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GEOMETRIC SOLUTIONS TO THE LAGRANGIAN SEMI-GEOSTROPHIC EQUATIONS

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Geometric solutions to the Lagrangian Semi-Geostrophic Equations.

Summary

An algorithm is described which produces a solution to the two dimensional semi-geostrophic equations using piecewise constant data. Subject to the limitations of such data, the solution is exact and can be used as a reference for comparison with numerical models.

1. INTRODUCTION

This paper describes an algorithm to solve, for piecewise constant data, the two dimensional semi-geostrophic equations in conservation form, in the presence of the deformation field.

$u = -\alpha x, \; v = \alpha y$

The equations are as derived by Hoskins and Bretherton (1972);

$$\frac{DM}{Dt} + \alpha M = 0$$

$$\frac{D\theta}{Dt} = 0$$

$$\frac{DA}{Dt} + \alpha A = 0$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u\partial}{\partial x} + \frac{w\partial}{\partial z}$, $M = v + fx$,

θ is the potential temperature, and A is the area of a fluid element in the (x, z) plane.

The equations are solved for a pressure variable ϕ, such that

$$\phi = \phi g + {}^1/_2 fx^2$$

where ϕg is the geopotential. Thus the hydrostatic equation and the equation for cross front geostrophic balance give

$$\frac{\partial \phi}{\partial z} = \theta$$

$$\frac{\partial \phi}{\partial x} = M$$

In Purser and Cullen (1983), (referred to hereafter as PC), the symetric, inertial and static stability of the solution is shown to be analagous to the convexity of the function ϕ up until the formation of a discontinuity. However for a solution involving a discontinuity, the analytic stability criterion which involves derivatives cannot be defined. Therefore stability is redefined purely in terms of the convexity of the solution surface of ϕ. This allows the calculation to be taken beyond the formation of the front.

Thus the problem is reduced to a geometric construction of a convex surface, ϕ (x,z), as follows.

The initial data consists of a bounded region Ω, in (x, z) space over which the problem is considered;

$$\Omega = \{(x,z) | X_{min} \leq x \leq X_{max}, Z_{min} \leq z \leq Z_{max}\}$$

Ω is partitioned into a given number, N, of fluid elements Ωi, for each of which the potential momentum M_i, the potential temperture θ_i and A_i, the area of the element are prescribed.

Since

$$M_i = \frac{\partial \phi}{\partial x} \quad \text{and} \quad \theta_i = \frac{\partial \phi}{\partial z} \quad \text{for } (x,z) \in \Omega_i$$

we have

$$\phi = x.M_i + z.\theta_i + S_i \text{ for } (x,z) \in \Omega_i$$

where S_i is a constant of integration.

Define

$$\phi_i = x.M_i + z.\theta_i + S_i \text{ for all } (x,z) \in \Omega$$

Given the initial data and some first guess,

$$\{S_i | i=1,..,N\}$$

a shell ϕ' can be constructed such that

$$\phi'(x,z) = \max_{i=1,..,N} \{\phi_i\ (x,z)\} \text{ for } (x,z) \in \Omega$$

For each i, this defines an area

$$A_i' = \iint_{\Omega_i} dxdz \text{ where } \Omega_i' = \{(x,z) | \phi'(x,z) = \phi(x,z)\}$$

We now seek to obtain the correct areas A_i, by modifying the S_i. If there is some i such that $A_i' \neq A_i$, then adjust $\{S_i | i=1,..,N\}$ and rederive $\phi'(x,z)$ until $A_i' = A_i$ for all $i=1,...,N$.

It is known that there is a unique way of achieving this (PC), therefore;

$$\phi'(x,z) = x.M_i + z.\theta_i + S_i \quad \Leftrightarrow (x,z) \in \Omega i$$

Then let $\phi(x,z) = \phi'(x,z)$ and the solution is obtained.

In order to describe the algorithm, the following definitions are made:

The shell is divided into N segments;

$$F_i = \{(x,z,\phi'(x,z)) | \phi'(x,z) = \phi_i(x,z)\}$$

A fluid element Ω_i is the projection of F_i onto the plane ϕ=o;

$$\Omega_i = \{(x,z,o) | (x,z,\phi'(x,z)) \in F_i\}$$

(see Figure 1).

Such points on ϕ = o will henceforth be described by the coordinates (x,z) which should be read as (x,z,o).

The intersection between two elements defines an edge;

$$E_{ij} = \Omega_i \cap \Omega_j \text{ for distinct } i,j$$

and the intersection of three elements defines a vertex

$$V_{ijk} = \Omega_i \cap \Omega_j \cap \Omega_k \text{ for distinct } i,j,k.$$

Partition the boundary of Ω, $\partial\Omega$ into the following

$$\partial_1 = \{(x,z) \mid x = X_{min},\ Z_{min} \leqslant z \leqslant Z_{max}\}$$

$$\partial_2 = \{(x,z) \mid z = Z_{min},\ X_{min} \leqslant x \leqslant X_{max}\}$$

$$\partial_3 = \{(x,z) \mid x = X_{min},\ Z_{min} \leqslant z \leqslant Z_{max}\}$$

$$\partial_4 = \{(x,z) \mid z = Z_{max},\ X_{min} \leqslant x \leqslant X_{max}\}$$

(see Figure 2).

2. GEOMETRIC CONSTRUCTION

This section describes how the convex shell is constructed, given values $\{S_i \mid i=1,..,N\}$. The strategy followed is to consider recursively, a set of elements B_r, moving outwards until the whole shell is covered. Having previously obtained B_r, B_{r+1} comprises all adjacent elements to one or more members of B_r which are not themselves members and whose areas have not previously been calculated.

Let ΣA be a cumulative total of the areas calculated. Then when

$$\Sigma A = \iint_{\Omega} dxdz$$

the construction is complete.

In order to initialise the recursive procedure, an arbitrary point (x_o, z_o) is chosen on $\partial\Omega$. The element to which this point belongs is found by

$$(x_o, z_o) \in \Omega_i \Leftrightarrow \phi_i(x_o, z_o) = \max\{\phi_j(x_o, z_o)\}$$

If Ω_i is not unique, then another (x_o, z_o) is chosen until the point belongs uniquely to some element.

This element then becomes the sole member of B_o. B_1 comprises the neighbours of Ω_i, all of which are isolated in the calculation of Ω_i. The recursion then proceeds as previously stated. See diagram 1.

CONSTRUCTING AN ELEMENT

To calculate the area of an element, its geometric location must first be described. The members of Ω_i are uniquely defined by the members of the finite set of vertices of that element:

$$C_i = \{(x_{ij}, z_{ij}) | j=1,...,N_i\}$$

$$\subset \{V_{ijk} | j,k=1,..N; j \neq k; \ j,k, \neq i\} \cup \{E_{ij} \cap \partial\Omega | i \neq j; j=1,..,N\}$$

In order to construct C_i, one of its members, (x_1, z_1) say, must be known. Some edge of Ω_i incident upon (x_1, z_1), whether an edge with an adjacent segment, or part of $\partial\Omega$, must also be specified. In the general case, if $\Omega_i \in B_r$, both items of data can be determined from information derived in the calculation of elements which are now members of B_{r-1}. In the initial case when $r = 0$, (x_1, z_1) is determined by the intersection of some edge E_{ij} with the boundary on which (x_o, z_o) lies.

Given (x_1, z_1) and some specified side of Ω_i, this side is followed until the next vertex (x_2, z_2) is reached, and the next side to be followed becomes the edge or boundary whose intersection with the first side defines (x_2, z_2). This procedure is followed until some $(x_m, z_m) = (x_1, z_1)$ when the calculation of C_i is complete. See diagram 2.

Having found (x_m, z_m), the side of Ω_i specified, ℓ say, is either part of the boundary or an edge E_{ij} for some $j \in B_{r-1}$.

Consider the case when $\ell = E_{ij}$ (see Figure 3).

We must first find the point $(x',y') \in \partial\Omega$, which, should no intersection with an edge exist, would be the next vertex. The two points (x_n',z_n') $n=1,2$, where E_{ij} intersects $\partial\Omega$ are located. If the previous side, ℓ', was ∂a, some a, then (x_m,z_m) will be one of these points and hence (x',z') is trivially determined.

If ℓ' was E_{ik}, for some $k \neq i, j$; then $(x',z') = (x_n',z_n')$ satisfying the inequality

$$\phi_i(x_n',z_n') > \phi_k(x_n',z_n')$$

However if (x_m,z_m) is the initial point considered on the segment, then k is not known. Such a k can be found by comparing (x_m,z_m) and $V_{ijk'}$; $k'=1,..,N$; $k'\neq j,i$.

The next step is to consider the intersections with E_{ij} of all lines of the form E_{ih} for $h=1,..,N$; $h\neq i,j$.

First isolate those edges $\{E_{ih} \mid n=1,..,T\}$ some T, such that V_{ijh} lies between (x_m,z_m) and (x',z'), or equals (x',z'). If there exists an h' such that

$$|(x_m,z_m) - V_{ijh'}| < |(x_m,z_m) - (x,z)|$$

for all $(x,z) \in \{V_{ijh_n} \mid n=1,..,T;\ h_n \neq h'\} \cup \{(x',z')\}$

then $(x_{m+1}, z_{m+1}) = V_{ijh'}$ and the next edge to consider is $E_{ih'}$. If $|(x_m, z_m) - V_{ijh_n}| > |(x_m, z_m) - (x', z')|$ for all h_n, n=1,..,T then $(x_{m+1}, z_{m+1}) = (x', z')$ and the next side of the element is the boundary upon which (x', z') lies.

However if (x', z') is the intersection of two boundaries, then comparing ϕ_i and ϕ_j at some point of known displacement along one of the boundaries will determine that which should be followed.

The other possible outcome is that of a multiple vertex. That is when $V_{ijk} = V_{ijl}$ for distinct i,j,k,l; or $V_{ijk} = \partial_a \cap \partial_b$.

In this case the basic strategy is to compare the gradients $(\partial z / \partial x)$ of the edges and boundaries considered. The problem can then be split into three cases. The case where the side of the element being followed is a boundary is relatively simple. If however the side is some E_{ij}, then it is first determined in which direction the next side is to be followed i.e. is the element being constructed in a clockwise or an anticlockwise sense. This is done by rotating (x_m, z_m) about (x', z') to the point (x_r, z_r) say, and then comparing the values of $\phi_i(x_r, z_r)$ and $\phi_j(x_r, z_r)$. The cyclic orders of the gradients are then compared to obtain the required edge or boundary.

The above describes the construction of a side, ℓ, of Ω_i when that side is E_{ij} for some j. If $\ell = \partial a$ for some a=1,..,4 then the method is similar to that above.

3. ITERATIVE SCHEME

Initially an approximation to the data is made which allows an analytic solution to be obtained. This is used to initialise the set

$K = \{S_i | i=1,..,N\}$

Having constructed the shell ϕ', say the areas of the fluid elements are;

$\{A_i' | i=1,..,N\}$

If $A_i'=A_i$ for all $i=1,..,N$, then the solution has been obtained. If there is some i such that $A_i' \neq A_i$, then K must be adjusted by some means.

The problem is non-linear as the rate of change of area of an element with s, $\partial A_i / \partial s_i$ changes as its neighbours change. Therefore a simple matrix inversion will not suffice.

The method used is a two step iteration. Find j such that

$|A_j'-A_j| \geq |A_i'-A_i|$ for all $i=1,..,N$. If $A_j'=0$ or $\sum_{i=1}^{N} A_i$, then S_j is successively incremented or decremented respectively until this is not the case.

Then, in order to obtain a linearized approximation to the rate of change of area with s, s_j is incremented by C, a multiple of $(A_j - A_j')$. The new shell is then constructed. The approximation is then made;

$$\frac{\partial A_i}{\partial s_i} = \frac{A_j'_{new} - A_j'_{old}}{C} \qquad \text{for all } i=1,..,N$$

and an increment is calculated for each S_i;

$$\text{Increment} = \lambda \,.\, \frac{A_i - A_i'_{new}}{\partial A_i/\partial s_i}$$

for some constant λ.

The geometry is then calculated and the interation is repeated until

$|A_j' - A_j| < \epsilon$ for all $j=1,..,N$ for some predefined ϵ.

4. RESULTS

The method was applied to a model of a front using an initial distribution of θ following an arctangent curve with respect to x, where the x axis bisects Ω. M is considered linear in x.

We have

$$\theta = 300 + \tan^{-1}x \qquad (1)$$

$$M = 10^5.f.x \qquad (2)$$

for $x \in [-5,5]$

The data was converted to piecewise constant in a manner such that element boundaries represented isotherms at intervals of 0.1. From (1)

$$x = \tan(\theta - 300) \qquad (3)$$

therefore for any element, Ω_i, a value of x can be ascribed to each of its bounding isotherms, say x_L, x_R.

Then θ_i is calculated by averging θ over Ω_i,

$$\theta_i = \frac{1}{x_R - x_L} \int_{x_L}^{x_R} \theta \, dx$$

This can be solved analytically using (1).

Thus from (2) and (3), a value of M can be calculated for the element

$$M_i = 10^5.f.\tan(\theta_i - 300)$$

The area of the element is taken as

$$A_i = (X_R - X_L).H$$

where H is the height of Ω.

See Figure 4.

Subsequently at time t, A_i and M_i are multiplied by $\exp(-\alpha t)$ where α is the deformation constant of the field. Ω is made to shrink at the same rate by moving the left and right boundaries inwards.

Figures 5, 6, 7 show the initial distribution of the fluid elements and then progressive deformation of the field causing the formation of two distinct discontinuities within the fluid.

These can be compared with results from a semi-geostrophic finite difference model (see Cullen and Purser 1984).

REFERENCES

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Purser and Cullen; "Some General Properties of the semi-geostrophic equations inferred by Geometrical Considerations".
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To be published JAS, 1984.

DIAGRAM 1

<u>CONSTRUCTION OF Ω</u>

Find some Ω_i and let $B_o=\{\Omega_i\}$

Construct members of B_o, thus

defining B_1. Let r=o.

Let r=r+1

Construct members of B_r

Let $B_{r+1} = \{\Omega_i$ adjacent to members of $B_r\}$

No.——Is Ω covered by elements constructed so far?

Yes

Construction of Ω complete.

DIAGRAM 2

CONSTRUCTION OF $\Omega_I \in B_r$

Starting from (x_1, z_1) follow

E_{ij} where $j \in B_{r-1}$

Find closest intersection

(x_m, z_m)

Is $(x_m, z_m) = (x_1, z_1)$? — Yes $\longrightarrow$ Construction of Ω_i complete

No

Is $(x_m, z_m) = E_{ij} \cap E_{ik}$? or $(x_m, z_m) = E_{ij} \cap \partial_\ell$?

Some k | Some ℓ

Follow edge E_{ik} | Follow boundary ∂_ℓ

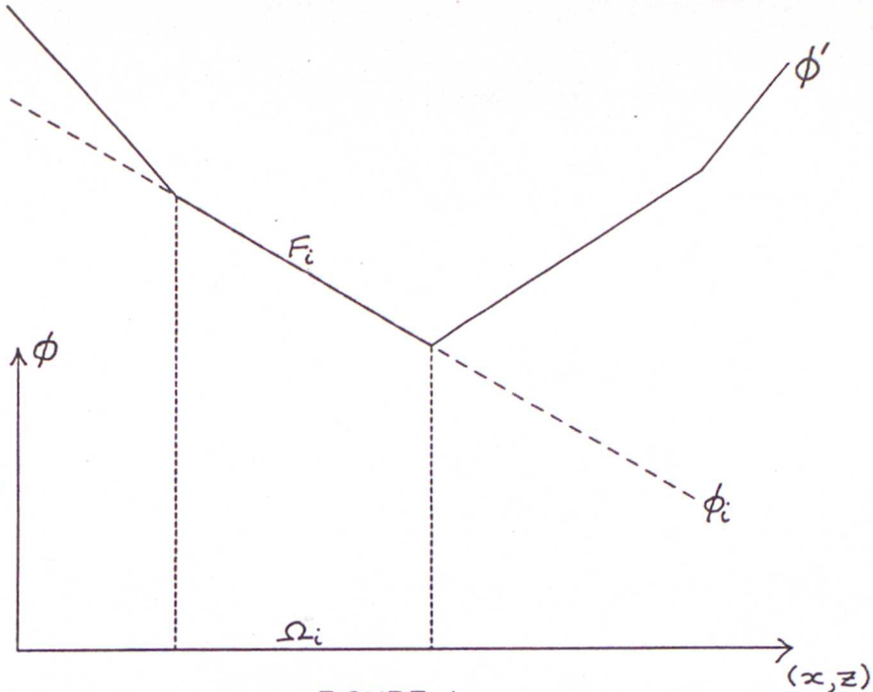

FIGURE 1

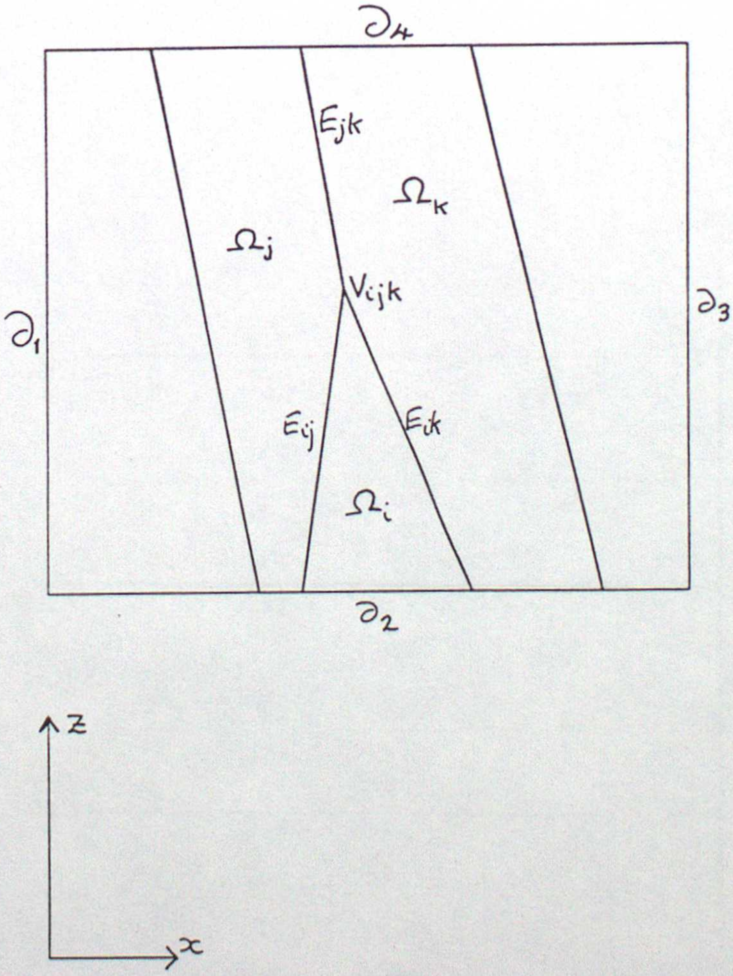

FIGURE 2

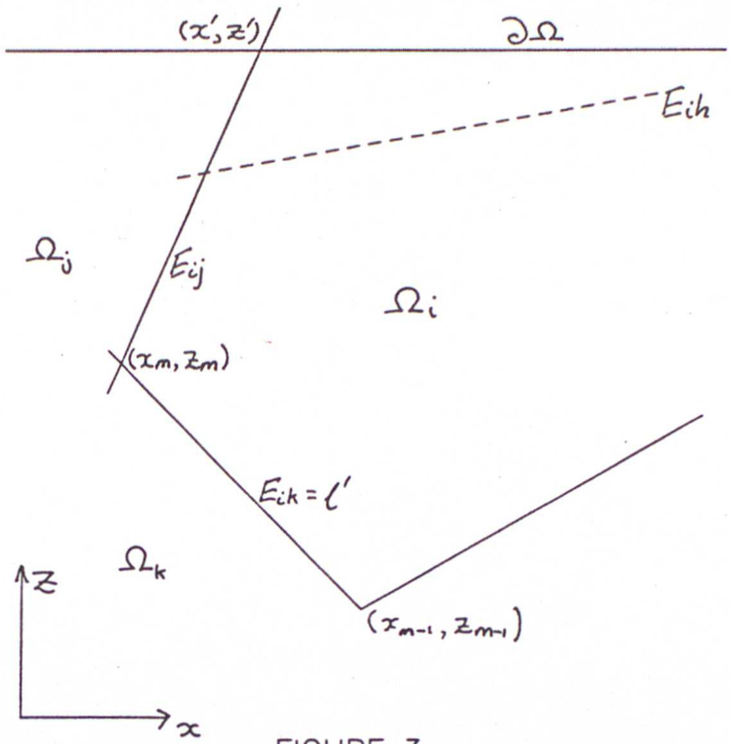

FIGURE 3

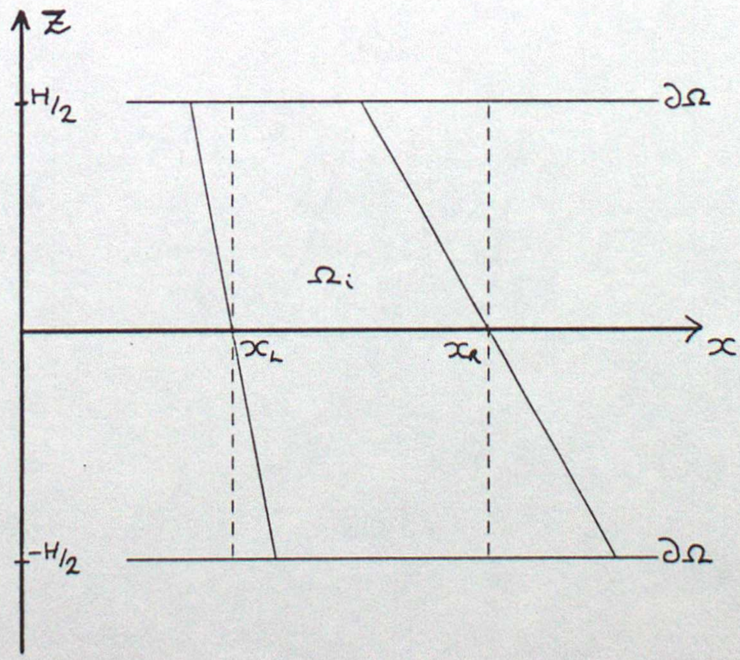

FIGURE 4

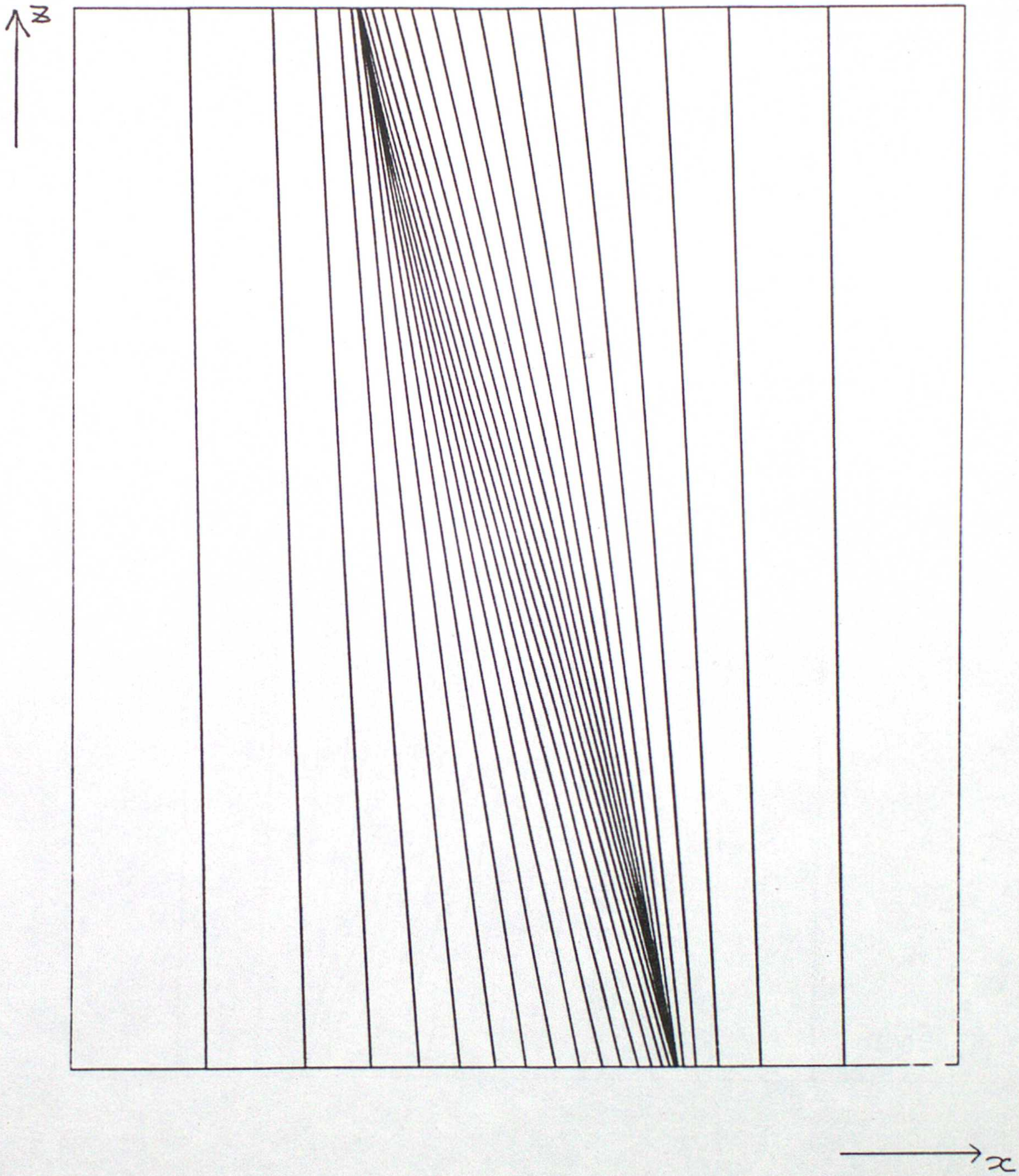

FIGURE 5 TIME = 0 SECS.

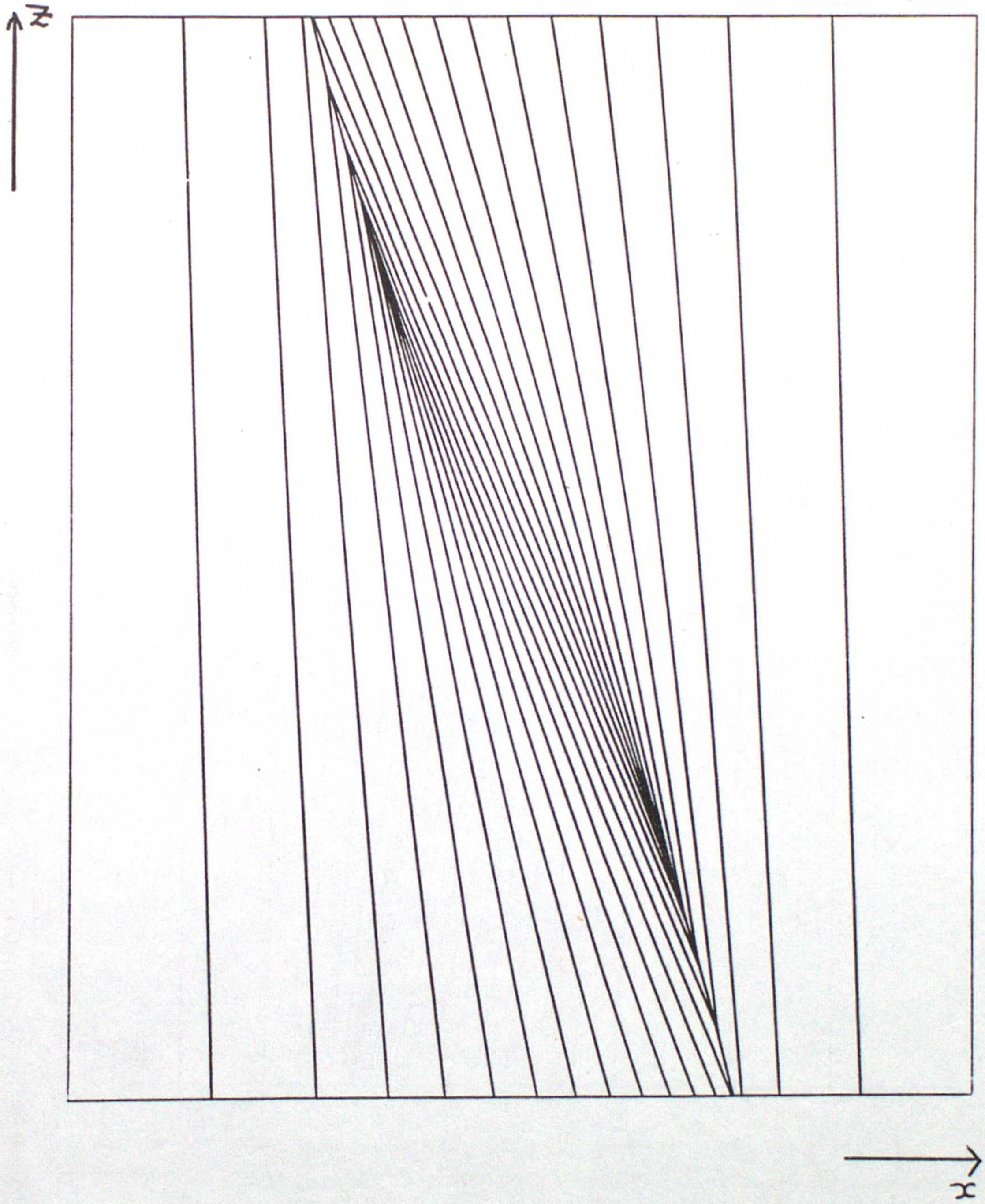

Figure 6 Time = 20,000 Secs.

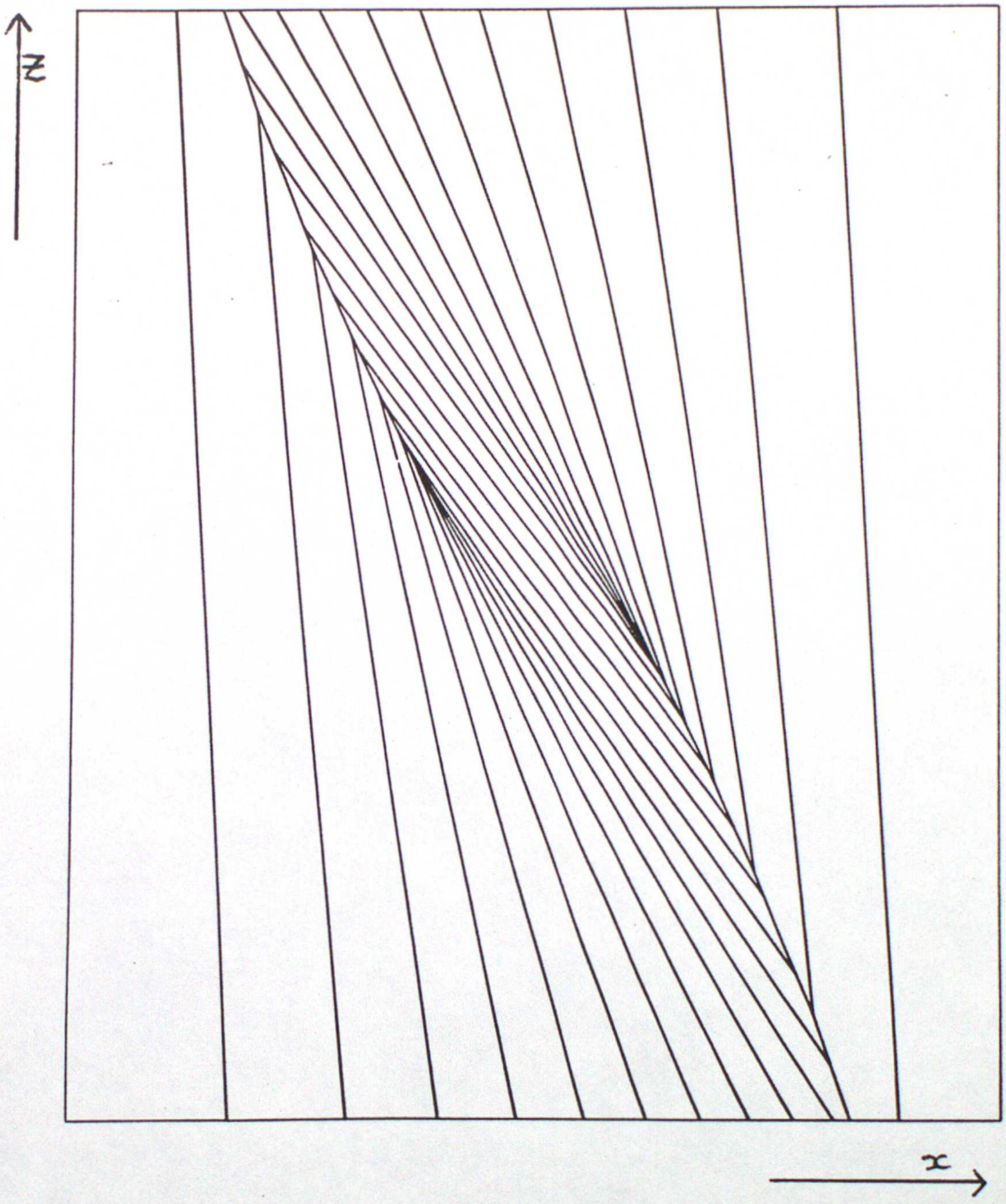

FIGURE 7 TIME = 50,000 SECS.