MET O 11 TECHNICAL NOTE NO 200

A Note on the Properties of the Primitive Hydrostatic equations of motion

By

J. Norbury and M.J.P. Cullen

(Dr. J. Norbury - Maths Institute- Oxford)

Met O 11 (Forecasting Research)
Meteorological Office
London Road,
Bracknell,
Berkshire RG12 2SZ
England.

March 1985.

A NOTE ON THE PROPERTIES OF THE PRIMITIVE HYDROSTATIC EQUATIONS OF MOTION

By J NORBURY

Mathematics Institute, 24-29 St Giles, Oxford

and M. J. P. CULLEN

Meteorological Office, Bracknell

SUMMARY

Discussion of the proper boundary conditions to use in limited area forecasting models requires knowledge of the properties of the governing equations. The theory of Oliger and Sundstrom states that the primitive hydrostatic equations commonly used are not hyperbolic and no local boundary conditions can be chosen. This paper shows their result to be incorrect, the equations are hyperbolic but not all the characteristics involve the time variable. Implications for the boundary conditions are discussed.

1. INTRODUCTION

Oliger and Sundstrom (1978) analysed the properties of various systems of equations used in numerical weather prediction. They showed that the compressible non-hydrostatic equations of motion are hyperbolic, as are the shallow water equations. They derived appropriate boundary conditions for both sets. The hydrostatic and anelastic systems were shown, however, not to be hyperbolic. In the former case no choice of local boundary conditions could be made which allowed a solution to exist; in the latter case it was still possible to choose correct boundary conditions. In order to derive these negative results, an attempt was made to derive periodic eigensolutions and solve for their frequency in time. In this paper we show that the hydrostatic equations are in fact hyperbolic, but that not all the characteristics involve the time variable. This is why the method used by Oliger and Sundstrom was unable to find them. We present a simple version of the analysis in this paper and discuss the practical implications. A fuller analysis will be published elsewhere (Cullen et al (1985)).

2. THEORY

We write the governing equations using the z-coordinate system of Hoskins (1975) and make the Boussinesq and hydrostatic approximations. This gives, in standard notation:

$$\frac{Du}{Dt} - fv + \frac{\partial\phi}{\partial x} = 0 \qquad (1)$$

$$\frac{Dv}{Dt} + fu + \frac{\partial\phi}{\partial y} = 0 \qquad (2)$$

$$\frac{D\theta}{Dt} = 0 \qquad (3)$$

$$-g\theta/\theta_0 + \frac{\partial\phi}{\partial z} = 0 \qquad (4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (5)$$

An upper boundary condition is needed at z = H, corresponding to zero pressure. In this coordinate system z=0 is a constant pressure surface near the ground; so that it behaves mathematically like a free boundary.

The dependent variable in (1) to (5) is $\underline{u} = (u,v,\theta,w,\phi)$ which can be written as a function $\underline{u} = \underline{u}\,(x,y,z,t)$. The system (1) to (5) is fully hyperbolic if there exist five characteristic surfaces, say $C(x,y,z,t) = 0$, with normals

$$\underline{n} \equiv (\alpha,\ \beta,\ \gamma,\ \delta) \equiv \left[\ \frac{\partial C}{\partial x},\quad \frac{\partial C}{\partial y},\quad \frac{\partial C}{\partial z},\quad \frac{\partial C}{\partial t}\ \right] \qquad (6)$$

such that, if $\underline{u}$ is given on $C(x,y,z,t) = 0$, then $\underline{u}$ cannot be extended uniquely into the rest of (x,y,z,t) space. This definition is discussed in detail in Courant and Hilbert (1962) (Volume 2, Section 6.3, pp 578-602). Their results show that this system is fully hyperbolic if the determinant of the matrix A vanishes for five normal directions, where

$$A = \begin{pmatrix} \delta + u\alpha + v\beta + w\gamma & 0 & 0 & 0 & \alpha \\ 0 & \delta + u\alpha + v\beta + w\gamma & 0 & 0 & \beta \\ 0 & 0 & \delta + u\alpha + v\beta + w\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma \\ \alpha & \beta & 0 & \gamma & 0 \end{pmatrix} \qquad (7)$$

Each row of this matrix corresponds to one equation of the system (1) to (5), and is obtained by writing it in the form:

$$A\ \frac{\partial \underline{u}}{\partial \underline{n}} = \underline{s} \qquad (8)$$

where $\underline{s}$ involves only $\underline{u}$ and its derivatives tangential to $C = 0$. If det(A) is non zero, then (8) can be solved uniquely, and hence $\underline{n}$ may be continued from the surface $C = 0$ into the rest of (x,y,z,t) space. On the other hand, if det(A) is zero, we can only solve (8) provided $\underline{n}$ is orthogonal to the eigenvector subspace, and even then the solution $\frac{\partial \underline{u}}{\partial n}$ is not unique. Thus the surface C=0 is in this case the familiar characteristic surface of a hyperbolic system. Data cannot be specified arbitrarily on the characteristic surface and the solution $\underline{u}$ may have jump discontinuities across it. We note that the system is actually a symmetric hyperbolic system because A is symmetric; and also that z is the most "time-like" variable with (x,y,t) being most "space-like". This is because only the g component of the normal direction defined in (6) enters one of the rows of the matrix A, and it is clear that det(A) is zero if γ=0, which gives a characteristic direction parallel to the z axis and independent of t. This solution is excluded by Oliger and Sundstrom's analysis which only seeks solutions proportional to $e^{i\mu t}$.

The complete solution for the conditions under which det(A) is zero is

$$\delta + u\alpha + v\beta + w\gamma = 0 \text{ (three times)} \qquad (9)$$

or

$$\gamma = 0 \qquad \text{(twice)} \qquad (10)$$

as may be verified by elementary expansion of the determinant. Note that in case (9) the normal to the characteristic surface C=0 is orthogonal to (u,v,w,1); further, the projection of the normal onto (x,y,z) space is orthogonal to the velocity vector. Therefore the projection of C=0 onto (x,y,z) space contains instantaneous streamlines, and C=0 consists of particle paths in (x,y,z,t) space (see Courant and Hilbert (1962), Vol II, Section 6.3a for a related calculation in compressible gas flow). Comparison of this solution with that for compressible non-hydrostatic flow given by Oliger and Sundstrom (1978) shows the same three solutions of the form (9), but the two solutions corresponding to sound waves are replaced by the two solutions (10). Note also that there are no characteristics corresponding to gravity waves, these only appear if the equations are written in free surface form, or as equations for a multi-layer fluid system with layer depths as variables.

The symmetric matrix A thus leads to three characteristic surfaces, each consisting of particle paths. This means that there are three <u>ordinary</u> differential equations, holding for three quantities along the particle paths. One is obviously $\frac{D\theta}{Dt} = 0$ (equation (3)). The others reduce to local ordinary differential equations for $\partial u/\partial z$ and $\partial v/\partial z$. Because (1) and (2) are nonlinear, these quantities are not actually conserved along characteristics. Any combination of these three quantities, such as the potential vorticity, can be used instead.

There are two ordinary differential equations parallel to the z axis, one is equation (4) for $\partial\phi/\partial z$, the other is an equation for $D/Dt\ (\partial w/\partial z)$ which is not a conservation law because of the nonlinearity of the system.

3. IMPLICATIONS

This analysis gives a much simpler prescription for the correct initial and boundary data for a primitive equation model, most of which is in line with current practice. Only three variables can be given initially since only three characteristics intersect t=0. These can simply be u,v, and θ. Two conditions must be given in z, to correspond to the two characteristics parallel to the z axis. These are normally ϕ at z=0 and w at z=H, corresponding to zero pressure. In order to incorporate the conditions of no flow through the earth's surface, the boundary z=0 must be treated as free. An extra prognostic equation is written for the height of the pressure surface $z=0$ allowing an extra boundary condition to be imposed on w.

On lateral boundaries (parallel to the z axis) only three boundary conditions can be given on inflow and none on outflow. The complete solution is thus determined by giving u,v, and θ where the flow is into the domain in (x,y,t), either in across the lateral boundaries or as initial conditions, and two conditions on the upper and lower boundaries to allow ϕ and w to be determined.

The other implication is that discontinuities in components of $\underline{u}$ or its derivatives may propagate along characteristic surfaces. Fronts could be considered as discontinuities propagating along particle paths, though real fronts do not take such a simple form as sharp shear layers are unstable.

The analysis presented here suggests that the provision of correct boundary conditions for the primitive hydrostatic equations is straightforward and in line with some current practice as reviewed by Haltiner and Williams (1980) (chapter 7). Only the fluid velocity has to be considered in determining inflow and outflow points. Internal gravity waves do not appear in the analysis because the spatial derivatives associated with horizontal advection are given equal weight with those associated with divergence and vertical advection. The usual linearisation from which internal gravity waves are derived assumes strong stratification and simple vertical structure so that the internal wave speed is still large compared with the advection speed. These assumptions are clearly restrictive.

This theory does not give any guarantee of the good behaviour of the solution. Since lower order terms are ignored, the effects of rotation and gravity are omitted. They have no effect on the basic mathematical nature of the equations but a large effect on the actual solutions obtained. Different methods are required to determine what boundary conditions will allow a balanced solution or will absorb particular kinds of waves, many such are reviewed in standard textbooks.

REFERENCES

Courant, R. and Hilbert, D. 1962 'Methods of mathematical physics'. Vol. II, Interscience, New York, 830 pp.

Cullen, M.J.P., Norbury, J. and Purser, R.J. 1985 'Solutions of a Lagrangian conservation law model of atmospheric motions', in preparation.

Haltiner, S.J. and Williams, R.T. 1980 "Numerical prediction and dynamic meteorology', Wiley, New York, 477 pp.

Hoskins, B.J. 1975 "The geostrophic momentum approximation and the semi-geostrophic equations', J. Atmos. Sci., 32, 233-242.

Oliger, J. and Sundstrom, A. 1978 "Theoretical and practical aspects of some initial boundary value problems in fluid dynamics', SIAM J. Appl. Math. 35, 419-446.