MET O 11 TECHNICAL NOTE NO 138

THE ACCURACY OF FINITE DIFFERENCE APPROXIMATIONS TO DIFFERENTIAL EQUATIONS

by

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March 1980

1. INTRODUCTION

Gadd (1978) has reviewed the Lax-Wendroff approximation to the advection equation (equation (1)) and described a modification that is simple, but results in significantly improved phase speed errors. The new scheme is only

$$\frac{\partial \theta}{\partial t} = -u \frac{\partial \theta}{\partial x} \qquad (1)$$

second order accurate, but compares favourably with Crowley's (1968) fourth order accurate extension of the Lax-Wendroff scheme and with a fourth order accurate leap frog approximation. The usual approach to the concept of accuracy centres on the calculation of individual terms e.g. $\partial\theta/\partial x$ in equation (1). There is no prima facie reason to suppose that high order accuracy in this sense will lead to improved solutions in all circumstances, and Gadd's results confirm that it does not. It is natural to seek a definition of order of accuracy that is directly related to the effect of an approximation on the solutions of the equations. In this note I point out that

(i) there is such a definition of "order of accuracy"

(ii) Gadd's new approximation is second order accurate, but close to an approximation that is third order accurate.

(iii) there exists a leapfrog, or time centred, approximation to the two-dimensional equation that is fourth order accurate in this new sense. The two-dimensional version is not a naive combination of two one-dimensional versions. As in the case of Gadd's modification of the Lax-Wendroff approximation, this new approximation is stable for longer time steps than those possible when the spatial derivatives are calculated with fourth order accuracy.

2. THE ACCURACY OF GADD'S APPROXIMATION

The accuracy of a finite difference approximation to an equation can be assessed using the dispersion relation, equation (2), provided only that a dispersion relation exists. The dispersion relation is usually obtained by substituting solutions of the form exp $i(\sigma t - kx)$, where, for simplicity, I have supposed that there are only two independent variables x and t , and deriving

$$\sigma = \sigma(k) \qquad (2)$$

For equation (1)

$$\sigma = uk \qquad (3)$$

Any general solution to the equation and boundary conditions can be written as a linear combination of such Fourier functions, and it is reasonably to hope that a definition of accuracy based on the effect of an approximation on the dispersion rleation is more useful than one based on the accuracy of approximations to individual terms in the equations.

When functions of the form exp $i(\tilde{\sigma} t - kx)$ are substituted in the finite difference equation $\tilde{\sigma}$ will only appear on the product $\tilde{\sigma}\delta t$ and k as the product $k\delta x$. The dispersion relation takes the form

$$\tilde{\sigma}\delta t = \delta t\,\tilde{\sigma}(k\delta x) \qquad (4)$$

where, as in the continuous case, the function $\tilde{\sigma}$ depends on other variables e.g. u , but now also δx and δt . The approximation is nth order accurate if

$$\tilde{\sigma} = \sigma + O((k\delta x)^{n+1}) \qquad (5)$$

The expansion is in the non-dimensional variable $k\delta x$, which is small for waves exp ikx that are well resolved (i.e. have many gird points within a wavelength). Since the derivation of equation (4) involves all terms in the original equations, it is clear that this approach combines the effect of time and space (t and x) truncation, and this advantage is brought out in the discussion of Gadd's modification to the Lax-Wendroff approximation. It is possible to achieve higher orders of accuracy in the dispersion relation using only second order approximations for the individual terms of the equation.

Gadd approximates equation (1) in two steps

$$\theta_{j+\frac{1}{2}}^{n+\frac{1}{2}} = (\overline{\theta}^{x})_{j+\frac{1}{2}}^{n} - \frac{\mu}{2}(\delta_x \theta)_{j+\frac{1}{2}}^{n} \tag{6}$$

$$\theta_j^{n+1} = \theta_j^n - \mu\{(1+a)(\delta_x\theta)_j^{n+\frac{1}{2}} - a(\delta_{3x}\theta)_j^{n+\frac{1}{2}}\} \tag{7}$$

where the notation follows Gadd i.e. where n labels the time level ($t = t_0 + n\delta t$), j labels the spatial grid point ($x = x_0 + j\delta x$), and $\mu = u\delta t/\delta x$. Gadd showed that this approximation is stable for $\mu \leq 1$ if

$$a \leq 3/4\,(1-\mu^2) \tag{8}$$

and arbitrarily (or with a view to reducing the dissipation) chose the limiting case. Substituting exp $i(\tilde{\sigma}t - kx)$ in equations 6 and 7 gives

$$\begin{aligned} \exp(i\tilde{\sigma}\delta t) = {} & 1 - 2\mu^2\sin^2\alpha(1 + 4/3\,a\sin^2\alpha) \\ & + 2i\mu\sin\alpha\cos\alpha(1 + 4/3\,a\sin^2\alpha) \end{aligned} \tag{9}$$

where $\alpha = k\delta x/2$. (Equation (9) is equation (11) of Gadd (1978)). The series expansion of $\tilde{\sigma}$ for small α gives

$$\begin{aligned} \tilde{\sigma}\delta t = {} & 2\mu\alpha - 4/3\,\mu\alpha^3(1-\mu^2-2a) \\ & + 2i\mu^2\alpha^4(1-\mu^2-4/3\,a) + O(\alpha^5) \end{aligned} \tag{10}$$

The exact dispersion relation, equation 3, can be re-written

$$\tilde{\alpha} \delta t = 2\mu\lambda \tag{11}$$

so $\tilde{\alpha} = \alpha + O(\lambda^4)$ only if

$$a = 1/2\,(1-\mu^2) \tag{12}$$

For this choice of a there is an error in the λ^4 term, and the scheme is third order accurate and fourthorder dissipative. Tables 1 and 2 give the damping per time step (exp $(-I_m \tilde{\alpha}\,\delta t)$) and the relative phase speeds ($Re\,\tilde{\alpha}/\alpha$) as functions of μ and wavelength and are directly comparable with Tables 1 and 2 of Gadd (1978). It can be seen that equation (12) gives a significant improvement in the phase speeds but, as expected, it is more dissipative than Gadd's choice. Figures 1 and 2 show the result of advecting a step function through 100 time steps using the two schemes. It is worth commenting that equation (12) gives fourth order accuracy in the phase speed, and that the effect of the dissipation of either scheme is not always undesirable.

3. AN ACCURATE LEAPFROG APPROXIMATION TO THE TWO-DIMENSIONAL ADVECTION EQUATION

The approach described above can be used to derive a fourth order accurate approximation to the two-dimensional advection equation

$$\frac{\partial \theta}{\partial t} = -u\frac{\partial \theta}{\partial x} - v\frac{\partial \theta}{\partial y} \tag{13}$$

The new approximation is

$$\theta_{ij}^{n+1} - \theta_{ij}^{n-1} = -2\mu\left[(1-\nu^2+a)\delta_{2x}\theta - a\,\delta_{4x}\theta + \nu^2\delta_{2x}\overline{\theta}^{2y}\right]_{ij}^{n} - 2\nu\left[(1-\mu^2+b)\delta_{2y}\theta - b\,\delta_{4y}\theta + \mu^2\delta_{2y}\overline{\theta}^{2x}\right]_{ij}^{n} \tag{14}$$

where $\nu = v\,\delta t/\delta y$, $a = 1/3(1-\mu^2)$ and $b = 1/3(1-\nu^2)$. Figure 3 shows the result of advecting a step function through 100 time steps using this scheme in one-dimension, and can be compared with figures 1 and 2.

Equation (15) is stable for

$$|\mu| + |\nu| \leq 1 - \varepsilon \tag{15}$$

where ε = 0.009. $\varepsilon = 0$ if μ and ν are both less than 0.856. $\partial\theta/\partial x$ and $\partial\theta/\partial y$ are calculated with fourth order accuracy if $\nu^2 = \mu^2 = 0$ and $a = b = 1/3$, but the advantage of this accuracy is lost if time truncation is significant (μ or ν not small), and these choices give a scheme that is only stable for $|\mu| + |\nu| \leq 3/4$.

Wavelength in grid lengths (L)

μ	2	3	4	5	6	7	8	9	10
1.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.83	0.92	0.97	0.99	0.99	1.00	1.00	1.00	1.00
0.8	0.59	0.83	0.94	0.97	0.99	0.99	1.00	1.00	1.00
0.7	0.31	0.76	0.92	0.97	0.99	0.99	1.00	1.00	1.00
0.6	0.03	0.74	0.92	0.97	0.99	0.99	1.00	1.00	1.00
0.5	0.25	0.77	0.93	0.97	0.99	0.99	1.00	1.00	1.00
0.4	0.50	0.82	0.95	0.98	0.99	1.00	1.00	1.00	1.00
0.3	0.71	0.89	0.97	0.99	0.99	1.00	1.00	1.00	1.00
0.2	0.87	0.95	0.98	0.99	1.00	1.00	1.00	1.00	1.00
0.1	0.97	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 1. Damping per timestep as a function of the non-dimensional advecting velocity (μ) and the wavelength in grid lengths (L) for $\alpha = 1/2(1-\mu^2)$.

Wavelength in grid length (L)

μ	2	3	4	5	6	7	8	9	10
1.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	1.11	1.03	1.01	1.00	1.00	1.00	1.00	1.00	1.00
0.8	1.25	1.03	1.01	1.00	1.00	1.00	1.00	1.00	1.00
0.7	1.43	1.00	0.99	0.99	1.00	1.00	1.00	1.00	1.00
0.6	1.67	0.93	0.97	0.98	0.99	0.99	1.00	1.00	1.00
0.5	0.00	0.85	0.94	0.97	0.98	0.99	0.99	1.00	1.00
0.4	0.00	0.77	0.91	0.96	0.98	0.99	0.99	1.00	1.00
0.3	0.00	0.70	0.88	0.95	0.97	0.98	0.99	0.99	1.00
0.2	0.00	0.65	0.87	0.94	0.97	0.98	0.99	0.99	1.00
0.1	0.00	0.63	0.85	0.93	0.97	0.98	0.99	0.99	1.00

Table 2. Relative phase speed as a function of the non-dimensional advecting velocity and the wavelength in grid lengths (L) for $\alpha = \frac{1}{2}(1-\mu^2)$.

Figure Captions

1. A step function after advection through 100 time steps for 5 values of the non-dimensional advecting velocity $\mu = u\delta t/\delta x$. Gadd's modification of the Lax-Wendroff scheme was used with $\alpha = 3/4(1-\mu^2)$.

2. A step function after advection through 100 time steps for 5 values of the non-dimensional advecting velocity $\mu = u\delta t/\delta x$. Gadd's modification of the Lax-Wendroff scheme was used, but with $\alpha = 1/2(1-\mu^2)$ rather than the value proposed by Gadd.

3. A step function after advection through 100 time steps for 5 values of the non-dimensional advecting velocity $\mu = u\delta t/\delta x$. The advection scheme is leap frog and leads to a fourth order accurate dispersion relation. A small amount of fourth order dissipation gives results very similar to figure 2.

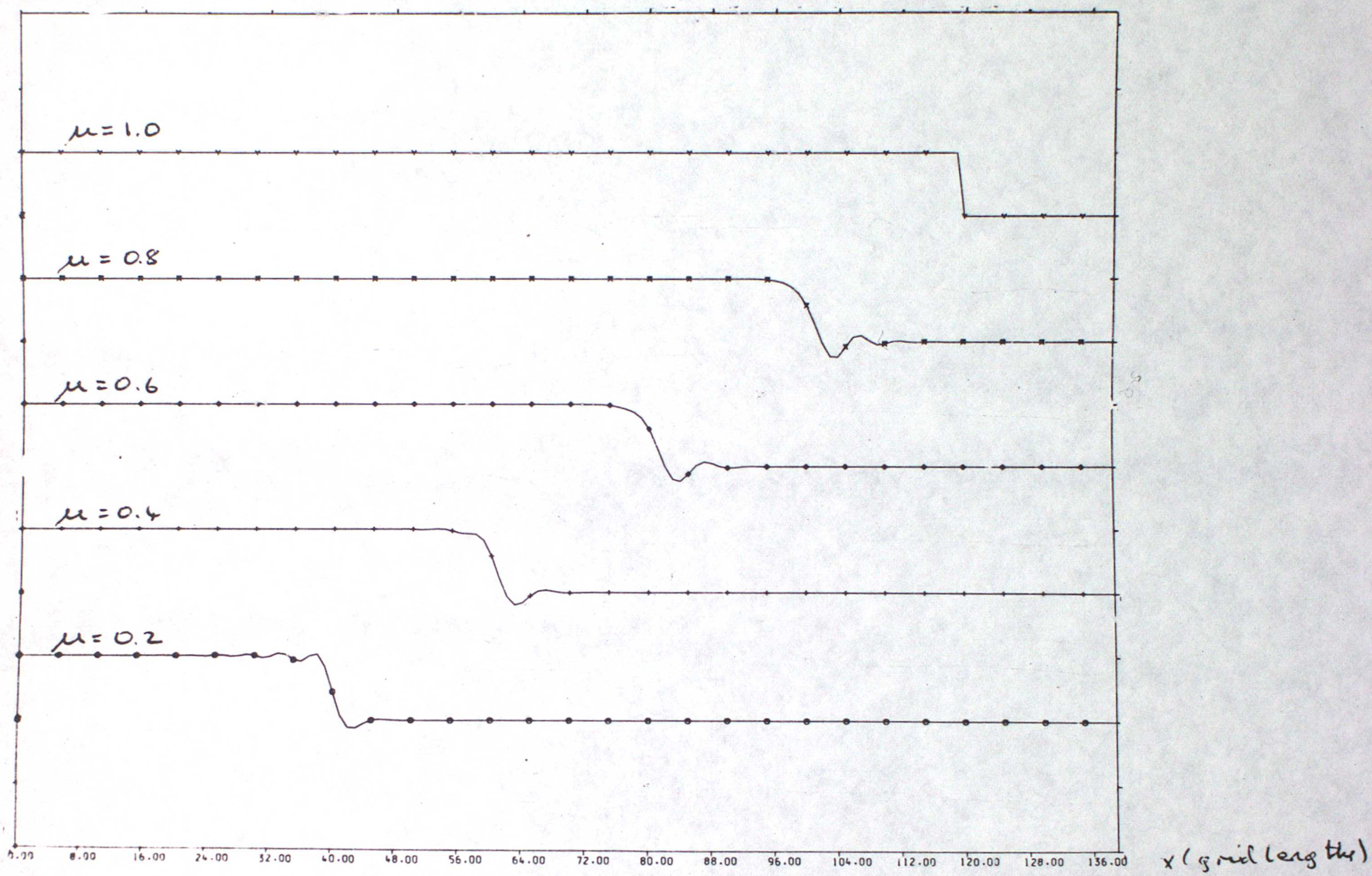

FIG. 1 A step function after advection through 100 time steps for 5 values of the non-dimensional advecting velocity μ. Gadd's modification of the Lax-Wendroff scheme was used with $a = 3/4(1-\mu^2)$. The marks on the lines have no significance.

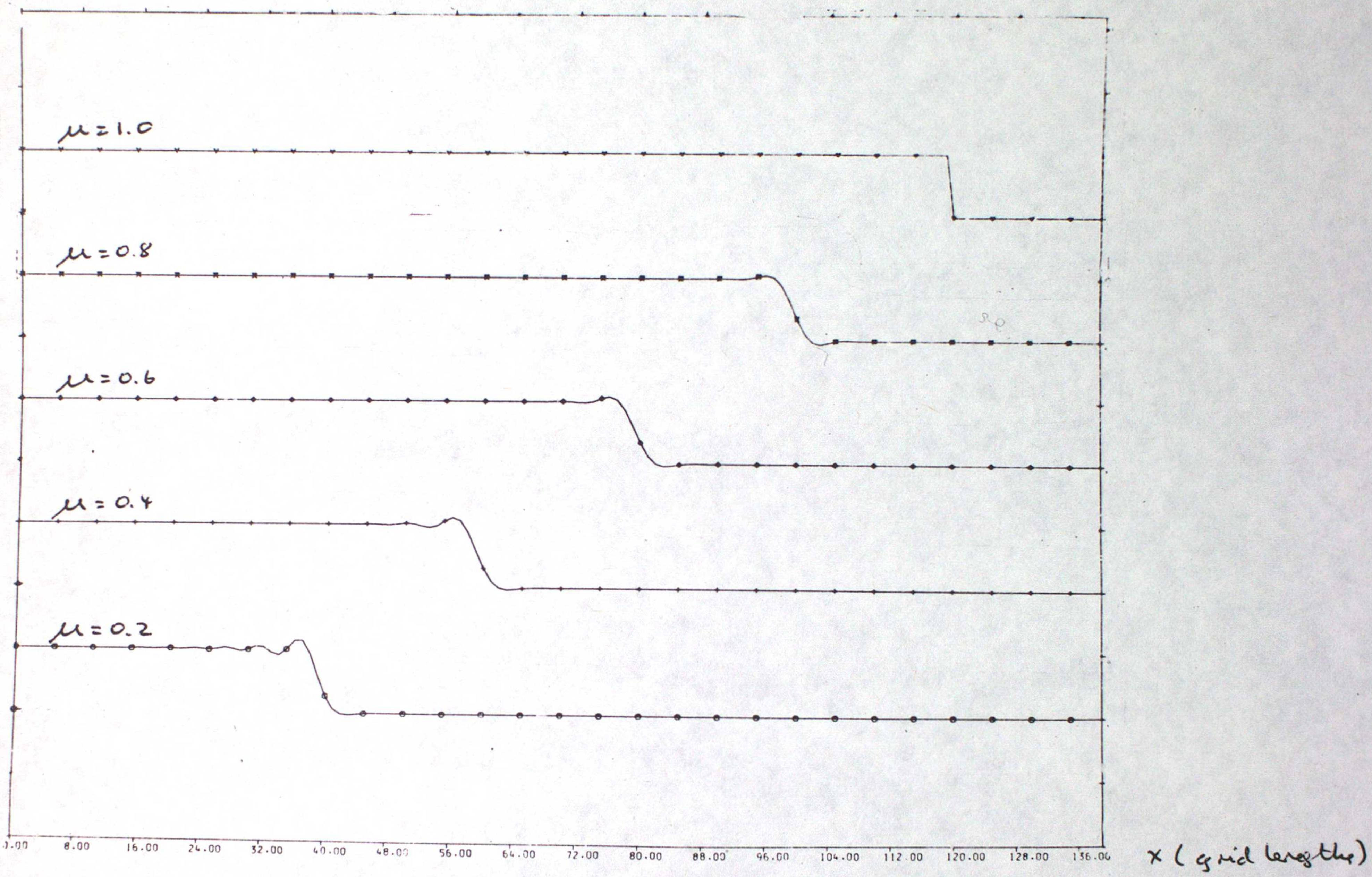

FIG. 2 A step function after advection through 100 time steps for 5 values of the non-dimensional advecting velocity μ. Gadd's modification of the Lax-Wendroff scheme was used, but with $a = 1/2(1-\mu^2)$ rather than the value proposed by Gadd. The marks on the lines have no significance.

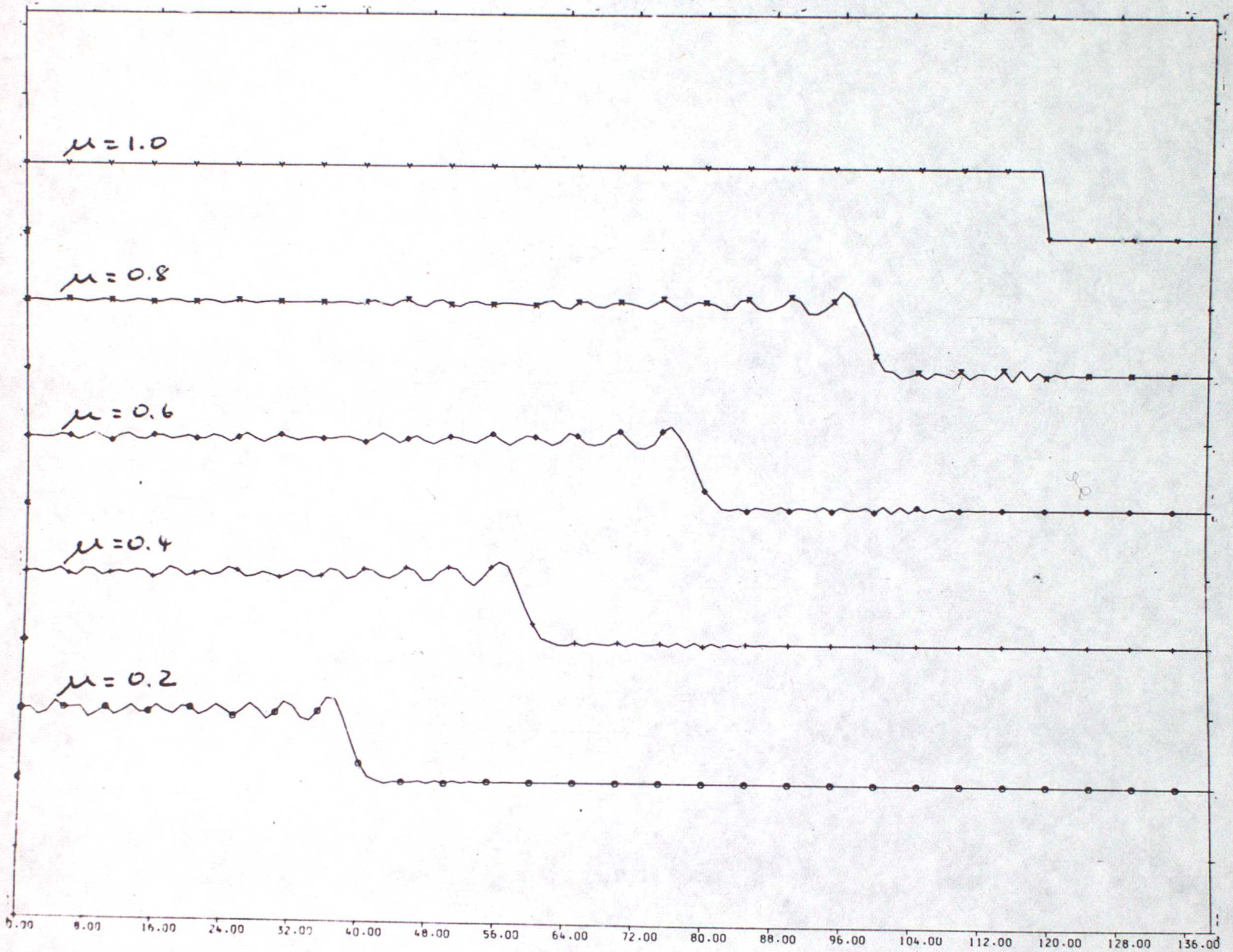

FIG 3 A step function after advection through 100 time steps for 5 values of the non-dimensional advecting velocity $\mu = u\delta t/\delta x$. The advection scheme is leap frog and leads to a fourth order accurate dispersion relation. A small amount of fourth order dissipation gives results very similar to Fig 2. The marks on the lines can be ignored.