdot}5 \,.\, 10^3$

i.e. $\delta = 8{\cdot}0 \,.\, 10^{-7}$ cm./sec.

The total dew deposition in the 7 hours between sunset and sunrise is therefore 0·20 mm., which agrees very well with our previous estimate.

A simple calculation shews that the heat liberated by this would be

$$\Delta = 600 \,.\, 8{\cdot}0 \,.\, 10^{-7}$$
$$= 4{\cdot}8 \,.\, 10^{-4} \text{ gm. cal./sq. cm./sec.}$$

Thus $R_N' - \Delta = 1{\cdot}62 \,.\, 10^{-3}$ gm. cal./sq. cm./sec.

and hence the fall of temperature at 4 ft. above the ground would be

$$\frac{1{\cdot}62}{2{\cdot}1} \,.\, 13{\cdot}0 = 10{\cdot}0^\circ\text{C}.$$

which is in good agreement with the observed fall.

Thus unless either the rate of deposition of dew is excessive or the value of R_N at Benson is a serious underestimate it would appear that in June at Leafield, where presumably the soil is dry and the thermal conductivity of the ground is low, the conduction of heat upwards from the earth is negligible.

When the soil is wet (as in winter) the thermal conductivity of the ground is considerably increased, and it is probable that the conduction of heat upwards from the lower layers of the earth is an important factor. The small range of temperature in the screen at 4 ft. found by N. K. Johnson at Leafield in December suggests this. Simultaneous observations of temperature at various levels in the air and also in the ground together with measurement of radiation, and of the rate of deposition of dew are desirable.

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