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Scientific Paper No. 2

Conservation of Vorticity at 100 millibars

by J. R. PROBERT-JONES, B.A.

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Contents

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INTRODUCTION

The purpose of the experiment described in this paper has been to ascertain whether the flow at 100 millibars can be treated as essentially non-divergent. The method of attack has been to determine whether sequences of 100-millibar charts can be so drawn that (geostrophic) absolute vorticity is conserved in the motion. The graphical technique developed by Fjørtoft[1] has been used.

THEORY

The vorticity equation in non-divergent flow, neglecting vertical motion, is

$$\frac{\partial \eta}{\partial t} + \mathbf{V} . \nabla \eta = 0 \qquad . \ . \ . \ (1)$$

where η is the absolute vorticity.

If we consider geostrophic motion in an isobaric surface,

$$\eta = \frac{g}{l} \nabla^2 z + l$$

and

$$\mathbf{V} = \frac{g}{l} \mathbf{k} \wedge \nabla z,$$

where z is the height of the isobaric surface, l is the Coriolis parameter, and **k** the unit vertical vector, equation (1) becomes:

$$\frac{\partial}{\partial t} \nabla^2 z = -J(z, \eta). \qquad . \ . \ . \ (2)$$

If Z_i are the values of z at the four points of a square mesh of side $2h$ surrounding a point O, a simple finite difference approximation to $\nabla^2 z$ is

$$\nabla^2 z = \frac{1}{h^2} \left(\sum_{i=1}^{4} Z_i - 4Z_o \right) = \frac{4}{h^2} \left(\bar{Z} - Z_o \right),$$

where

$$\bar{Z} = \frac{1}{4} \sum_{i=1}^{4} Z_i.$$

Equation (2) gives:

$$\frac{\partial}{\partial t}(Z - \bar{Z}) = -\frac{l}{m_2} \mathbf{V} . \nabla \left\{ \frac{m^2}{l} (Z - \bar{Z}) \right\} + \mathbf{V} . \frac{lh^2}{4gm^2} \nabla l, \qquad . \ . \ . \ (3)$$

where m is the scale factor of the map projection used. Putting

$$G = \int_0^\theta \frac{l^2 h^2}{4gm^2} \cot \theta \, d\theta,$$

* The superscript figures refer to the bibliography on p. 7.

θ being the latitude, and neglecting $(Z - \bar{Z}) \nabla \frac{m^2}{l}$ in comparison with $\frac{m^2}{l} \nabla (Z - \bar{Z})$, equation (3) becomes:

$$\frac{\partial}{\partial t}\left(Z - \bar{Z} - G\right) = -\frac{g}{l} J\,(Z,\, Z - \bar{Z} - G)$$

$$= -\frac{g}{l} J\,(\bar{Z} + G,\, Z - \bar{Z} - G). \qquad \ldots (4)$$

Equation (4) shows that in non-divergent geostrophic flow the vorticity approximation $Z - \bar{Z} - G$ is conserved in the space-mean field $\bar{Z} + G$. This latter field is known from

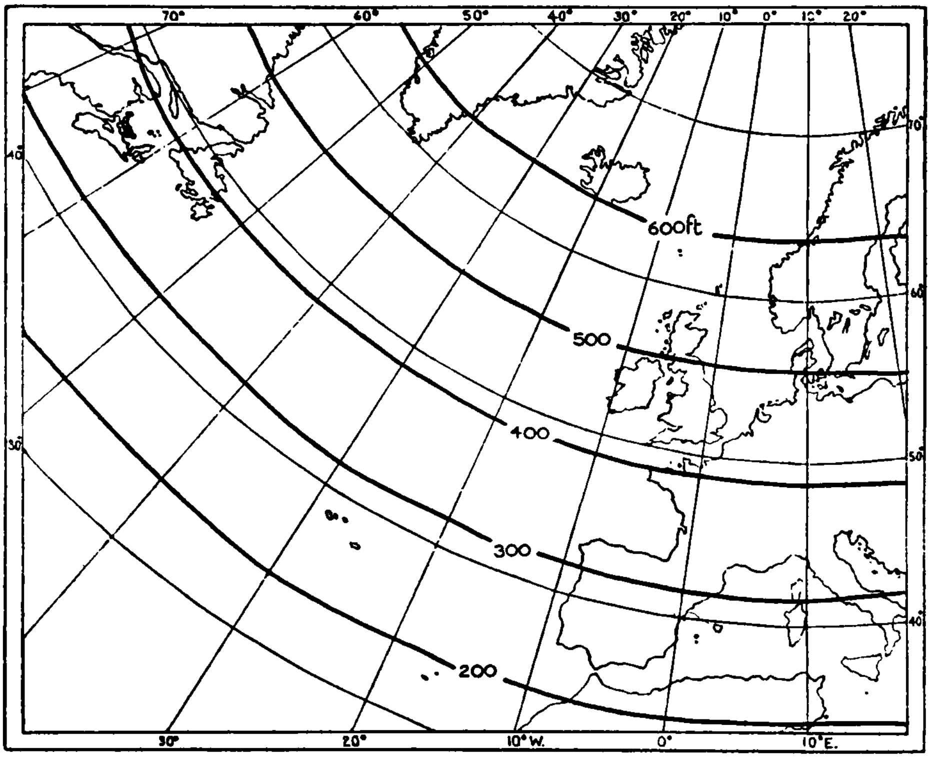

FIGURE 1. Contours of the G field.

The contours of G are valid for a conical orthomorphic projection with standard parallels at 65° and 45°N. and for a grid length of 600 miles.

experiment to be more conservative than the Z field, and is assumed sufficiently constant to warrant displacement of the vorticity field in a single time-step of 12 hours.

Contours of G are shown in Figure 1.

It can be shown that the maximum value of $(Z - \bar{Z}) \nabla \frac{m^2}{l}$, the term neglected above, is about one-fifth of $\frac{m^2}{l} \nabla (Z - \bar{Z})$.

DESIGN OF THE EXPERIMENT

It has been shown above that if the flow at 100 millibars is non-divergent a certain function $Z - \bar{Z} - G$, representing the vorticity, will be conserved in the $\bar{Z} + G$, or smoothed flow, field. For convenience throughout this paper the $Z - \bar{Z} - G$ field will be referred to as the vorticity field, although it is only a graphical approximation to this field.

Johnson[2,3] finds that the root mean square random error of observation of height of the 100-millibar surface over the British Isles is 150 feet, and that the vector root mean square error for wind observations is less than 6 knots. These errors and the scarcity of observations, particularly of wind, at this level, allow considerable latitude in the drawing of 100-millibar contours. Therefore it was decided to examine a few sequences of routine 100-millibar charts and to attempt to redraw them to be consistent with the hypothesis that absolute vorticity was conserved following the motion and yet retain consistency with the observations. Thus the vorticity given by any one chart should not differ, by more than a stipulated amount at any point, from the vorticity obtained by advection, forwards or backwards in time, from its immediate neighbour. It was anticipated that more than one modification to any one chart might be necessary.

If the charts could be so redrawn, a comparison could subsequently be made between the fitting of the wind and height reports in the original and modified drawings.

METHOD

100-millibar charts for the six months November to April for the winters 1953–54, 1954–55 and 1955–56 were examined for sequences in which a fairly strong flow existed in the neighbourhood of the British Isles together with a well marked feature in the flow pattern which would provide significant vorticity variations. A reasonable cover of observations was also sought. Five sequences were extracted and are detailed below.

(i) 1500 G.M.T., 16 December to 1500 G.M.T., 19 December 1955. A trough to the west of the British Isles in the strong flow moved east and was replaced by a quasi-stationary ridge over the Atlantic.

(ii) 1500 G.M.T., 5 February to 1500 G.M.T., 8 February 1954. A trough developed to the west of Ireland and moved east slowly to lie over central Europe.

(iii) 0300 G.M.T., 4 November to 1500 G.M.T., 6 November 1955. A deep trough with its axis from Iceland to the Azores remained quasi-stationary and developed a closed vortex.

(iv) 0300 G.M.T., 14 February to 0300 G.M.T., 17 February 1954. In the strong flow a ridge from the Azores to Iceland and a trough over the east of America moved east to cover the British Isles and the central Atlantic respectively.

(v) 0300 G.M.T., 8 November to 1500 G.M.T., 12 November 1955. A trough to the west of Ireland decreased in amplitude while a ridge off Labrador moved east slowly.

A preliminary inspection of routine working-charts suggested that too much emphasis had been placed on height values and too little on winds and that some alteration was allowable. This was therefore carried out for all sequences, care being taken to ensure continuity in time throughout each sequence. In the remainder of this paper these redrawn charts will be referred to as "original" charts. The last three of these charts for sequence (ii) are shown in Figures 3, 7 and 11.

Estoque[4] has detailed the method of obtaining graphically the required vorticity and advection fields. A grid length of 600 miles was used, and isopleths drawn at 100-foot intervals. One alteration was made to his procedure in the present case. Instead of forming the $\bar{Z} + G$ field by adding the G field to the $\bar{Z}$ field, a very inaccurate process at the 100-millibar level, the $Z - G$ field was formed as an intermediate step. From this the $Z - \bar{Z} - G$ field and thence the $\bar{Z} + G$ field is easily obtained.

The field of $-(Z - \bar{Z} - G)$, corresponding to the vorticity, for the first chart of each sequence, was advected for a period of 12 hours. On comparing this with the pattern of $-(Z - \bar{Z} - G)$ for the second chart it was apparent that significant differences existed, of the order of 300 feet in places, and the patterns definitely did not resemble each other. Since the $Z - \bar{Z} - G$ isopleths were drawn at 100-foot intervals, it was decided to attempt to achieve correspondence within a tolerance of 100 feet which is equivalent to a divergence of about 2×10^{-6} sec^{-1}.

An attempt was then made to modify the second chart within the limits of the accuracy of the observations by altering the shear and curvature, to give a vorticity pattern similar to the advected one and within the 100-foot tolerance. In the first two sequences this was not possible in one step, and changes in the first chart in each sequence had to be made, using advection of vorticity backwards in time from a modified second chart, in order to keep within the 100-foot tolerance.

The vorticity of the modified second chart was then advected in the space-mean flow field and the third chart modified to give a vorticity pattern similar to the advected one. This procedure was continued for the rest of the sequence. The original and modified height charts, and the original, modified and advected vorticity charts for the last part of sequence (ii), from 1500 G.M.T., 7 February to 1500 G.M.T., 8 February 1954, are shown in Figures 3–14.

The method requires that the advective field can be treated as constant over the period of advection. Examination of these fields at 12-hour intervals showed that this was satisfied to a high order of accuracy.

RESULTS

It was found possible to redraw the five sequences of 100-millibar charts so that the change in each 12-hour period implied a divergence of less than 2×10^{-6} sec^{-1}, which is negligible on the synoptic scale. A visual comparison of the original, advected and modified vorticity fields clearly revealed the great improvement given by the modified fields in comparison with the original fields in corresponding with the advected fields.

An indication of the relative accuracy of the two sets of 100-millibar charts is given by a comparison of their smoothness in time. Values of height and geostrophic wind were read off at points of a grid shown in Figure 2 covering that part of the chart where the drawing was considered most reliable. Values of the statistics $|Z_{12} - Z_0|$, the modulus of the 12-hour height change, and $|\mathbf{V}_{12} - \mathbf{V}_0|$, the modulus of the 12-hour vector wind change, are given in Table I, where $|\overline{\mathbf{V}}|$ is the mean of the wind speeds.

A comparison between reported winds and heights, and those read off the two sets of charts at the seventeen stations: Stornoway, Aldergrove, Hemsby, Camborne, Flensburg, Wiesbaden, Munich, Gibraltar and Malta, and Ocean Weather Stations 'A', 'B', 'C', 'D', 'E', 'I', 'J' and 'K', is given in Table II, where Z_r, $\mathbf{V}_r$ refer to the heights and geostrophic

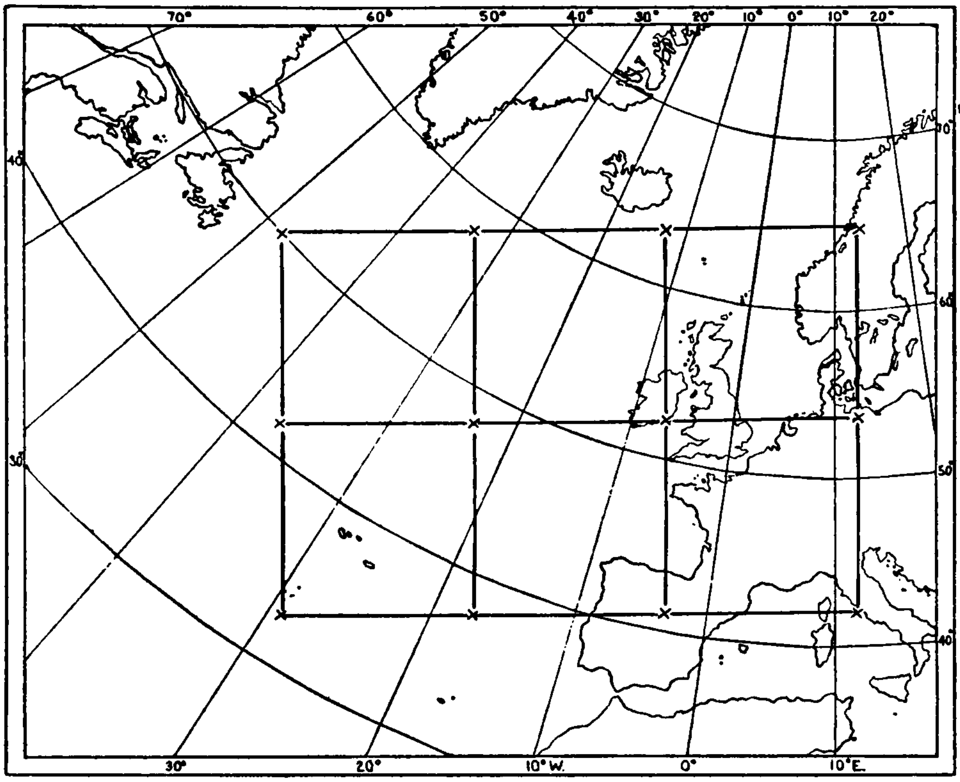

FIGURE 2. Grid used in testing the time variations of contour height and geostrophic wind.

winds read off the charts, and Z_0, V_0 the observed or reported values from the radio-sonde ascents. No deterioration in the fitting of the observations during the course of any sequence was found.

Johnson[2] shows that the root mean square random error of observation of height at 100 millibars from British radio-sondes is 150 feet, and this figure probably applies to the

TABLE I. Time variation in height and geostrophic wind from grid

Sequence	$\lvert Z_{12}-Z_0 \rvert$ Average value ft.	r.m.s. value ft.	$\lvert V_{12}-V_0 \rvert$ Average value kt.	r.m.s. value kt.	$\lvert \overline{V} \rvert$ kt.	No. of cases
I Original	184	241	16	20	44	72
I Modified	160	206	12	14	48	
II Original	167	211	14	18	39	72
II Modified	176	217	9	10	37	
III Original	158	183	12	15	29	60
III Modified	154	171	8	9	30	
IV Original	203	261	17	19	42	72
IV Modified	165	222	11	13	40	
V Original	178	207	10	12	29	108
V Modified	157	183	6	7	29	

radio-sondes of the other stations used in Table II. The results of the Payerne trials (1956), however, show a significant systematic error between the radio-sondes of different countries. Errors due to chart drawing and reading off the charts are small in comparison, and the expected value of $|Z_r - Z_0|$ can be taken as greater than 150 feet by perhaps 20 to 30 feet.

Upper limits to the root mean square error of wind observations from British and British Ocean Weather Stations of 6 and 9 knots are given by Johnson.[3] Errors due to chart drawing, ageostrophic motion and estimation of the geostrophic wind are not negligible, but it is probable that the expected value of $|\mathbf{V}_r - \mathbf{V}_0|$ does not exceed 10 knots.

TABLE II. Comparison between observed and measured heights and winds

Sequence	$\lvert Z_r - Z_0\rvert$ Average value ft.	$\lvert Z_r - Z_0\rvert$ r.m.s. value ft.	No. of cases	$\lvert \mathbf{V}_r - \mathbf{V}_0\rvert$ Average value kt.	$\lvert \mathbf{V}_r - \mathbf{V}_0\rvert$ r.m.s. value kt.	$\lvert \overline{\mathbf{V}}_0\rvert$ kt.	$\lvert \overline{\mathbf{V}}_r\rvert$ kt.	No. of cases
I Original	114	157	98	11	14	45	43	73
I Modified	158	196		10	12		42	
II Original	131	174	92	10	12	39	38	38
II Modified	167	210		11	12		34	
III Original	83	120	85	9	11	27	28	62
III Modified	98	134		9	11		26	
IV Original	101	148	78	13	15	40	40	48
IV Modified	136	187		11	12		38	
V Original	95	129	137	9	12	25	26	99
V Modified	109	145		8	10		24	

CONCLUSIONS

The 12-hour root mean square variation of height and geostrophic wind of the original set of 100-millibar charts is 222 feet and 17 knots respectively; for the set modified to be consistent with vorticity advection the figures are 200 feet and 11 knots. The observations of height and wind have been fitted with root mean square errors for the original set of 146 feet and 13 knots and for the modified set of 175 feet and 11 knots respectively. The modified set is therefore smoother in time with respect to height and significantly smoother with respect to wind than the original set. The observed winds are fitted better by the modified drawings, and although the heights are fitted better by the original set, the discussion of the errors in the height values given above would seem to indicate that the fit of the original set is too good. On the basis of these figures it is suggested that the modified set provides a more realistic approximation to the true flow than the original set.

Any generalization to conditions at the 100-millibar level at other times is not strictly justified. Since, however, the sequences selected show very varying conditions and probably contain representative types of flow found at this level, it can be concluded that the evidence supports the hypothesis of approximate non-divergent flow at the 100-millibar level.

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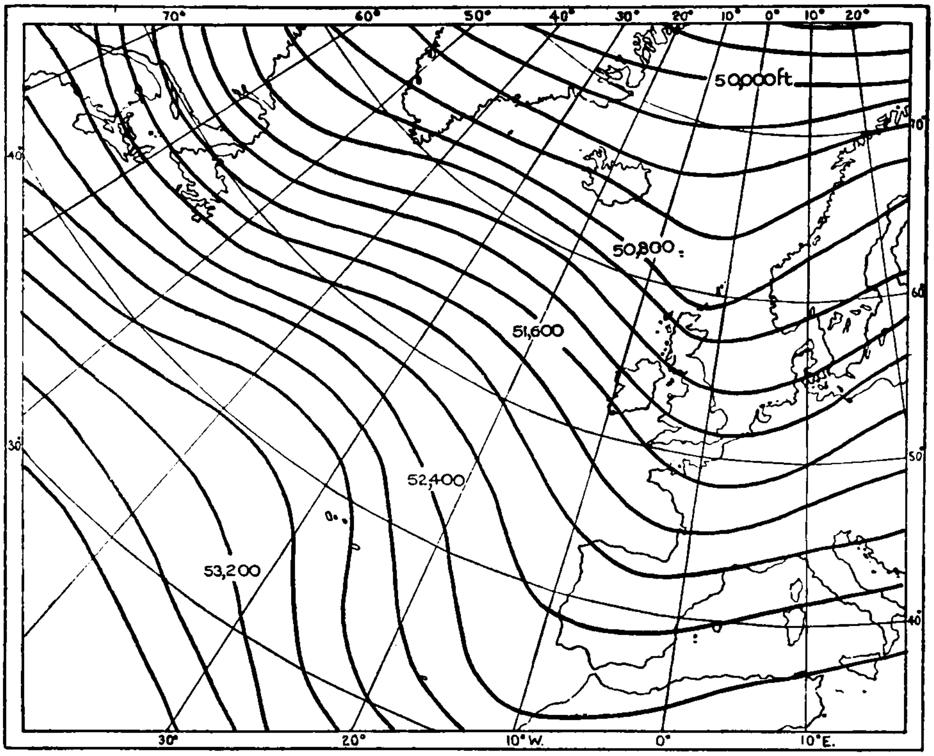

FIGURE 3. 'Original' Z field (100-mb contours), 1500 G.M.T., 7 February 1954.

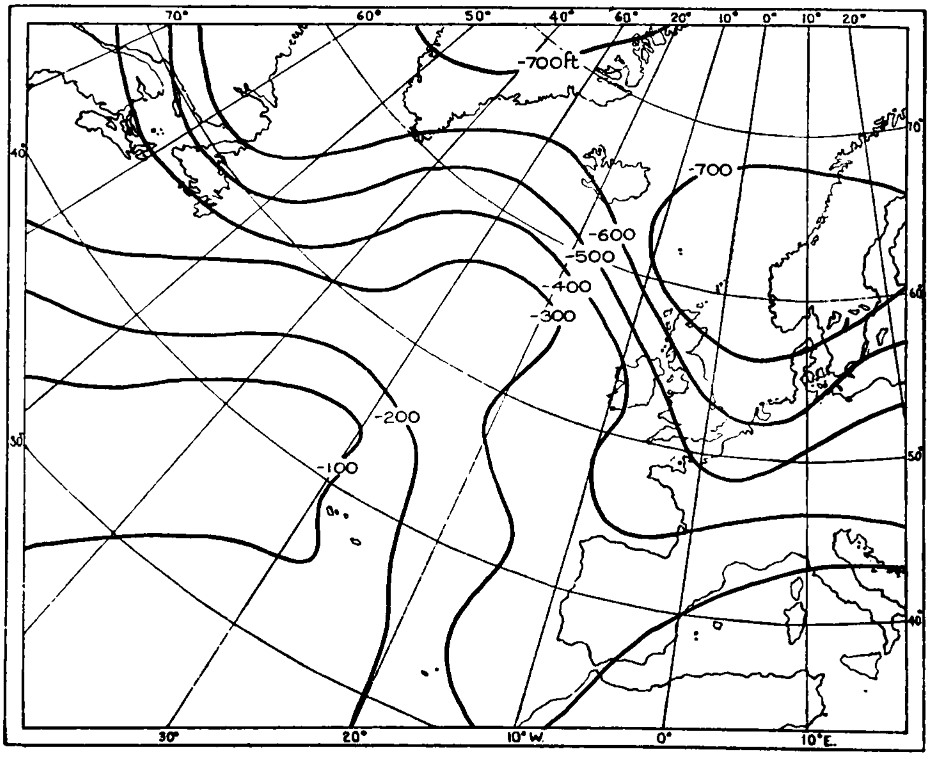

FIGURE 4. 'Original' $Z - \bar{Z} - G$ field, 1500 G.M.T., 7 February 1954.

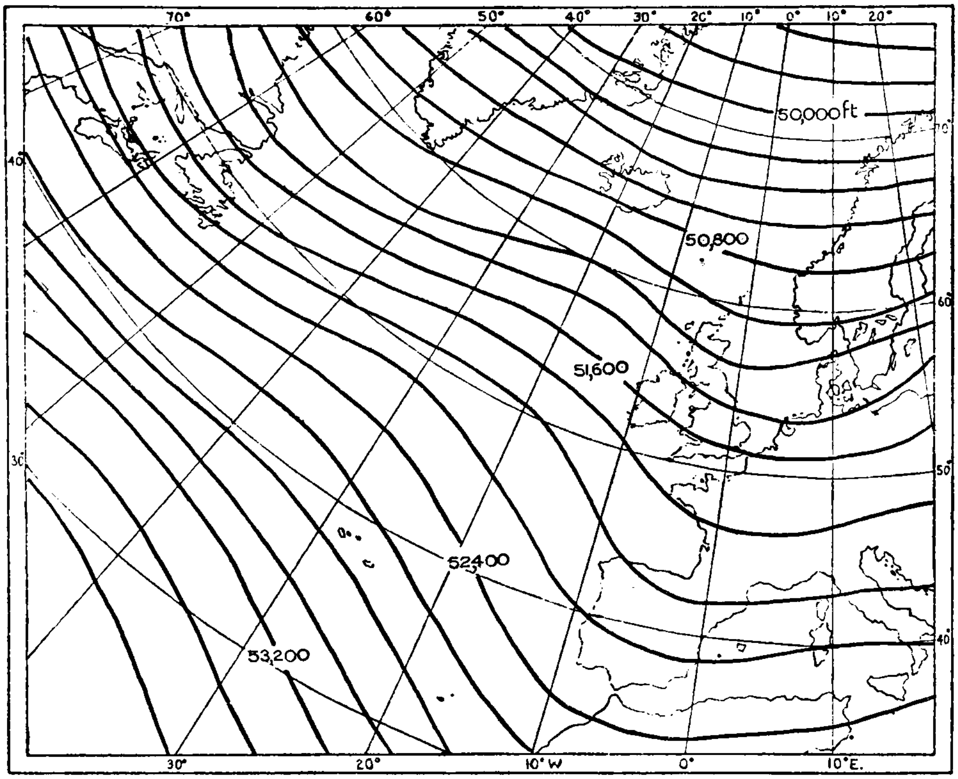

FIGURE 5. Modified Z field (100-mb contours), 1500 G.M.T., 7 February 1954.

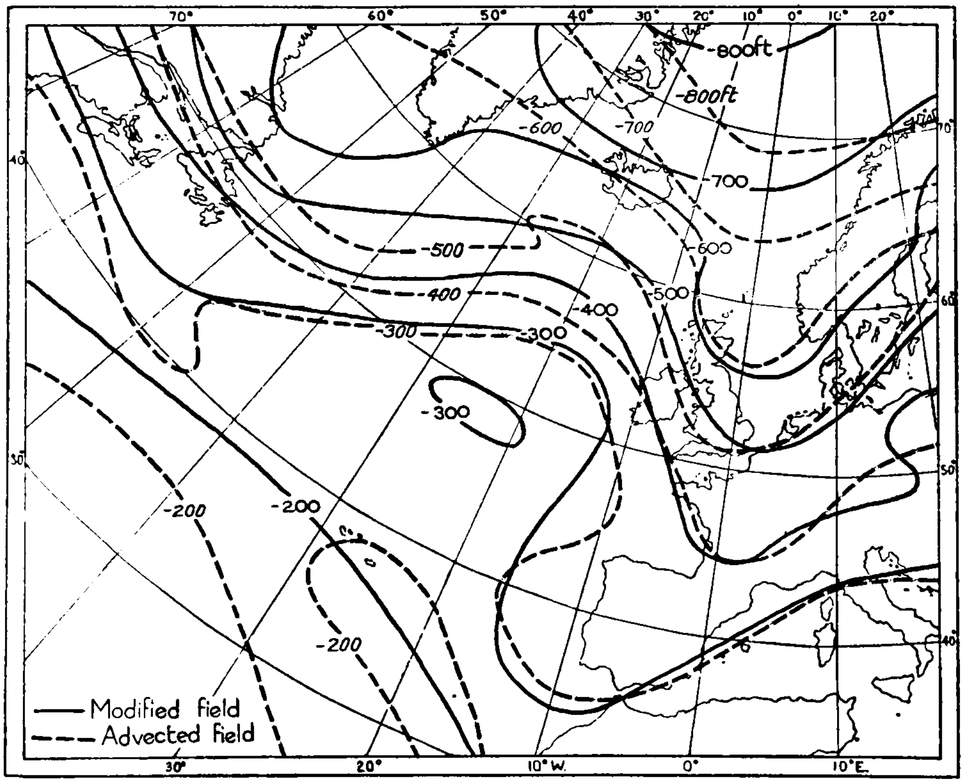

FIGURE 6. Modified and advected $Z - \mathcal{Z} - G$ fields, 1500 G.M.T., 7 February 1954.

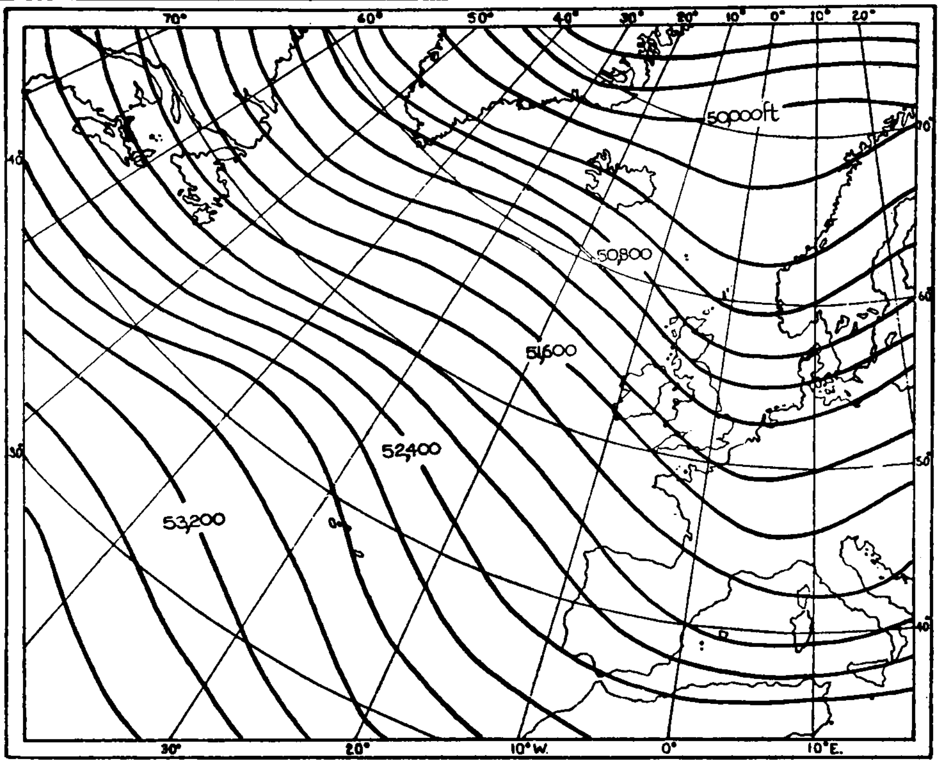

FIGURE 7. 'Original' Z field (100-mb contours), 0300 G.M.T., 8 February 1954.

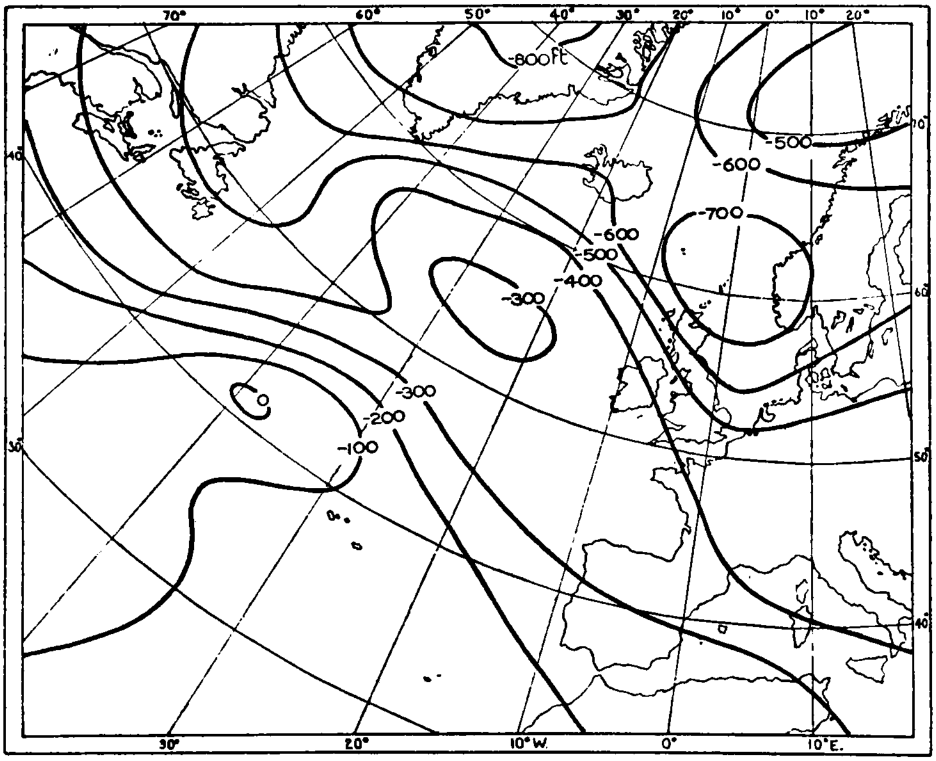

FIGURE 8. 'Original' $Z - \bar{Z} - G$ field, 0300 G.M.T., 8 February 1954.

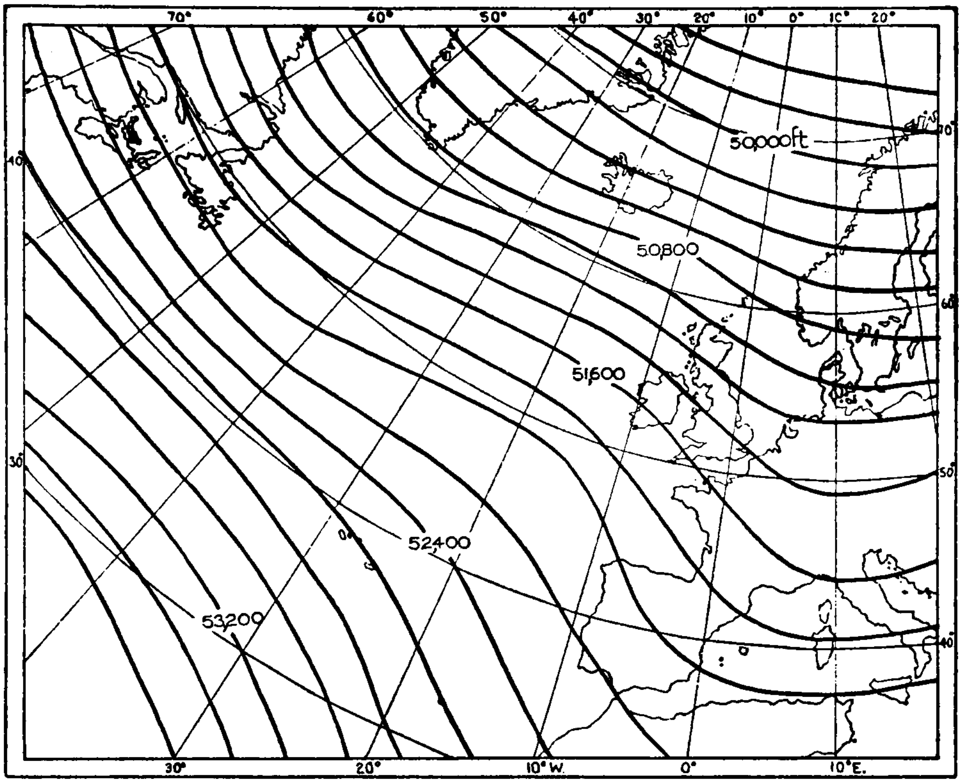

FIGURE 9. Modified Z field (100-mb contours), 0300 G.M.T., 8 February 1954.

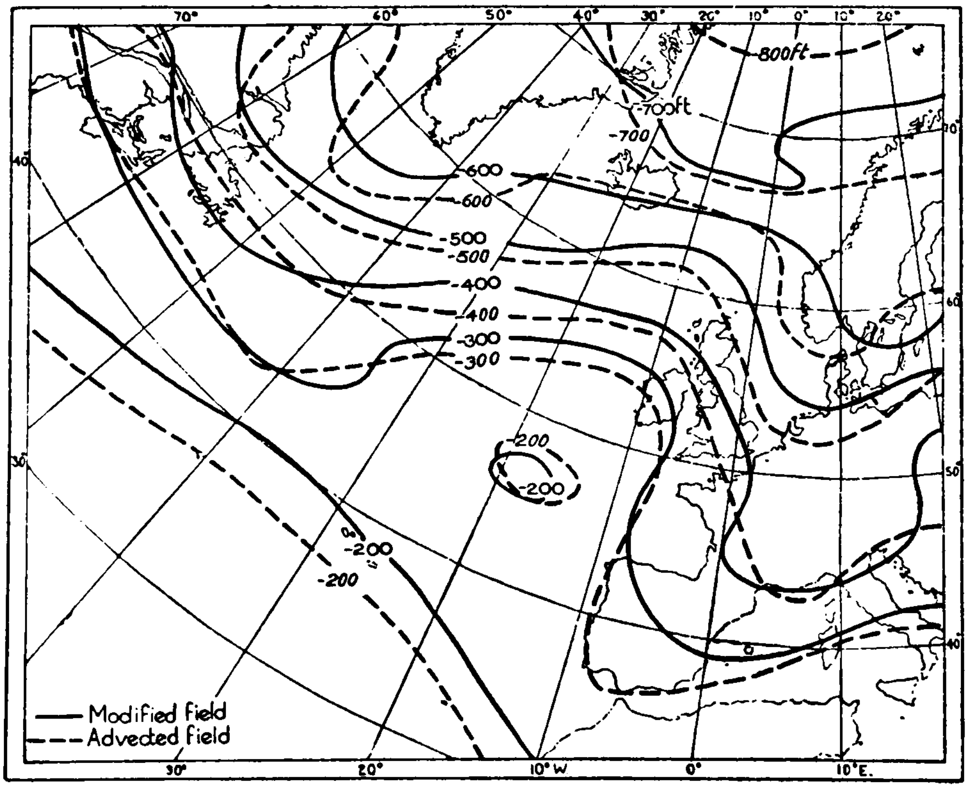

FIGURE 10. Modified and advected $Z - \mathcal{Z} - G$ fields, 0300 G.M.T., 8 February 1954.

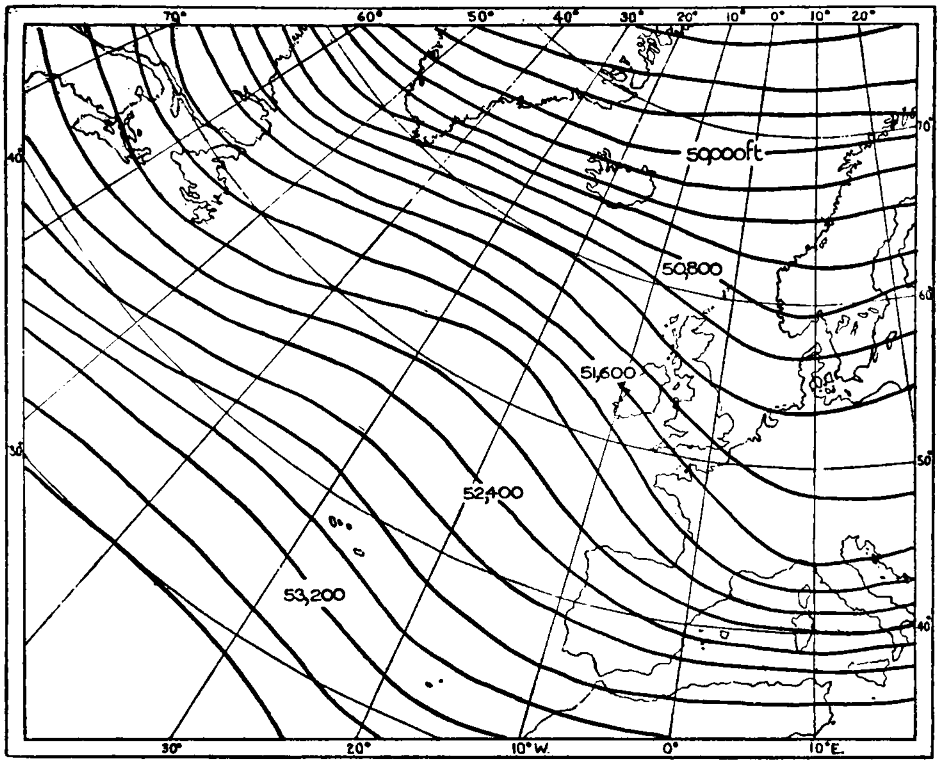

FIGURE 11. 'Original' Z field (100-mb contours), 1500 G.M.T., 8 February 1954.

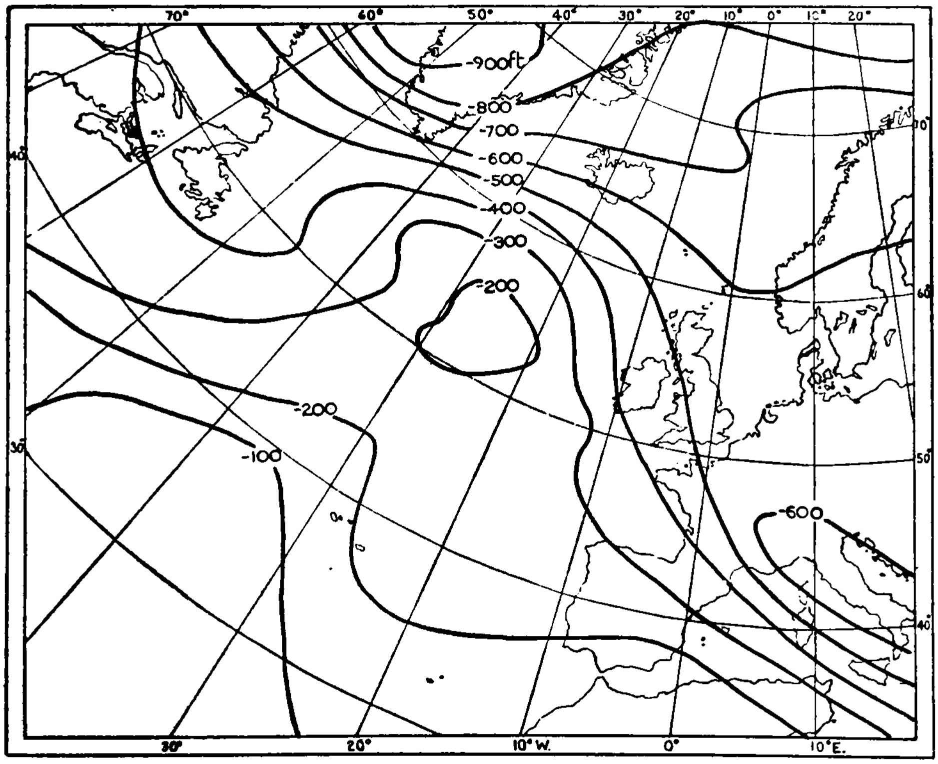

FIGURE 12. 'Original' $Z - \mathbf{Z} - G$ field, 1500 G.M.T., 8 February 1954.

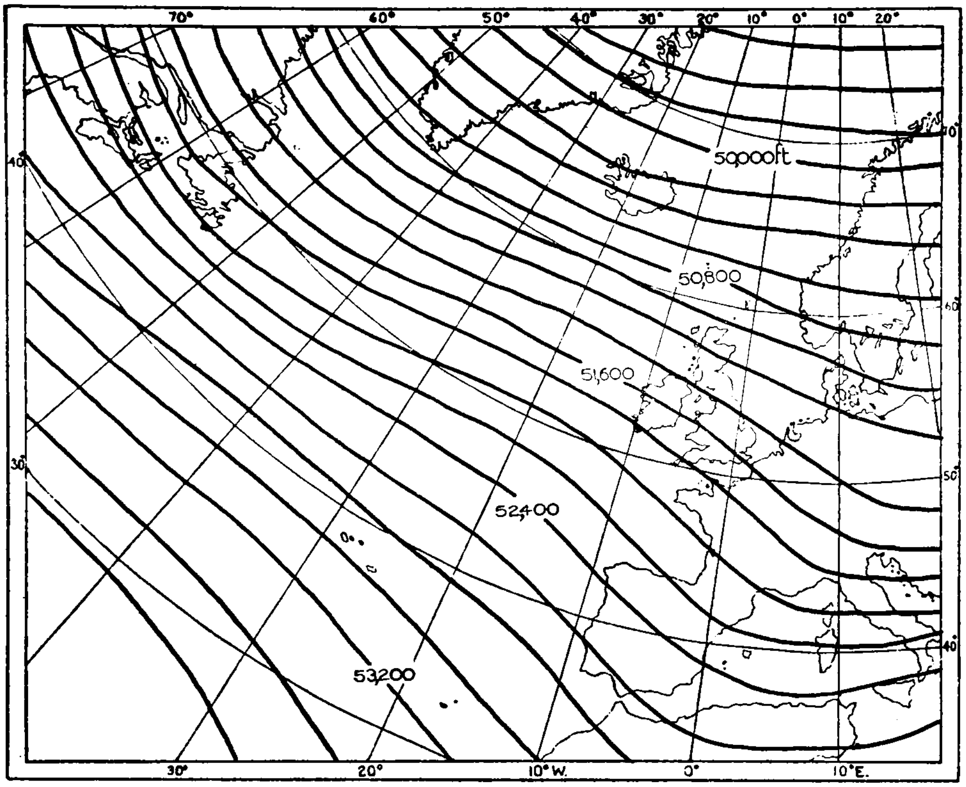

FIGURE 13. Modified Z field (100-mb contours), 1500 G.M.T., 8 February 1954.

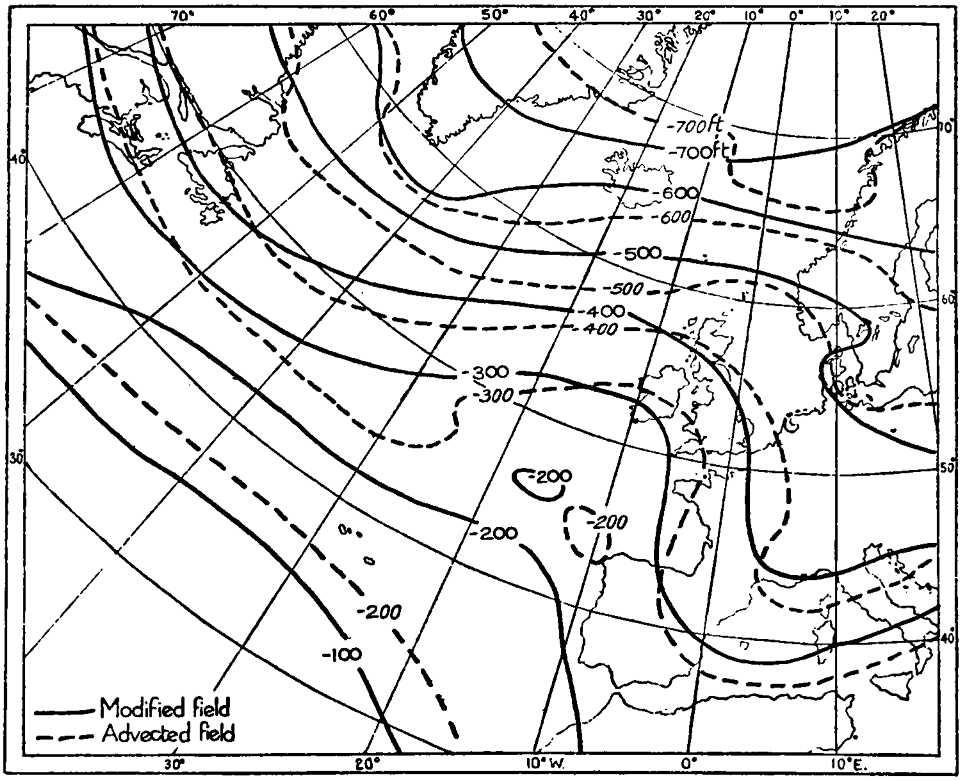

FIGURE 14. Modified and advected $Z - \bar{Z} - G$ fields, 1500 G.M.T., 8 February 1954.

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