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THE CORIOLIS FORCE IN GLOBAL ATMOSPHERIC MODELS:

I. THE NAVIER-STOKES EQUATION AND THE HYDROSTATIC PRIMITIVE EQUATIONS

by

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Summary

The spherical polar components of the Coriolis force consist of terms in $\sin\phi$ and terms in $\cos\phi$, where ϕ is latitude (referred to the frame rotation vector as polar axis). The $\cos\phi$ Coriolis terms are not retained in the usual hydrostatic primitive equations of numerical weather prediction and climate simulation, their neglect being consistent with the shallow atmosphere approximation and the simultaneous exclusion of various small metric terms. Scale analysis for diabatically-driven, synoptic-scale motion in the tropics suggests that the $\cos\phi$ Coriolis terms may attain magnitudes of order 10% of those of key terms in the hydrostatic primitive equations. It is argued that the $\cos\phi$ Coriolis terms should be included in global simulation models, but a review of the conservation properties of the hydrostatic primitive equations suggests the high standards against which any more accurate approximation of the Navier-Stokes equations should be judged.

1. Introduction

Numerical weather prediction and global climate models seek to simulate the behaviour of the atmosphere by using accurate representations of the governing equations of motion, thermodynamics and continuity. These equations contain terms describing intrinsic fluid dynamical processes such as advection and the pressure gradient force, and terms representing sources or sinks of momentum and heat. The latter may be referred to as forcing terms; they represent the divergence of subgridscale fluxes, and diabatic forcing due to radiative flux divergence and latent heating.

Over the past 30 years much attention has been paid to improving the representation of the forcing processes, especially in the context of climate simulation using global circulation models. Over the same period much progress has also been made in refining time integration schemes and in accommodating increased spatial resolutions. The emergence of competitively economic spectral models has been a notable development. There has, however, been no change in the assumed representation of the intrinsic fluid dynamical processes: almost all global models are based on the hydrostatic primitive equations (HPEs). (The exceptions are models based on various geostrophically balanced approximations of the HPEs. These models are used for the important scientific purpose of developing comprehension of atmospheric behaviour, and the more sophisticated of them may offer forecast accuracy rivalling that of HPE models.)

The HPEs are simpler in several respects than the complete equations of motion. In addition to the neglect of vertical accelerations in the momentum balance, the HPEs use a spherical approximation to the spheroidal geometry of geopotential surfaces and assume the shallow atmosphere approximation. Various metric terms are neglected. Of greater quantitative importance, the Coriolis terms involving $2\Omega\cos\phi$ which appear in the zonal and vertical components of the momentum equation are omitted. (Here Ω is the rotation rate of the Earth and ϕ is latitude). The omission of these <u>$\cos\phi$ Coriolis terms</u> is the "traditional approximation", which has been a matter of controversy in the background of dynamical meteorology for many years (see, for example, Eckart 1960, Phillips 1966, 1968, 1973, Veronis 1968 and Wangsness 1970). (As stressed in section 3, a close relation exists between the shallow atmosphere approximation, the neglect of certain metric terms and the omission of the $\cos\phi$ Coriolis terms. Taken together, but not individually, they constitute a dynamically consistent approximation which implies satisfactory analogue forms of energy, angular momentum and potential vorticity conservation. The term "traditional approximation" is customarily reserved for the omission of the $\cos\phi$ Coriolis terms, however).

The $\cos\phi$ Coriolis terms have recently been considered in a number of studies in meteorology, oceanography and geophysical fluid dynamics. Leibovich and Lele (1985) included them in a comprehensive investigation of Ekman layer stability, and Mason and Thomson (1987) retained them in a numerical simulation of boundary layer eddies. Garwood, Gallacher and Muller (1985) considered the importance of the

$\cos\phi$ Coriolis terms in the turbulent kinetic energy budget of the oceanic surface mixed layer. The model of planetary geostrophic motion proposed by Shutts (1989) includes the terms after consistent approximation of the Lagrangian function. Burger and Riphagen (1991) retained the terms (and others not included in the HPEs) in a study of the equations of motion expressed in an arbitrary vertical coordinate system. Draghici (1987), (1989) has argued that the $\cos\phi$ Coriolis terms represent the most important nonhydrostatic effect in mesoscale atmospheric systems. He proposed ways of including the $\cos\phi$ Coriolis terms in formulations which use tangent-plane and other geometric approximations to the Earth's sphericity; analogues of energy and potential vorticity conservation laws are retained.

In this paper we examine the importance of the $\cos\phi$ Coriolis terms for synoptic scale motion in the tropics, and conclude that they may not be negligible if diabatic processes are of first order importance in the thermodynamic equation and are balanced by vertical advection of potential temperature. We recommend that consideration be given to the inclusion of the $\cos\phi$ Coriolis terms in simulation models of the global atmosphere, but emphasize that the good conservation properties of the HPEs set a high standard against which any new set of equations should be judged. Ways of representing the $\cos\phi$ Coriolis terms in acoustically-filtered models of a global, compressible atmosphere are described in Part II of this study (White and Bromley 1994).

A list of symbols is given in section 2. Components of the Navier-Stokes equation and the corresponding HPE forms are reviewed in section 3, with special attention to conservation properties. Section 4 considers the quantitative and qualitative importance of the $\cos\phi$ Coriolis terms, and a concluding discussion follows in section 5.

2. List of symbols and definitions

a Earth's mean radius

c_v Specific heat at constant volume

c_p Specific heat at constant pressure

f Coriolis parameter: $2\Omega\sin\phi$

$\underline{g}$ Acceleration due to apparent gravity

g Magnitude of $\underline{g}$

$\underline{i}$ Unit vector in zonal direction

$\underline{j}$ Unit vector in meridional direction

$\underline{k}$ Unit vector in direction of apparent vertical (direction of $-\underline{g}$)

p Pressure

p_o A constant reference surface pressure

$\underline{r}$ Position vector relative to centre of Earth

r Distance from centre of Earth

t Time

$\underline{u}$ Velocity (relative to Earth)

u Zonal component of $\underline{u}$

$\underline{v}$ Horizontal part of $\underline{u}$ (= (u, v, 0))

v Meridional component of $\underline{u}$

w Vertical component of $\underline{u}$

z Height above mean sea level

D Scale height of reference atmosphere (= RT_s/g)

E $2\Omega U\cos\phi/g$

$\underline{F} = (F_\lambda, F_\phi, F_r)$ Frictional force per unit mass

$\underline{F}_h$ Horizontal part of $\underline{F}$ (=$(F_\lambda, F_\phi, 0)$)

H Vertical space scale

L Horizontal space scale

N Buoyancy frequency: $((g/\theta)d\theta/dz)^{1/2}$

Q Diabatic heating rate per unit mass

R Gas constant per unit mass

Ro Rossby number: U/fL

T Temperature

U Horizontal velocity scale

W Vertical velocity scale

$\underline{Z} = (Z_\lambda, Z_\phi, Z_r)$ Absolute vorticity (see (3.12))

$\underline{\zeta}$ Absolute vorticity in HPE models (see (3.26))

θ Potential temperature: $T(p_o/p)^{R/c_p}$

λ Longitude

π A typical atmospheric pressure

ρ Density

τ Time scale

ϕ Latitude

Φ Geopotential

$\underline{\Omega}$ Angular velocity of Earth's rotation

∇ Shallow atmosphere gradient operator (in height coordinates): see (3.20), (3.24) and (3.26).

3. Components of the Navier-Stokes equation and the hydrostatic primitive equations

The components of the Navier-Stokes equation in a spherical polar coordinate system are derived in standard texts (see, for example, Gill (1982)). The HPEs and their conservation properties are discussed in detail by Lorenz (1967), Phillips (1973) and Hoskins *et al* (1985), as well as in textbooks. Only those aspects which are important in the present study are reviewed here.

(a) The Navier-Stokes equation

If velocities $\underline{u}$ are measured relative to a system rotating with angular velocity $\underline{\Omega}$, the Navier-Stokes equation is

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega}x\underline{u} - \underline{g} + \frac{1}{\rho}\,\mathrm{grad}p = \underline{F} \qquad (3.1)$$

Here $\underline{g}$ is apparent gravity (true gravity plus the centrifugal term $\underline{\Omega}$ x ($\underline{\Omega}$ x $\underline{r}$), where $\underline{r}$ is position relative to the centre of the Earth); $\underline{g} = -\,\mathrm{grad}\Phi$, where Φ is the geopotential. The three spherical polar components of (3.1) are

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) + \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} = F_\lambda \qquad (3.2)$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial\phi} = F_\phi \qquad (3.3)$$

$$\frac{Dw}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)u\cos\phi - \frac{v^2}{r} + g + \frac{1}{\rho}\frac{\partial p}{\partial r} = F_r \qquad (3.4)$$

Here

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial r} = \frac{\partial}{\partial t} + \underline{u}.\mathrm{grad} \qquad (3.5)$$

and u, v, w are the components of $\underline{u}$ in the λ (longitude), ϕ (latitude) and r (radial) directions; the polar axis is in the direction of $\underline{\Omega}$. See Fig. 1 for the coordinate configuration and section 2 for a list of symbols.

In fact, some approximations have been made in writing Eqs (3.3) and (3.4). The magnitude, g, of apparent gravity ($\underline{g}$) appears in Eq. (3.4), and no component of $\underline{g}$ in Eq. (3.3), because the adopted spherical coordinate system is an approximation to the spheroidal system defined by geopotential surfaces in the rotating frame. (See Phillips (1973), Gill (1982) and White (1982)). The replacement of spheroidal geopotentials by spheres is conceptually important, but it involves only a slight geometric distortion in the terrestrial case since $\Omega^2 r/g \leq 3 \times 10^{-3}$ in the troposphere and stratosphere.

When taken together with the continuity and thermodynamic equations

$$\frac{D\rho}{Dt} + \rho\ \mathrm{div}\ \underline{u} = 0 \qquad (3.6)$$

$$\frac{D\theta}{Dt} = \left(\frac{\theta}{Tc_p}\right) Q \qquad (3.7),$$

Eqs. (3.2)-(3.5) imply the following conservation laws for axial angular momentum, energy and potential vorticity:

$$\rho\frac{D}{Dt}\left[(u + \Omega r\cos\phi)r\cos\phi\right] = \rho F_\lambda r\cos\phi - \frac{\partial p}{\partial\lambda} \tag{3.8}$$

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\underline{u}^2 + \Phi + c_v T\right) + \mathrm{div}\,(p\underline{u}) = \rho(Q + \underline{u}.\underline{F}_h) \tag{3.9}$$

$$\rho\frac{D}{Dt}\left(\frac{\underline{Z}.\mathrm{grad}\theta}{\rho}\right) = \underline{Z}.\mathrm{grad}\,\frac{D\theta}{Dt} + \mathrm{grad}\,\theta.\mathrm{curl}\,\underline{F} \tag{3.10}$$

Perfect gas behaviour, $p = \rho RT$, has been assumed. In Eq. (3.10), $\underline{Z}$ is the absolute vorticity, curl $\underline{u} + 2\underline{\Omega}$. The components of $\underline{Z}$ in the $(\lambda,\ \phi,\ r)$ system are

$$\left.\begin{aligned} Z_\lambda &= \frac{1}{r}\frac{\partial w}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rv) \\ Z_\phi &= 2\Omega\cos\phi + \frac{1}{r}\frac{\partial}{\partial r}(ru) - \frac{1}{r\cos\phi}\frac{\partial w}{\partial\lambda} \\ Z_r &= 2\Omega\sin\phi + \frac{1}{r\cos\phi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\phi}(u\cos\phi)\right) \end{aligned}\right\} \tag{3.11}$$

The divergence of a vector field $\underline{A} = (A_\lambda, A_\phi, A_r)$ is

$$\mathrm{div}\,\underline{A} = \frac{1}{r\cos\phi}\left(\frac{\partial A_\lambda}{\partial\lambda} + \frac{\partial}{\partial\phi}(A_\phi\cos\phi)\right) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) \tag{3.12}$$

and the gradient of θ is the vector

$$\mathrm{grad}\,\theta = \left(\frac{1}{r\cos\phi}\frac{\partial\theta}{\partial\lambda},\ \frac{1}{r}\frac{\partial\theta}{\partial\phi},\ \frac{\partial\theta}{\partial r}\right) \tag{3.13}$$

The notation 'grad', 'div' and 'curl' will be reserved for the quantities defined by (3.13), (3.12) and (3.11). Shallow atmosphere versions of the respective operators are indicated by '∇' (see below).

(b) Height coordinate forms of the HPEs

The hydrostatic primitive equations (HPEs) which correspond to Eqs. (3.2)-(3.4) are

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{a\cos\phi}\right) v\sin\phi + \frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda} = F_\lambda \tag{3.14}$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{a\cos\phi}\right) u\sin\phi + \frac{1}{\rho a}\frac{\partial p}{\partial\phi} = F_\phi \tag{3.15}$$

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \tag{3.16}$$

with

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \underline{u}.\nabla \tag{3.17}$$

The terms omitted are those in $2\Omega\cos\phi$ (the $\cos\phi$ Coriolis terms), four metric terms, the vertical acceleration Dw/Dt and the vertical component F_r of the frictional force per unit mass. The shallow atmosphere approximation has also been made: r has been replaced by a, the Earth's mean radius, except in the derivative terms $\partial/\partial r$ which are retained as $\partial/\partial z$, z being height above mean sea level. In Eq. (3.16), g is to be understood as an appropriate mean magnitude of $\underline{g}$, independent of position and time.

The remaining Coriolis terms in (3.14) and (3.15) may be written in the familiar form $f\underline{k}\times\underline{v}$ if a horizontal vector equation is constructed. Here $f = 2\Omega\sin\phi$, $\underline{k}$ is unit vector in the local vertical and $\underline{v}$ is the horizontal part of $\underline{u}$. The HPE continuity and thermodynamic equations are

$$\frac{D\rho}{Dt} + \rho\, \nabla.\underline{u} = 0 \tag{3.18}$$

$$\frac{D\theta}{Dt} = \left(\frac{\theta}{Tc_p}\right) Q \tag{3.19}$$

Here D/Dt is defined as in (3.17), and

$$\nabla.\underline{u} = \frac{1}{a\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right) + \frac{\partial w}{\partial z} \tag{3.20}$$

Analogues of the conservation properties (3.8)-(3.10) are implied by the HPEs. The axial angular momentum principle is

$$\rho\frac{D}{Dt}\left[(u + \Omega a\cos\phi)\, a\cos\phi\right] = \rho F_\lambda a\cos\phi - \frac{\partial p}{\partial\lambda}\ , \tag{3.21}$$

the energy conservation law is

$$\rho\frac{D}{Dt}\left(\ \frac{1}{2}\,\underline{v}^2 + gz + c_vT\ \right) + \nabla.(p\underline{u}) = \rho(Q + \underline{v}.\underline{F}_h)\ , \tag{3.22}$$

and the potential vorticity law is

$$\rho\frac{D}{Dt}\left(\frac{\underline{\zeta}.\nabla\theta}{\rho}\right) = \underline{\zeta}.\nabla\left(\frac{D\theta}{Dt}\right) + \nabla\theta.\nabla\mathrm{x}\underline{F}_h\ . \tag{3.23}$$

Here $\quad \text{grad}\ \theta = \left(\frac{1}{a\cos\phi}\frac{\partial\theta}{\partial\lambda}\ ,\ \frac{1}{a}\frac{\partial\theta}{\partial\phi}\ ,\ \frac{\partial\theta}{\partial r}\right) \qquad (3.24)$

and $\quad \underline{\zeta} \equiv 2\Omega\underline{k}\sin\phi + \nabla\mathrm{x}\underline{v}\ , \qquad (3.25)$

with $\nabla\mathrm{x}\underline{v} \equiv \left(-\frac{\partial v}{\partial z},\ \frac{\partial u}{\partial z},\ \frac{1}{a\cos\phi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\phi}(u\cos\phi)\right)\right) \qquad (3.26)$

A proof of the potential vorticity law (3.23) is given in the Appendix. Although (3.23) is a celebrated result (see, for example, Hoskins _et al_ (1985)) so far as we are aware no direct derivation of it from the HPEs (3.14)-(3.19) has previously appeared in the literature.

The conservation properties (3.21), (3.22) and (3.23) depend on the simultaneous application of the shallow atmosphere approximation, the neglect of four metric terms and the neglect of the $\cos\phi$ Coriolis terms. Consider, for example, the angular momentum conservation law (3.21). The action of the shallow atmosphere operator D/Dt, defined by (3.17), on the shallow atmosphere zonal angular momentum (per unit mass), $(u + \Omega a\cos\phi)a\cos\phi$, gives

$$a\cos\phi \left(\frac{Du}{Dt} - 2\Omega v\sin\phi - \frac{uv\sin\phi}{a} \right)$$

which accounts for the Coriolis and metric terms in (3.14). On the other hand, the action of the unapproximated operator D/Dt, defined by (3.5), on the unapproximated zonal angular momentum per unit mass, $(u + \Omega r\cos\phi)r\cos\phi$, gives

$$r\cos\phi \left(\frac{Du}{Dt} - 2\Omega v\sin\phi - \frac{uv\sin\phi}{r} + 2\Omega w\cos\phi + \frac{uw}{r} \right)$$

which accounts for all the Coriolis and metric terms in (3.2). The height variation of the planetary angular momentum $\Omega r^2\cos^2\phi$ gives rise to the term $2\Omega w\cos\phi$ in (3.2). This height variation is neglected in the shallow atmosphere form $\Omega a^2\cos^2\phi$, and so the term $2\Omega w\cos\phi$ does not appear in (3.14). It is very difficult, if not impossible, to accommodate the term $2\Omega w\cos\phi$ in a shallow atmosphere formulation so as to imply a consistent angular momentum principle.

The omission of the $\cos\phi$ Coriolis terms from the HPEs is motivated largely by a desire to preserve good conservation properties when the shallow atmosphere approximation is made.

4. The importance of the cosΦ Coriolis terms

Inaccuracies are inevitably introduced by the omissions (and geometric distortions) made in replacing the Navier-Stokes equations by the HPEs. Under a wide range of conditions the cosΦ Coriolis terms are by far the largest of the omitted terms.

(a) Scale analysis of the zonal momentum balance

Consider the term $2\Omega w\cos\Phi$ in Eq. (3.2) in relation to the material derivative Du/Dt. In quasi-hydrostatic motion, continuity sets an upper bound on vertical velocities as

$$W \lesssim UH/L \tag{4.1}$$

where W and U are vertical and horizontal velocity scales, and H and L are vertical and horizontal length scales. Taking $Du/Dt \sim U^2/L$, gives $|2\Omega w\cos\Phi|/|Du/Dt| \lesssim 2\Omega H\cos\Phi/U$, which is independent of L. The condition for the neglect of $2\Omega w\cos\Phi$ in comparison with Du/Dt in Eq. (3.2) is thus

$$2\Omega H\cos\Phi/U \ll 1 \tag{4.2}$$

or $$2\Omega U\cos\Phi/g \ll U^2/gH \tag{4.3}$$

With $\Omega = 2\pi$ per day, $H = 10^4$m and $U = 10$ ms^{-1}, the quantity $2\Omega H\cos\Phi/U$ takes a value of about $0.14\cos\Phi$.

The upper bound (4.1) will be attained on the synoptic scale ($L \simeq 10^6$m) only if diabatic processes play a dominant role; standard scale analysis shows that free synoptic scale motion is quasi-nondivergent in the sense that $W \ll UH/L$ (Charney 1963, Phillips 1963, Pedlosky 1987). In the tropics - even on the synoptic scale -

diabatic effects may determine the vertical velocity, and the upper bound (4.1) may be approached (see Holton 1972, McBride and Gray 1980, Webster 1983 and Hoskins 1987). Synoptic scale systems of this type are presumably very important in the thermodynamics and dynamics of the tropical atmosphere, and probably of the entire circulation. Even accepting the magnitude of likely errors in estimating the forcing term F_λ in Eq. (3.2), it would appear that retention of $2\Omega w\cos\phi$ is necessary for accurate and reliable simulation.

The ratio $2\Omega H\cos\phi/U$ (see (4.2)) occurs if the effects of the height variation of the planetary angular momentum per unit mass, $\Omega r^2\cos^2\phi$, are considered. As noted in section 3, the term $2\Omega w\cos\phi$ in (3.2) arises from this height variation. Suppose that a parcel of fluid is initially at rest relative to the Earth's surface at latitude ϕ, and that it then rises a vertical distance H. It will acquire a zonal velocity $\Delta U = -2\Omega H\cos\phi$ in the absence of zonal pressure gradients or other zonal forces, if the height variation of the planetary angular momentum is taken into account. Thus, relative to a typical zonal velocity U,

$$\left|\frac{\Delta U}{U}\right| \simeq 2\Omega H\cos\phi/U$$

Dr G J Shutts (private communication) has pointed out that values of U are useful measures of the importance of the term $2\Omega w\cos\phi$ in (3.2). At the equator, a 15 km ascent from surface to tropopause will be associated with upper level easterly flow of about 2 ms^{-1}. This effect does not seem small enough to be easily neglected in simulation models, but it is not described by the HPEs. We note that the above arguments suggest that the effect may not be entirely negligible even in middle latitudes.

The scale estimate (4.1) also allows a time scale τ to be associated with the term $2\Omega w\cos\phi$ in Eq. (3.2). Taking $Du/Dt \sim U/\tau$, and assuming a balance between Du/Dt and $2\Omega w\cos\phi$ gives $\tau \sim L/2\Omega H$. With $L/H = 100$ we find $\tau \sim 10$ days. Although this is large compared with dissipation time scales in the tropics, the effect is not obviously negligible in numerical simulations (especially those designed to reproduce oscillations having a period of 40 days or more).

The term $2\Omega w\cos\phi$ in Eq. (3.2) may also be assessed in relation to the other Coriolis term, $-2\Omega v\sin\phi$. Large scale motion in the tropics may to a first approximation be described by a Sverdrup-type balance of planetary vorticity advection and vortex stretching (Gill 1980, Hoskins and Karoly 1981):

$$\frac{2\Omega v\cos\phi}{a} = 2\Omega\sin\phi\,\frac{\partial w}{\partial z}$$

Hence

$$\left|\frac{2\Omega w\cos\phi}{2\Omega v\sin\phi}\right| \sim \frac{H}{a}\cot^2\phi \tag{4.4}$$

If $H \sim 10^4$m, $(H/a)\cot^2\phi$ takes a value of about 0.1 at $\phi = 6^\circ$, while at $\phi = 2^\circ$ it approaches unity. The clear suggestion is that $2\Omega w\cos\phi$ cannot comfortably be neglected in Eq. (3.2) when applied to synoptic scale motion in the tropics. We note that the equatorial Rossby radius of deformation, which is the natural latitudinal scale for equatorially trapped motions, is typically equivalent to 6°- 12° of latitude.

(b) Scale analysis of the vertical momentum balance

If $2\Omega w\cos\Phi$ is retained in Eq. (3.2), then $-2\Omega u\cos\Phi$ must be retained in Eq. (3.4) in order to preserve consistent energetics. It is nevertheless helpful to carry out a scale analysis of Eq. (3.4), irrespective of energy consistency with Eq. (3.2). The quantity

$$E = \frac{2\Omega U\cos\Phi}{g} \tag{4.5}$$

takes a value of about $1.4 \times 10^{-4}\cos\Phi$ (assuming terrestrial values of parameters and $U = 10\ \mathrm{ms}^{-1}$, as in the previous section) and so it might appear that the term $-2\Omega U\cos\Phi$ is insignificant in Eq. (3.4). But E is not a universal measure of the importance of $-2u\Omega\cos\Phi$; deviations from a spatial mean hydrostatic balance are much smaller than g, and it is these deviations which affect the horizontal motion through the horizontal pressure gradient terms - see, for example, Holton (1972). If a mean, hydrostatically balanced state is introduced, then the pressure p and density ρ may be expressed as

$$\left.\begin{aligned} p &= p_o(r) + p' \\ \rho &= \rho_o(r) + \rho' \end{aligned}\right\} \tag{4.6}$$

with $$\frac{dp_o}{dr} = -\rho_o g \tag{4.7}$$

Eq. (3.4) can then be written as

$$\frac{Dw}{Dt} - 2\Omega u\cos\Phi - \frac{u^2 + v^2}{r} + \frac{g\rho'}{\rho} + \frac{1}{\rho}\frac{\partial p'}{\partial r} = 0 \tag{4.8}$$

upon removal of the mean state balance. The importance of $2\Omega u\cos\Phi$ in Eq. (4.8) may be gauged by a comparison with $(1/\rho)(\partial p'/\partial r)$ (which contributes to the deviation hydrostatic balance). We assume tropical scaling, with the Rossby number $Ro \equiv U/fL \sim 1$. From (3.2), if $Ro \sim 1$,

$$\left| \frac{1}{\rho} p' \right| \sim U^2$$

Hence
$$\frac{2\Omega u\cos\phi}{\left| \frac{1}{\rho} \frac{\partial p'}{\partial r} \right|} \sim \frac{2\Omega H\cos\phi}{U}$$

The $\cos\phi$ Coriolis term in Eq. (3.4) is thus negligible compared with the horizontally varying part of the vertical pressure gradient term only to the extent that

$$\frac{2\Omega H\cos\phi}{U} \ll 1 \qquad (4.9)$$

or

$$E \ll \frac{U^2}{gH} \qquad (4.10)$$

which are the same as the conditions ((4.2) or (4.3)) for neglect of the $\cos\phi$ Coriolis term in Eq. (3.2). We expect therefore that neglect of $-2\Omega u\cos\phi$ in Eq. (3.4) will lead to errors of up to 10% in the horizontally varying balance which affects the horizontal motion through Eqs. (3.2) and (3.3). (We repeat that this argument applies to tropical balances. In middle latitudes, at least on the synoptic scale, geostrophic control gives larger variations of Φ' for a given horizontal velocity scale U: $|\Phi'| \sim U^2/Ro$, $Ro \ll 1$. The $\cos\phi$ Coriolis term in Eq. (3.4) is negligible in this case to the extent that

$$\frac{2\Omega D Ro}{U} \left(\approx \frac{D}{L}\right) \ll 1 \qquad (4.11)$$

Condition (4.10) is typically well satisfied in middle latitude motion, for which $Ro \sim 10^{-1}$ and $D/L \sim 10^{-2}$).

It is interesting to note that the derivation of conditions (4.8) and (4.9) does not depend on the upper bound (4.1) to the vertical velocity. Eq. (4.1) may be used, however, to show that Dw/Dt is negligible compared with $-2\Omega u\cos\phi$ on the synoptic scale:

$$\left| \frac{Dw}{Dt} \right| \sim \frac{UW}{L} \lesssim \frac{U^2H}{L^2}$$

so
$$\frac{|Dw/Dt|}{|2\Omega u|} \lesssim \frac{UH}{2\Omega L^2} = \frac{H}{L} Ro \sim 10^{-3} ,$$

assuming $L = 10^6$m, $U = 10$ ms^{-1} and $H = 10^4$m. Draghici (1987), (1989) notes that $-2\Omega u\cos\phi$ dominates Dw/Dt for a range of mesoscale motions also, and thus apparently represents the most important nonhydrostatic effect in such cases.

(c) Previous adiabatic analyses

The above scale analysis suggests that the treatment of dynamically important balances in Eqs. (3.2) and (3.4) may be subject to errors of about 10% (at least in the tropics) if the $\cos\phi$ Coriolis terms are neglected. Phillips (1968) and Gill (1982) have considered these terms to be less important. From the dispersion relation for linearized waves on an atmosphere at rest, Phillips identified $4\Omega^2 \ll N^2$ (where N is the buoyancy frequency) as the criterion for neglect of the $\cos\phi$ Coriolis terms. Gill gave the more stringent condition $2\Omega \ll N$ after a scale analysis of the linearized, equatorial β-plane equations. If N is of order 10^{-2} s^{-1} (a typical tropospheric value) then $2\Omega/N \sim 10^{-2}$, and the $\cos\phi$ Coriolis terms appear negligible even according to Gill's criterion. (In the deep ocean, $N \approx 10^{-3}$ s^{-1} or less (Gill 1982), so $2\Omega/N \sim 10^{-1}$; in this case the $\cos\phi$ Coriolis terms are not entirely negligible by Gill's criterion).

However, both Phillips' and Gill's analyses assume adiabatic motion, vertical velocities being related to horizontal density fluctuations and the buoyancy frequency. In deriving the condition

$2\Omega H\cos\Phi/U \ll 1$ for the neglect of the $\cos\Phi$ Coriolis term in Eq. (3.2) (see Eq. (4.2)) we have estimated vertical velocities from the continuity equation and have thus used an upper bound which may be approached in regions of strong diabatic heating. This seems a suitable treatment for tropical synoptic-scale convective complexes in which diabatic heating plays an important role in the dynamics (including the determination of phase speeds). We consider that the $\cos\Phi$ Coriolis terms must play a small but not negligible part in the dynamics of such convective systems.

Even if Gill's adiabatic criterion $2\Omega \ll N$ is considered appropriate to the tropical atmosphere, it seems undesirable that the HPEs do not remain a physically acceptable approximation as the static stability (and hence N) tends to zero.

(d) Some further considerations

Two theoretical arguments for the inclusion of the $\cos\Phi$ Coriolis terms may also be noted.

(i) From the Navier-Stokes equation (3.1) the natural definition of hydrostatic and geostrophic balance is

$$2\underline{\Omega} \times \underline{u} - \underline{g} + \frac{1}{\rho}\,\mathrm{grad}\ p = 0 \qquad (4.11)$$

In some theoretical treatments (see, for example, Hide (1977) and references therein) Eq. (4.11) is the starting point for systematic study of slowly evolving motions in rotating systems. Eq. (4.11) is also the definition of balance used by Shutts (1989) in his study of planetary geostrophic motion. The HPEs cannot represent (4.11) intact because of their neglect of the $\cos\phi$ Coriolis terms. It seems undesirable that basic theory and the dynamical equations used by global simulation models should part company at such an elementary stage.

(ii) Neglect of the $\cos\phi$ Coriolis terms leads to a change in the direction of the Coriolis force as well as its magnitude. The true Coriolis force, $2\underline{\Omega} \times \underline{u}$, lies in the plane perpendicular to the Earth's rotation axis, but the HPE version, $f\underline{k} \times \underline{v}$, corresponds to an "approximation" in which the force lies in local horizontal planes. See Fig. 2. The omission of the term $-2\Omega u\cos\phi$ from the vertical component of the momentum balance is responsible for the change of direction. The usual justification for this omission is that

$$g \gg \Omega^2 r \gg 2\Omega u\cos\phi$$

but we have seen that the horizontally fluctuating part of the vertical balance requires a more discriminating treatment (section 4(b)).

5. Discussion

In this paper we have examined the importance of the $\cos\phi$ Coriolis terms which appear in the zonal and vertical components of the momentum equation. These terms are unimportant in synoptic-scale, quasi-adiabatic motion in middle latitudes: they attain magnitudes of only 1%, or less, of the Lagrangian rate of change in the zonal component equation and of the horizontally-varying part of the pressure gradient term in the vertical component equation. In the tropics, where the Rossby number may be of order unity and vertical velocities may be determined by diabatic heating even on the synoptic scale, the $\cos\phi$ Coriolis terms may attain magnitudes of about 10%, in the above senses, in both the zonal and vertical component equations. Similar magnitudes may also be approached in middle latitudes in frontal zones - where diabatic heating may be large and isentropic slopes steep (Draghici 1987, 1989).

Terms attaining such magnitudes might reasonably be considered negligible in theoretical treatments and models aimed at developing conceptual understanding of atmospheric behaviour. But it seems inconsistent with the rationale of numerical flow simulation to omit known terms which attain these magnitudes. In view of the sophistication attained in modern weather prediction and climate simulation models, we consider that the $\cos\phi$ Coriolis terms should now be retained in such models. Certainly, scale analysis suggests that their retention is more comfortable than their omission. There are also several theoretical reasons why retention of the $\cos\phi$ Coriolis terms is desirable (see section 4(d)).

The hydrostatic primitive equations do not include the $\cos\phi$ Coriolis terms, but they imply good conservation properties (as reviewed in section 3(b)). Indeed, the theoretical acceptability of the hydrostatic primitive equations probably rests as much on the existence and nature of the conservation laws (3.21)-(3.23) as on a conviction that the omitted terms are small (and the geometric distortions negligible). The HPEs set a demanding standard against which any proposed extended forms should be judged. If the conservation laws are to be fully respected, the $\cos\phi$ Coriolis terms cannot be included in the HPEs without making other changes to the equations at the same time. Several extended forms of the HPEs are described in Part II of this study (White and Bromley 1994).

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Appendix

Derivation of the HPE potential vorticity conservation law (3.23)

Eqs. (3.14) and (3.15) may be written (if $F_\lambda = F_\phi = 0$) as

$$\frac{D\hat{u}}{Dt} = 2\Omega\hat{v}y - \left(\frac{1-y^2}{\rho}\right)\frac{\partial p}{\partial x} \tag{A1}$$

$$\frac{D\hat{v}}{Dt} = 2\Omega\hat{u}y - \left(\frac{\hat{u}^2 + \hat{v}^2}{1-y^2}\right) - \left(\frac{1-y^2}{\rho}\right)\frac{\partial p}{\partial y} \tag{A2}$$

where $\hat{u} = u\cos\phi$, $\hat{v} = v\cos\phi$

$$y = \sin\phi\ ,\ \frac{\partial}{\partial x} = \frac{1}{\cos^2\phi}\frac{\partial}{\partial\lambda}$$

Also, $\frac{D}{Dt} = (\frac{\partial}{\partial t} + \underline{\hat{u}}.\hat{\nabla})$,

with $u = (\hat{u}, \hat{v}, w)$, and $\hat{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

Eq. (3.18) becomes $\frac{D\rho}{Dt} + \rho\hat{\nabla}.\underline{\hat{u}} = 0.$ (A3)

Define $\underline{\hat{v}} = (\hat{u}, \hat{v}, 0)$ and

$$\underline{\hat{\zeta}} = (\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3) = 2\,\Omega y\underline{k} + \hat{\nabla}x\underline{\hat{v}}$$

$$= (-\frac{\partial\hat{v}}{\partial z}, \frac{\partial\hat{u}}{\partial z}, f + \frac{\partial\hat{v}}{\partial x} - \frac{\partial\hat{u}}{\partial y})$$

Note that $\underline{\hat{\zeta}}.\hat{\nabla}\theta = \underline{\zeta}.\nabla\theta$ (A4)

(see Eqs. (3.24) and (3.25)).

The operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ do not commute:

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} - \frac{2y}{(1-y^2)}\frac{\partial}{\partial x} \tag{A5}$$

Hence, for any β,

$$
\left.
\begin{aligned}
\frac{D}{Dt}\left(\frac{\partial\beta}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{D\beta}{Dt}\right) - \frac{\partial\hat{\underline{u}}}{\partial x}.\hat{\nabla}\beta + \frac{2y\hat{v}}{(1-y^2)}\frac{\partial\beta}{\partial x} \\
\frac{D}{Dt}\left(\frac{\partial\beta}{\partial y}\right) &= \frac{\partial}{\partial y}\left(\frac{D\beta}{Dt}\right) - \frac{\partial\hat{\underline{u}}}{\partial y}.\hat{\nabla}\beta - \frac{2y\hat{u}}{(1-y^2)}\frac{\partial\beta}{\partial x} \\
\frac{D}{Dt}\left(\frac{\partial\beta}{\partial z}\right) &= \frac{\partial}{\partial z}\left(\frac{D\beta}{Dt}\right) - \frac{\partial\hat{\underline{u}}}{\partial z}.\hat{\nabla}\beta
\end{aligned}
\right\} \qquad (A6)
$$

From (A6) and the horizontal components (A1), (A2) it follows that

$$
\left.
\begin{aligned}
\frac{D}{Dt}\hat{\zeta}_1 &= \frac{D}{Dt}\left(-\frac{\partial\hat{v}}{\partial z}\right) = (\hat{\underline{\zeta}}.\hat{\nabla})\hat{u} - \hat{\zeta}_1\hat{\nabla}.\hat{\underline{u}} + \frac{y}{(1-y^2)}\frac{\partial}{\partial z}(\hat{u}^2 + \hat{v}^2) + \ldots. \\
\frac{D}{Dt}\hat{\zeta}_2 &= \frac{D}{Dt}\left(\frac{\partial\hat{u}}{\partial z}\right) = (\hat{\underline{\zeta}}.\hat{\nabla})\hat{v} - \hat{\zeta}_2\hat{\nabla}.\hat{\underline{u}} + \ldots. \\
\frac{D}{Dt}\hat{\zeta}_3 &= \frac{D}{Dt}\left(\frac{\partial\hat{v}}{\partial x} - \frac{\partial\hat{u}}{\partial y}\right) + 2\Omega\hat{v} = (\hat{\underline{\zeta}}.\hat{\nabla})w - \hat{\zeta}_3\hat{\nabla}.\hat{\underline{u}} + \ldots.
\end{aligned}
\right\} \qquad (A7)
$$

(Here + ... indicates pressure gradient terms. They are of no special interest since it is readily shown that they vanish in the subsequent manipulations).

Put $\beta = \theta$ in (A6) (assuming $D\theta/Dt = 0$) and multiply by $\alpha\hat{\zeta}_i$ (where $\alpha = 1/\rho$):

$$
\left.
\begin{aligned}
\alpha\hat{\zeta}_1\frac{D}{Dt}\left(\frac{\partial\theta}{\partial x}\right) &= -\alpha\hat{\zeta}_1\frac{\partial\hat{\underline{u}}}{\partial x}.\hat{\nabla}\theta - \frac{\alpha y}{(1-y^2)}\frac{\partial}{\partial z}(\hat{v}^2)\frac{\partial\theta}{\partial x} \\
\alpha\hat{\zeta}_2\frac{D}{Dt}\left(\frac{\partial\theta}{\partial y}\right) &= -\alpha\hat{\zeta}_2\frac{\partial\hat{\underline{u}}}{\partial y}.\hat{\nabla}\theta + \frac{\alpha y}{(1-y^2)}\frac{\partial}{\partial z}(\hat{u}^2)\frac{\partial\theta}{\partial x} \\
\alpha\hat{\zeta}_3\frac{D}{Dt}\left(\frac{\partial\theta}{\partial z}\right) &= -\alpha\hat{\zeta}_3\frac{\partial\hat{\underline{u}}}{\partial z}.\hat{\nabla}\theta
\end{aligned}
\right\} \qquad (A8)
$$

Use (A3) in (A6) and multiply by $\partial\theta/\partial x$, $\partial\theta/\partial y$, $\partial\theta/\partial z$:

$$\left.\begin{aligned}
&\frac{\partial\theta}{\partial x}\frac{D}{Dt}(\alpha\,\hat{\zeta}_1) = \alpha\frac{\partial\theta}{\partial x}(\hat{\underline{\zeta}}.\hat{\nabla})\hat{u} + \frac{\alpha y}{(1-y^2)}\frac{\partial}{\partial z}(\hat{u}^2+\hat{v}^2)\frac{\partial\theta}{\partial x} +\ldots. \\
&\frac{\partial\theta}{\partial y}\frac{D}{Dt}(\alpha\,\hat{\zeta}_2) = \alpha\frac{\partial\theta}{\partial y}(\hat{\underline{\zeta}}.\hat{\nabla})\hat{v} +\ldots. \\
&\frac{\partial\theta}{\partial z}\frac{D}{Dt}(\alpha\,\hat{\zeta}_3) = \alpha\frac{\partial\theta}{\partial z}(\hat{\underline{\zeta}}.\hat{\nabla})w +\ldots.
\end{aligned}\right\} \qquad (A9)$$

Upon addition of the 6 equations (A8) and (A9), the third order terms involving $\hat{\zeta}_i$ or $(\hat{\underline{\zeta}}.\hat{\nabla})$ sum to zero, as also do the terms in $y/(1-y^2)$. Hence

$$\frac{D}{Dt}\left(\frac{\hat{\underline{\zeta}}.\hat{\nabla}\theta}{\rho}\right) = 0$$

which - in view of (A4) - establishes (3.23) in the case $F_\lambda = F_\phi = D\theta/Dt = 0$. The general form (3.23) may be obtained by repeating the derivation with the term $(1-y^2)^{1/2}\,F_\lambda$ included on the r.h.s. of (A1), $(1-y^2)^{1/2}\,F_\phi$ on the r.h.s. of (A2) and $\alpha\hat{\underline{\zeta}}.\hat{\nabla}(\frac{D\theta}{Dt})$ on the r.h.s. of (A8).

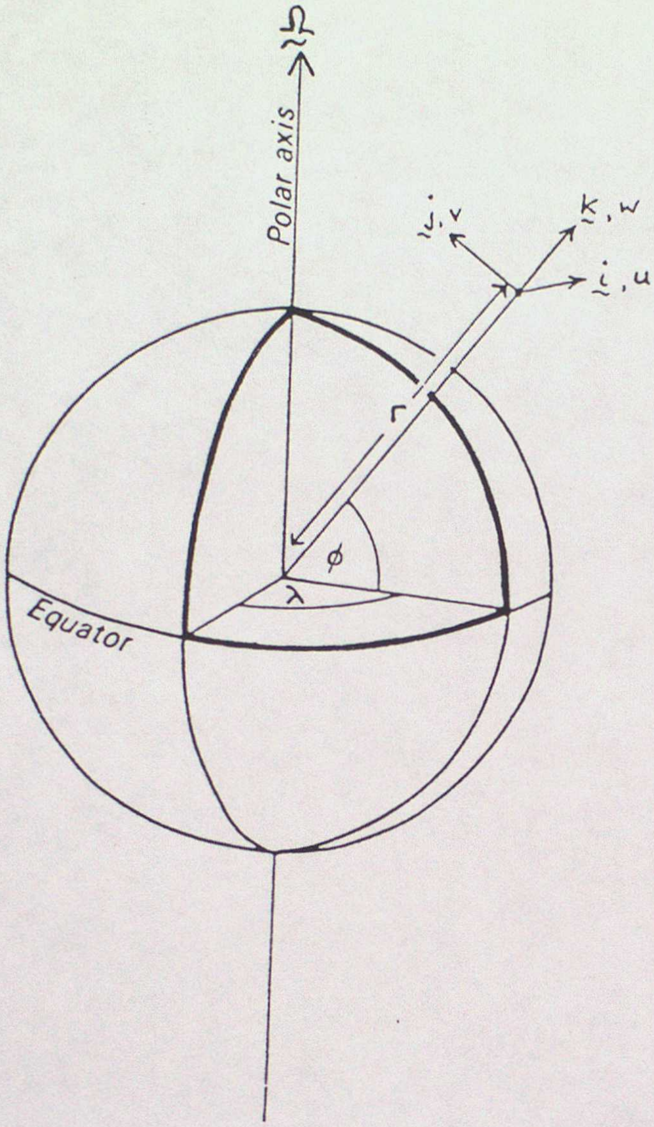

Figure 1 The (λ,ϕ,r) spherical polar system.

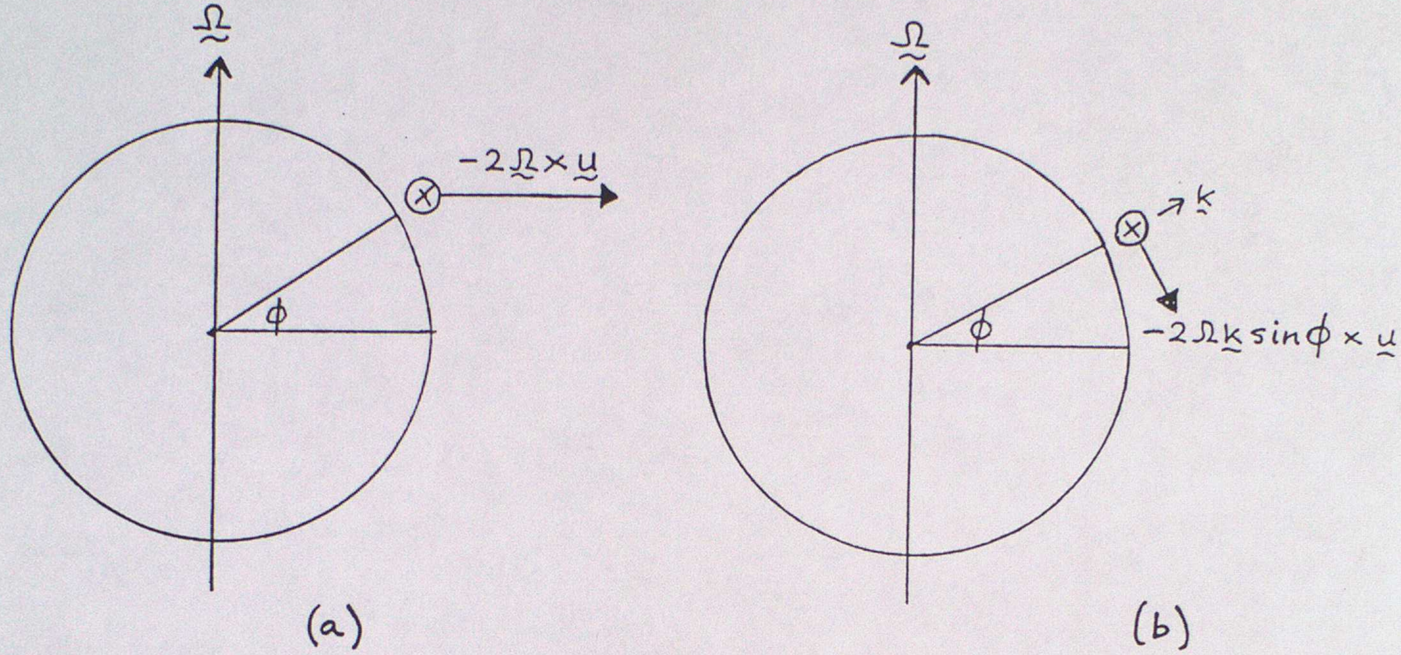

Figure 2 (a) Showing the direction of the Coriolis force acting on a particle moving zonally with velocity $\underline{u}$ at latitude ϕ.

(b) As (a), but showing the direction of the Coriolis force as represented in the HPEs.