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II. DYNAMICALLY CONSISTENT QUASI-HYDROSTATIC EQUATIONS

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Summary

Global, quasi-hydrostatic models having a complete representation of the Coriolis force are proposed. The vertical component of the momentum equation remains in diagnostic form in each case. The simplest proposed models conserve energy but do not imply exact axial angular momentum principles or analogues of Ertel's potential vorticity conservation law. Conservation of axial angular momentum and potential vorticity (as well as energy) is achieved by a more elaborate formulation in which all metric terms are retained and the shallow atmosphere approximation is relaxed. Distance from the centre of the Earth is replaced by a pseudo-radius which is a function of pressure only. This model is put forward as a more accurate alternative to the traditional hydrostatic primitive equations; it preserves the desired conservation laws and may be integrated by broadly similar methods (although its implementation in an existing σ-coordinate spectral model would not be straightforward). Various possible extensions are discussed, including a global, acoustically-filtered model which retains a prognostic form of the vertical momentum equation. Isomorphisms with certain Boussinesq forms of the equations of motion are noted and exploited.

1. Introduction

Coriolis terms proportional to the cosine of the latitude, ϕ, occur in the zonal and vertical components of the Navier-Stokes equation when the motion is referred to a frame rotating about the polar axis. The importance of these cosϕ Coriolis terms, which are neglected in the familiar hydrostatic primitive equations, was examined in part I of this study (White, Bromley and Hoskins 199x, here cited as paper I). It was concluded that the cosϕ Coriolis terms may play a small but not negligible rôle in the dynamical balances of diabatically-driven, synoptic-scale motion, especially in the tropics. Their inclusion in global numerical models was therefore advocated, but a review of the conservation properties of the hydrostatic primitive equations (HPEs) demonstrated the high standards of dynamical consistency against which any more accurate approximation to the Navier-Stokes equations should be judged.

In this paper we propose various ways of including the cosϕ Coriolis terms in acoustically-filtered models of a global, compressible atmosphere. Varying degrees of dynamical consistency are achieved. Each proposed model involves relaxation of the hydrostatic approximation in the sense that certain terms other than those representing gravity and the vertical pressure gradient are retained in the vertical component of the momentum equation. This is accomplished (when pressure is used as vertical coordinate) through an adaptation of the procedure used by Miller (1974) to formulate a nonhydrostatic convection model in pressure coordinates.

Pressure coordinate forms of the HPEs, and their conservation properties, are briefly recalled in section 2. In section 3, modified HPE systems which include the $\cos\phi$ Coriolis terms within the usual shallow atmosphere framework are proposed. Good energy conservation properties are implied, but precise axial angular momentum and potential vorticity conservation principles are lacking. In section 4, by a partial relaxation of the shallow atmosphere approximation and the inclusion of all metric terms, we obtain pressure and sigma coordinate formulations which include the $\cos\phi$ Coriolis terms and possess good analogue forms of the conservation laws for axial angular momentum and potential vorticity, as well as energy. Possible extensions of these acoustically-filtered formulations, and various other issues, are discussed in the concluding section 5.

Unless otherwise stated, symbols have the same meanings as in paper I. A simplifying notation for differential operators in different vertical coordinate systems is adopted; details are given below, in section 2.

2. Pressure coordinate forms of the HPEs

The most important of the quasi-hydrostatic models proposed in sections 3 and 4 are those which use pressure (rather than height) as vertical coordinate. For comparison we lay out in this section the well-known pressure coordinate forms of the HPEs and note the corresponding versions of the various conservation properties. The pressure coordinate HPEs are of course precise transforms of the height coordinate equations (see Eqs (3.14)-(3.20) of paper I).

The zonal and meridional components of the HPE momentum balance may be written as

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{a\cos\phi}\right) v\sin\phi + \frac{1}{a\cos\phi}\frac{\partial\Phi}{\partial\lambda} = F_\lambda \tag{2.1}$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{a\cos\phi}\right) u\sin\phi + \frac{1}{a}\frac{\partial\Phi}{\partial\phi} = F_\phi \tag{2.2}$$

with

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + \omega\frac{\partial}{\partial p} \tag{2.3}$$

Here $\omega = Dp/Dt$, $\Phi = gz$ and the derivatives $\partial/\partial t$, $\partial/\partial\lambda$, $\partial/\partial\phi$ are each taken at constant pressure.

The continuity equation is

$$\nabla_p \cdot \underline{v} + \frac{\partial\omega}{\partial p} = 0 \tag{2.4}$$

in which

$$\nabla_p \cdot \underline{v} \equiv \frac{1}{a\cos\phi}\left\{\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right\} \tag{2.5}$$

and the thermodynamic equation may be written as

$$c_p \frac{DT}{Dt} - \frac{RT\omega}{p} = Q \tag{2.6}$$

(with D/Dt given by (2.3)).

For many purposes it is helpful to work in terms of the pressure-based but height-like coordinate $z_s(p)$ defined by

$$z_s(p) \equiv \int_p^{p_o} \frac{RT_s(p')dp'}{gp'} \tag{2.7}$$

in which $T_s = T_s(p)$ is some reference temperature profile and p_o is a constant reference surface pressure. In terms of z_s, Eqs (2.3) and (2.4) become

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + \tilde{w}\frac{\partial}{\partial z_s} \tag{2.8}$$

$$\nabla_p \cdot \underline{v} + \frac{1}{\rho_s}\frac{\partial}{\partial z_s}(\rho_s \tilde{w}) = 0 \tag{2.9}$$

in which $\tilde{w} = \dfrac{Dz_s}{Dt} = -\dfrac{\omega RT_s}{gp}$ (2.10)

and $\rho_s = p/RT_s$ (2.11)

Eq (2.9) is isomorphic to the continuity equation of an incompressible fluid having a mean density profile $\rho_s(z)$, where z is geometric height. The formal resemblance of the HPEs in pressure coordinates to quasi-Boussinesq equations in height coordinates is particularly clear when the transformation to $z_s(p)$ is made. Isomorphisms of this sort will be exploited in later sections.

The HPE conservation laws (3.21)-(3.23) of paper I take the following forms in pressure coordinates:

Axial angular momentum

$$\frac{D}{Dt}\left((u + \Omega a\cos\phi)a\cos\phi\right) = F_{\lambda} a\cos\phi - \frac{\partial\Phi}{\partial\lambda} \tag{2.12}$$

Energy

$$\frac{D}{Dt}\left(\frac{1}{2}\underline{v}^{2} + c_{p}T \right) + \nabla_{p}.(\underline{v}\Phi) + \frac{\partial}{\partial p}(\omega\Phi) = Q + \underline{v}.\underline{F}_{h} \tag{2.13}$$

Potential vorticity

$$\rho_{s}\frac{D}{Dt}\left(\frac{\underline{\zeta}.\nabla\theta}{\rho_{s}}\right) = (\underline{\zeta}.\nabla)\frac{D\theta}{Dt} + \nabla\theta.\nabla x\underline{F}_{h} \tag{2.14}$$

Here $\nabla \equiv \left(\frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}\ ,\ \frac{1}{a}\frac{\partial}{\partial\phi}\ ,\ \frac{\partial}{\partial z_{s}} \right)$ (2.15)

and $\underline{\zeta} = 2\Omega\underline{k}\sin\phi + \nabla x\underline{v}$ (2.16)

with $\nabla x\underline{v} = \left(-\frac{\partial v}{\partial z_{s}}, \frac{\partial u}{\partial z_{s}}, \frac{1}{a\cos\phi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\phi}(u\cos\phi)\right)\right)$ (2.17)

Since p is here held constant in the horizontal derivatives, the operators ∇ and ∇x - and hence $\underline{\zeta}$ - are differently defined than in the height-coordinate case (see Eqs(3.24) - (3.26) of paper I). We do not consider it desirable to introduce a special notation to emphasize this; both definitions will be used in later sections but it should be clear in each case which is appropriate.

3. Energy conserving extensions of the HPEs

As noted in paper I the term $2\Omega w\cos\phi$ in the zonal component of the Navier-Stokes equation ((3.2) of paper I) represents the rate of change of zonal velocity required purely by conservation of angular momentum when a particle is displaced vertically at latitude ϕ. It is not possible to include the $\cos\phi$ Coriolis terms in the shallow atmosphere HPEs whilst retaining full dynamical consistency. This section is motivated by the fact that energy consistency is readily achieved, and by the observation that some authoritative texts (e.g.Holton 1972, Gill 1982) cite shallow atmosphere forms of the components of the Navier-Stokes equation in which both of the $\cos\phi$ Coriolis terms and all of the metric terms are retained. The analysis also serves to introduce the technique for representing the necessary nonhydrostatic effects within a pressure coordinate framework.

(a) Height coordinate forms

If the $\cos\phi$ Coriolis terms are included, but no other changes are made, the height coordinate HPEs assume the following modified forms:

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{a\cos\phi}\right) v\sin\phi + 2\Omega w\cos\phi + \frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda} = F_\lambda \qquad (3.1)$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{a\cos\phi}\right) u\sin\phi + \frac{1}{\rho a}\frac{\partial p}{\partial\phi} = F_\phi \qquad (3.2)$$

$$- 2\Omega u\cos\phi + g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \qquad (3.3)$$

$$\frac{D\rho}{Dt} + \rho\nabla.\underline{u} = 0 \tag{3.4}$$

$$\frac{D\theta}{Dt} = \left(\frac{\theta}{Tc_p}\right) Q \tag{3.5}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z} \tag{3.6}$$

Eqs (3.1) - (3.5) imply the energy equation

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\underline{v}^2 + gz + c_vT\right) + \nabla.(p\underline{u}) = \rho(Q + \underline{v}.\underline{F}_h) \tag{3.7}$$

which is the same as the original HPE form (see (3.22) of paper I).

From the zonal component, (3.1), it is readily shown that

$$\rho\frac{D}{Dt}\left[(u + \Omega a\cos\phi)a\cos\phi\right] = \rho F_\lambda a\cos\phi - \frac{\partial p}{\partial\lambda} - 2\Omega\rho a w\cos^2\phi \tag{3.8}$$

and

$$\rho\frac{D}{Dt}\left[(u + \Omega r\cos\phi)r\cos\phi\right] = \rho F_\lambda r\cos\phi - \frac{r}{a}\frac{\partial p}{\partial\lambda}$$
$$+ \frac{\rho uw\cos\phi}{r} - 2\Omega\rho vr\left(\frac{r}{a} - 1\right)\sin\phi\cos\phi \tag{3.9}$$

(D/Dt is defined by (3.6)). Neither (3.8) nor (3.9) constitutes an acceptable angular momentum principle: Eq (3.8) does not reproduce the HPE form ((3.21) of paper I) and Eq (3.9) does not reproduce the Navier-Stokes form ((3.8) of paper I). However, at least some of the extra terms are expected to be small for motion in a shallow atmosphere; see Veronis (1968).

Eqs (3.1)-(3.5) imply the following law for the rate of change of potential vorticity in free, adiabatic motion:

$$\rho \frac{D}{Dt} \left(\frac{\underline{\zeta}^{*}.\nabla\theta}{\rho} \right) = - \frac{4\Omega \sin\phi}{a} (\underline{u}.\nabla\theta) \tag{3.10}$$

in which

$$\underline{\zeta}^{*} = \Bigg(- \frac{\partial v}{\partial z} , \ \frac{\partial u}{\partial z} + 2\Omega\cos\phi , \ 2\Omega\sin\phi + \frac{1}{a\cos\phi} \left(\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi} (u\cos\phi) \right) \Bigg)$$

The scalar quantity $\underline{\zeta}^{*}.\nabla\theta/\rho$ is thus a Lagrangian conserved quantity of the formulation (given free, adiabatic motion) only in steady flow. Eq (3.10) may be derived by an extension of the method outlined in the Appendix of paper I.

There are several other energetically consistent extensions of the HPEs which include the $\cos\phi$ Coriolis terms. Inclusion of $-u^2/a$ in the vertical component (3.3) gives the HPE energy equation (3.7) if uw/a is included in the zonal component (3.1). This extension retains the quantity $(2\Omega + u/a\cos\phi)$ wherever $(2\Omega + u/r\cos\phi)$ appears in the components of the Navier-Stokes equation (see (3.2)-(3.4) of paper I). The further inclusion of $-v^2/a$ in the vertical component (3.3) is energetically consistent if vw/a is included in the meridional component (3.2). Shallow atmosphere versions of the components of the Navier-Stokes equations have been discussed by Veronis (1968), Holton (1972) and Gill (1982); these too are energetically consistent. None of the possible extensions noted here gives satisfactory angular momentum and potential vorticity conservation laws.

(b) Pressure coordinate forms

Since Eq (3.3) is a nonhydrostatic form, it might be expected that pressure coordinate transforms of the equations considered in section 3(a) would not be useful. However, the hydrostatic approximation remains an accurate statement except where horizontal variations of the balance represented by (3.3) are relevant, and progress can be made using the technique applied by Miller (1974) to develop a pressure coordinate model of nonhydrostatic convective motion. (See Miller and White (1984) and White (1989a) for further justification and discussion of this technique.) A procedure similar to that described below could be applied to any of the extensions of the HPEs noted in the previous subsection.

We define

$$\varepsilon \equiv \frac{2\Omega u \cos\phi}{g} \tag{3.11}$$

(Scale values of ε are equal to the quantity E defined by Eq (4.5) of paper I.) Transformation to pressure coordinates of Eqs (3.1)-(3.5) gives

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{a\cos\phi}\right) v\sin\phi + 2\Omega\,\frac{Dz}{Dt}\cos\phi + \frac{(1-\varepsilon)}{a\cos\phi}\frac{\partial\Phi}{\partial\lambda} = F_\lambda \tag{3.12}$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{a\cos\phi}\right) u\sin\phi + \frac{(1-\varepsilon)}{a}\frac{\partial\Phi}{\partial\phi} = F_\phi \tag{3.13}$$

$$\frac{RT}{p} + (1-\varepsilon)\,\frac{\partial\Phi}{\partial p} = 0 \tag{3.14}$$

$$\frac{D}{Dt}\,(\ln(1-\varepsilon)) + \frac{1}{a\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}\,(v\cos\phi)\right) + \frac{\partial\omega}{\partial p} = 0 \tag{3.15}$$

$$c_p\,\frac{DT}{Dt} - \frac{RT\omega}{p} = Q \tag{3.16}$$

Here $\displaystyle \frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + \omega\,\frac{\partial}{\partial p}$

and all differentiations w.r.t. t, λ or ϕ are taken at constant p.

Eqs (3.12)-(3.16) are exact transforms of the height coordinate equations and thus imply the same conservation properties. They could, in principle, be used in numerical integrations, but simpler forms may be derived by exploiting the smallness of ε - as represented by the scale value E; see Eq (4.5) of paper I.

In addition to E, relevant non-dimensional quantities are a Froude number (F), a Rossby number (Ro) and an inverse Richardson number (Ri^{-1}). Table 1 gives their definitions and lists typical values for synoptic scale motion in the tropical and middle latitude tropospheres. These values will be assumed repeatedly in the following scale analysis. For both the tropics and middle latitudes, E, F, Ri^{-1} and Ro obey the key order-of-magnitude inequalities

$$E \ll F \ll Ri^{-1} \ll Ro \tag{3.17}$$

Consider first the continuity equation (3.15). The horizontal divergence term is of order $\frac{U}{L}$ Ro and so it dominates the material derivative term if

$$E \ll Ro \tag{3.18}$$

which is typically very well satisfied at all latitudes, and especially in the tropics (see Eq (3.17) or Table 1). This argument applies to middle-latitude synoptic-scale systems and to low-latitude systems in which diabatic forcing is dominant (and Ro $\sim$ 1). In quasi-adiabatic, synoptic-scale motion in low latitudes the horizontal divergence is of order $\frac{U}{L}(RiRo)^{-1}$ (Charney 1963). The condition for the neglect of the material derivative term in Eq (3.15) is therefore

$$E \ll Ri^{-1} \tag{3.19}$$

(since Ro $\sim$ 1) which is typically well satisfied, according to Eq (3.17) or Table 1. In summary, neglect of the material derivative term in Eq (3.15) is justified for synoptic scale motion at any latitude. The remnant continuity equation is of course the HPE form (2.4).

The ε terms in Eqs (3.12) and (3.13) may be neglected by comparison with Du/Dt or Dv/Dt to the extent that (3.17) is satisfied. A more demanding condition for the neglect of the ε term in Eq (3.12) is that it should be small compared with $2\Omega\cos\phi\,\frac{Dz}{Dt}$. (This is the first comparison which distinguishes the present case from that of the HPEs themselves.) It is readily shown that the relevant conditions are

$$F \ll 1 \quad \text{for tropical diabatic systems}$$

and $$F \ll Ri^{-1} \quad \text{for adiabatic systems.}$$

$F \ll 1$ is very well satisfied. $F \ll Ri^{-1}$ is satisfied to one order of magnitude at least (see Table 1) but is the most marginal of the conditions so far obtained. Note, however, that the term $2\Omega w\cos\phi$ is itself negligible in adiabatic, synoptic-scale motion (see section 4(a) of paper I).

To retain energy consistency it is necessary to make further approximations in Eqs (3.12) and (3.14). There are several ways to proceed, each of which exploits a form of the familiar approximation

$$\frac{Dz}{Dt} = w \simeq - \frac{\omega}{\rho g} \tag{3.20}$$

(which is valid if $\frac{N^2H}{g} \ll 1$ - a condition equivalent to $F \ll Ri^{-1}$; see Eq (3.17) and Table 1). Here we choose the simplest such approach, following that used by Miller (1974) in his nonhydrostatic convection problem. Reference profiles of temperature and geopotential, $T_s(p)$ and $\Phi_s(p)$, are introduced; these profiles represent a horizontally averaged, hydrostatically balanced mean state as a function of pressure. Eq (3.20) is applied in the form

$$\frac{Dz}{Dt} \approx \tilde{w} = - \frac{\omega RT_s}{gp} \tag{3.21}$$

(cf Eq (2.10), in which $T_s(p)$ was <u>any</u> chosen reference temperature profile). In the vertical component (3.14), $\varepsilon \frac{\partial\Phi}{\partial p}$ is replaced by $\varepsilon \frac{d\Phi_s}{dp} = - \varepsilon \frac{RT_s}{p}$. After incorporating all these approximations the set of pressure coordinate equations (3.12)-(3.16) becomes

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{a\cos\phi}\right) v\sin\phi + 2\Omega\tilde{w}\cos\phi + \frac{1}{a\cos\phi}\frac{\partial\Phi}{\partial\lambda} = F_\lambda \qquad (3.22)$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{a\cos\phi}\right) u\sin\phi + \frac{1}{a}\frac{\partial\Phi}{\partial\phi} = F_\phi \qquad (3.23)$$

$$\frac{RT}{p} + \frac{\partial\Phi}{\partial p} + \varepsilon\frac{RT_s}{p} = 0 \qquad (3.24)$$

$$\frac{1}{a\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right) + \frac{\partial\omega}{\partial p} = 0 \qquad (3.25)$$

$$c_p\frac{DT}{Dt} - \frac{RT\omega}{p} = Q \qquad (3.26)$$

with $\tilde{w}$ defined by Eq (3.21).

The energy equation implied by (3.22) - (3.26) is the same as the HPE, p-coordinate form (2.13). It is readily shown, however, that a satisfactory angular momentum principle is not produced. Furthermore, even for free, adiabatic flow the potential vorticity law takes the non-conservative form

$$\rho_s\frac{D}{Dt}\left(\frac{\underline{\zeta}'.\nabla\theta}{\rho_s}\right) = -\frac{4\Omega\sin\phi}{a}(\tilde{\underline{u}}.\nabla\theta)$$

in which $\tilde{\underline{u}} = (\, u,\ v,\ \tilde{w}\,)$

and $$\underline{\zeta}' = \left(-\frac{\partial v}{\partial z_s},\ \frac{\partial u}{\partial z_s} + 2\Omega\cos\phi,\ 2\Omega\sin\phi + \frac{1}{a\cos\phi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\phi}(u\cos\phi)\right)\right)$$

Lagrangian conservation of the quantity $\underline{\zeta}'.\nabla\theta/\rho_s$ is thus achieved only in free, diabatic steady flow. The conservation properties of Eqs (3.22)-(3.26) evidently parallel those of the height coordinate equations discussed in section 3(a).

The pressure coordinate equations (3.22) - (3.26) may be transformed to σ-coordinates ($\sigma = p/p_*$, where p_* = surface pressure) without approximation. The resulting system could be integrated in time essentially as the σ-coordinate HPEs are treated. Details will not be given here.

4. Energy, angular momentum and potential vorticity conserving extensions of the HPEs

The modifications of the HPEs which were proposed in section 3 retain the $\cos\phi$ Coriolis terms and imply consistent energetics. They do not possess exact angular momentum principles or analogues of Ertel's potential vorticity theorem. Although the inconsistencies in both respects may be quantitatively small, it is desirable that any proposed extension of the HPEs should possess comparably good conservation laws for angular momentum and potential vorticity as well as energy. Also, the conservation of angular momentum governs the form invariance of the energetics to transformation between co-rotating frames; it can be shown that energy conservation laws are frame-independent only if an appropriate angular momentum principle exists (White 1989b).

(a) A height coordinate model

Consider an approximation of the Navier-Stokes equation in which the horizontal components are retained unchanged but the material derivative and friction terms are omitted from the vertical component:

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) + \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} = F_\lambda \qquad (4.1)$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial\phi} = F_\phi \qquad (4.2)$$

$$-\left(2\Omega + \frac{u}{r\cos\phi}\right)u\cos\phi - \frac{v^2}{r} + g + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \qquad (4.3)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \phi} + w\frac{\partial}{\partial r} \tag{4.4}$$

(cf Eqs (3.2) - (3.5) of paper I).

Eqs (4.1) - (4.3) may be written in vector form as

$$\frac{D\underline{u}}{Dt} - \underline{k}\frac{Dw}{Dt} + 2\underline{\Omega} \times \underline{u} + g\underline{k} + \frac{1}{\rho}\,\mathrm{grad}p = \underline{F}_h \tag{4.5}$$

in which $\underline{F}_h = (F_\lambda, F_\phi, 0)$ represents the horizontal components of the external force per unit mass. Eq (4.5) implies conservation of axial angular momentum because its zonal component (4.1) is the unapproximated Navier-Stokes form.

The unapproximated continuity and thermodynamic equations are

$$\frac{D\rho}{Dt} + \rho\,\mathrm{div}\underline{u} = 0 \tag{4.6}$$

$$\frac{D\theta}{Dt} = \left(\frac{\theta}{Tc_p}\right) Q \tag{4.7}$$

(given as Eqs (3.6) and (3.7) in paper I). Eqs (4.5)-(4.7) imply the energy conservation law

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\underline{v}^2 + \Phi + c_vT\right) + \mathrm{div}\,(p\underline{u}) = \rho(Q + \underline{v}.\underline{F}_h) \tag{4.8}$$

which differs from the unapproximated form (Eq (3.9) of paper I) only in the absence of the contribution of $\frac{1}{2}w^2$ to the specific kinetic energy and of the contribution of ρwF_r to the rate of working of the external force $\underline{F}$.

The potential vorticity properties of the model (Eqs (4.5)-(4.7)) may be established by noting that

$$\mathrm{curl}\left(\underline{k}\, \frac{Dw}{Dt} \right) = \frac{D\underline{\xi}}{Dt} + \underline{\xi}\ \mathrm{div}\ \underline{u} - (\underline{\xi}.\mathrm{grad})\ \underline{u} \tag{4.9}$$

$$\text{where} \quad r\underline{\xi} \equiv \underline{i}\, \frac{\partial w}{\partial \phi} - \underline{j}\, \frac{1}{\cos\phi}\, \frac{\partial w}{\partial \lambda} \tag{4.10}$$

($\underline{i}$ and $\underline{j}$ are unit vectors in the zonal and meridional directions.) Eq (4.9) reflects the special symmetry of the spherical polar coordinate system; a sketch proof is given in Appendix A.

Using Eqs (4.9), (4.10) and the known properties of the Navier-Stokes equation it follows that Eq (4.5) implies the vorticity equation

$$\frac{D\underline{Z}'}{Dt} + \underline{Z}'\mathrm{div}\ \underline{u} - (\underline{Z}'.\mathrm{grad})\underline{u} + \mathrm{curl}(\frac{1}{\rho}\ \mathrm{grad}p) = \mathrm{curl}\ \underline{F}_h \tag{4.11}$$

Here $\underline{Z}' = \underline{Z} - \underline{\xi}$ is the absolute vorticity (cf Eq (3.11) of paper I) stripped of all terms involving w:

$$\left.\begin{aligned} Z'_\lambda &= -\frac{1}{r}\frac{\partial}{\partial r}(rv) \\ Z'_\phi &= 2\Omega\cos\phi + \frac{1}{r}\frac{\partial}{\partial r}(ru) \\ Z'_r &= 2\Omega\sin\phi + \frac{1}{r\cos\phi}\left(\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi}(u\cos\phi) \right) \end{aligned}\right\} \tag{4.12}$$

If the motion is frictionless and adiabatic it then follows from Eqs (4.6), (4.7) and (4.11) that

$$\frac{D}{Dt}\left(\frac{\underline{Z}'\cdot\mathrm{grad}\theta}{\rho}\right) = 0 \tag{4.13}$$

Thus Eqs (4.5) - (4.7) imply an analogue of Ertel's potential vorticity theorem.

Miller and Gall (1982) used a form similar to (4.5) in a numerical study of zonally symmetric motion of a Boussinesq fluid. They noted its consistent energetics but did not discuss the angular momentum or potential vorticity properties.

(b) A pressure coordinate model

The height-coordinate model discussed in section 4(a) may be transformed to pressure coordinates, and a tractable and consistent set of equations obtained by an extension of the method used in section 3. In the interests of brevity we omit the detailed steps, and simply present the proposed equations. Apart from the retention of all metric terms, the only new feature is that the shallow atmosphere approximation is not applied. Rather, r is replaced by the pseudo-radius $r_s(p)$ defined as

$$r_s(p) = a + \int_p^{p_o} \frac{RT_s(p')}{gp'}\,dp' \tag{4.14}$$

Hence $r_s(p)$ is the mean radius of the Earth plus the height $z_s(p)$ defined by Eq (2.7). $T_s(p)$ is to be interpreted as a profile representing the horizontally averaged, hydrostatically balanced state of the atmosphere (see section 3). According to Eq (4.14)

$$\frac{Dr_s}{Dt} = -\frac{RT_s(p)\omega}{gp} = \tilde{w} \tag{4.15}$$

(cf.(3.21)) which is a key element in the dynamical consistency of the new model. In (4.15), and below, it is assumed that

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r_s\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r_s}\frac{\partial}{\partial\phi} + \omega\frac{\partial}{\partial p} \tag{4.16}$$

all differentiations with respect to t, λ and ϕ being taken at constant p.

The proposed p-coordinate equations are:

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r_s\cos\phi}\right)(v\sin\phi - \tilde{w}\cos\phi) + \frac{1}{r_s\cos\phi}\frac{\partial\Phi}{\partial\lambda} = F_\lambda \tag{4.17}$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r_s\cos\phi}\right)u\sin\phi + \frac{v\tilde{w}}{r_s} + \frac{1}{r_s}\frac{\partial\Phi}{\partial\phi} = F_\phi \tag{4.18}$$

$$\frac{RT}{p} + \frac{\partial\Phi}{\partial p} + \mu\frac{RT_s}{p} = 0 \tag{4.19}$$

$$\tilde{\nabla}_p . v + \frac{1}{r_s^2}\frac{\partial}{\partial p}\left(r_s^2\,\omega\right) = 0 \tag{4.20}$$

$$\frac{D\theta}{Dt} = \left(\frac{\theta}{Tc_p}\right) Q \tag{4.21}$$

In the vertical component equation (4.19),

$$\mu \equiv (2\Omega ur_s\cos\phi + u^2 + v^2)/r_s g \tag{4.22}$$

which is an extension of the quantity introduced in section 3 (see Eq (3.11)). In the continuity equation (4.20),

$$\tilde{\nabla}_p.\underline{v} \equiv \frac{1}{r_s\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right) \tag{4.23}$$

Scale analysis similar to that used in section 3 may be applied to justify the neglect of terms involving μ in deriving (4.17) and (4.18). The simultaneous <u>inclusion</u> of small metric terms (such as $v\tilde{w}/r_s$ in (4.18)) cannot be justified by scale analysis, however. Their retention is to be regarded as a consistency requirement, since the angular momentum and potential vorticity conservation principles noted below are lost if these small terms are omitted.

By using (4.15) it is easily seen that the zonal component (4.17) implies the axial angular momentum principle

$$\frac{D}{Dt}\left[(u + \Omega r_s\cos\phi)r_s\cos\phi\right] = F_\lambda r_s\cos\phi - \frac{\partial\Phi}{\partial\lambda} \tag{4.24}$$

This is evidently a good analogue of the Navier-Stokes form (Eq (3.8) of paper I), though the omission of the very small term $-(\mu/r_s\cos\phi)(\partial\Phi/\partial\lambda)$ during the derivation of (4.24) should be noted.

The energy equation implied by Eqs (4.17)-(4.21) is

$$\frac{D}{Dt}\left(\frac{1}{2}\underline{v}^2 + c_pT\right) + \tilde{\nabla}_p.(\underline{v}.\Phi) + \frac{1}{r_s^2}\frac{\partial}{\partial p}(r_s^2\omega\Phi) = Q + \underline{v}.\underline{F}_h \tag{4.25}$$

which is reminiscent of the HPE form (2.13).

The potential vorticity properties of the new model may be established by noting an isomorphism to the height coordinate model described in section 4(a). From (4.14) and (4.15) it follows that

$$\frac{\partial}{\partial p} = - \frac{RT_s}{gp} \frac{\partial}{\partial r_s}$$

$$\text{and} \quad \omega \frac{\partial}{\partial p} = \tilde{w} \frac{\partial}{\partial r_s} .$$

Hence

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r_s \cos\phi} \frac{\partial}{\partial \lambda} + \frac{v}{r_s} \frac{\partial}{\partial \phi} + \tilde{w} \frac{\partial}{\partial r_s} \tag{4.26}$$

(cf (2.8)) and the vertical component (4.19) may be written as

$$- 2\Omega u \cos\phi - \frac{u^2 + v^2}{r_s} - \frac{gT}{T_s} + \frac{\partial \Phi}{\partial r_s} = 0 \tag{4.27}$$

The inertial, metric and Coriolis terms in Eqs (4.17) - (4.19) are thus isomorphic to the corresponding terms in the height coordinate forms (see section 4(a)). Further, the isomorphism extends to the commutation properties of the relevant operators. For frictionless, adiabatic flow Eqs (4.17)-(4.21) imply the potential vorticity conservation law.

$$\frac{D}{Dt} \left(\frac{\tilde{\underline{Z}} . \tilde{\nabla} \theta}{\rho_s} \right) = 0 \tag{4.28}$$

Here $\rho_s = p/RT_s$ (as in (2.11)) ,

$$\tilde{\nabla} \equiv \left(\frac{1}{r_s \cos\phi} \frac{\partial}{\partial \lambda} , \frac{1}{r_s} \frac{\partial}{\partial \phi} , \frac{\partial}{\partial r_s} \right) \tag{4.29}$$

and

$$\tilde{\underline{Z}} = \left(- \frac{1}{r_s} \frac{\partial}{\partial r_s} (v r_s) \; , \; 2\Omega \cos\phi + \frac{1}{r_s} \frac{\partial}{\partial r_s} (u r_s) \; , \right.$$

$$\left. 2\Omega \sin\phi + \frac{1}{r_s \cos\phi} \left(\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi} (u \cos\phi) \right) \right) \tag{4.30}$$

Vertically propagating acoustic modes are not implied by the new system. This follows from either the omission of Dw/Dt in the vertical component equation or the isomorphism of the continuity equation (4.20) to an incompressible fluid form. Horizontally propagating acoustic modes - the Lamb modes - will be present (as in the HPEs) unless the boundary condition $\omega = 0$ is applied on appropriate pressure levels (see, for example, Miller and White (1984)).

(c) <u>σ-coordinate forms</u>

Transformation of the pressure coordinate equations (4.17)-(4.22) to σ-coordinates ($\sigma = p/p_*$, p_* = surface pressure) gives:

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r_s\cos\phi}\right)(v\sin\phi - \tilde{w}\cos\phi) + \frac{1}{r_s\cos\phi}\left(\frac{\partial\Phi}{\partial\lambda} + \frac{R(T+\mu T_s)}{p_*}\frac{\partial p_*}{\partial\lambda}\right) = F_\lambda \tag{4.31}$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r_s\cos\phi}\right)u\sin\phi + \frac{v\tilde{w}}{r_s} + \frac{1}{r_s}\left(\frac{\partial\Phi}{\partial\phi} + \frac{R(T+\mu T_s)}{p_*}\frac{\partial p_*}{\partial\phi}\right) = F_\phi \tag{4.32}$$

$$\frac{\partial\Phi}{\partial\sigma} + \frac{R}{\sigma}(T + \mu T_s) = 0 \tag{4.33}$$

$$\frac{\partial p_*}{\partial t}\frac{\partial}{\partial\sigma}(\sigma r_s^2) + r_s^2\,\tilde{\nabla}_\sigma.(p_*\underline{v}) + p_*\frac{\partial}{\partial\sigma}(r_s^2\,\dot{\sigma}) = 0 \tag{4.34}$$

$$c_p\frac{DT}{Dt} - \frac{RT\omega}{p} = Q \tag{4.35}$$

All differentiations with respect to t, λ and ϕ are carried out at constant σ,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r_s\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r_s}\frac{\partial}{\partial\phi} + \dot{\sigma}\frac{\partial}{\partial\sigma} \quad , \tag{4.36}$$

and

$$\tilde{\nabla}_\sigma\cdot(p_* \underline{v}) = \frac{1}{r_s^2\cos\phi}\left(\frac{\partial}{\partial\lambda}(p_* u r_s) + \frac{\partial}{\partial\phi}(p_* v r_s\cos\phi)\right) . \tag{4.37}$$

The quantity μ which appears in Eqs (4.31)-(4.33) is defined by Eq (4.22). $r_s = r_s(p)$ is defined by Eq (4.14) (and so multiplication by $r_s(p)$ does not commute with the operators $\partial/\partial t$, $\partial/\partial\lambda$, $\partial/\partial\phi$ taken at constant σ; hence the form of the divergence operator in (4.37)). $\tilde{w}$ is defined by Eq (4.15), with

$$\frac{\omega}{p} = \left(\frac{\dot{\sigma}}{\sigma} + \frac{1}{p_*}\frac{Dp_*}{Dt}\right) .$$

Numerical time integration of (4.31)-(4.35) may be carried out essentially as for the HPEs. The continuity equation (4.34) is used in the integrated forms

$$r_s^2(p_*)\frac{\partial p_*}{\partial t} = -\int_0^1 r_s^2\,\tilde{\nabla}_\sigma\cdot(p_* \underline{v})\,d\sigma \tag{4.38}$$

$$\dot{\sigma} = -\frac{1}{p_*}\frac{\partial p}{\partial t} - \frac{1}{p_* r_s^2}\int_0^\sigma r_s^2\,\tilde{\nabla}_\sigma\cdot(p_* \underline{v})\,d\sigma \tag{4.39}$$

to find $\partial p_*/\partial t$ and then $\dot{\sigma}$. The vertical component (4.33) is then used to find Φ (knowing u, v and T from the previous time-step) in the same way as the hydrostatic relation is used to find Φ from T (and Φ_*) when integrating the HPEs.

Equations (4.31), (4.32) and (4.35) may be converted to a flux formulation through the application of the continuity equation, (4.34). Integration of the σ-coordinate equations in either formulation poses no special difficulties in a grid-point numerical model. However, the variation of $r_s(p)$ over sigma surfaces means that spectral methods could not be applied in the same way as in HPE models; indeed, it appears that substantial reformulation of an existing spectral HPE model would be needed. (The pressure coordinate forms given in the previous subsection are readily amenable to spectral representation, given the approximate lower boundary condition $\omega = 0$ at $p = p_o$. All the shallow atmosphere formulations discussed in section 3 are amenable to spectral representation using spherical harmonics.)

5. Discussion

In this paper we have proposed dynamically consistent ways of including the $\cos\phi$ Coriolis terms in quasi-hydrostatic, acoustically-filtered models of the global atmosphere. The models described in section 3 conserve energy, but not axial angular momentum or potential vorticity. The pressure-coordinate model described in section 4 reproduces all three conservation properties, and incorporates a limited relaxation of the shallow atmosphere approximation. This new model (which may be precisely transformed to σ-coordinates or other vertical coordinate systems) is put forward as a more accurate alternative to the traditional hydrostatic primitive equations. It has comparably good conservation properties and is as easy to integrate numerically using grid-point methods. A number of variants of the new model seem well worth exploring in future studies and will now be briefly discussed.

(a) A nonhydrostatic global model

It is straightforward to extend the p-coordinate equations (4.17)-(4.21) to include a prognostic form of the vertical component. If (4.19) is replaced by

$$\frac{\tilde{D}w}{Dt} - 2\Omega u\cos\phi - \frac{u^2 + v^2}{r_s} - \frac{gT}{T_s} + \frac{\partial\Phi}{\partial r_s} = 0 \qquad (5.1)$$

(cf.(4.27)) then the three components (4.17), (4.18), (5.1) may be written in vector form as

$$\frac{\tilde{D}\underline{u}}{Dt} + 2\underline{\Omega} \times \tilde{\underline{u}} - \underline{k}\frac{gT}{T_s} + \tilde{\nabla}\Phi = \underline{F}_h \qquad (5.2)$$

where $\tilde{\underline{u}} = (u, v, \tilde{w})$. The implied axial angular momentum principle is (4.24), as before. When taken together with Eqs (4.20) and (4.21), Eq (5.2) gives consistent energetics and a potential vorticity conservation law in terms of the vorticity

$$\left(\frac{1}{r_s} \frac{\partial \tilde{w}}{\partial \phi} - \frac{1}{r_s} \frac{\partial}{\partial r_s} (v r_s) \,,\; 2\Omega\cos\phi + \frac{1}{r_s} \frac{\partial}{\partial r_s} (u r_s) - \frac{1}{r_s\cos\phi} \frac{\partial \tilde{w}}{\partial \lambda} \,, \right.$$

$$\left. 2\Omega\sin\phi + \frac{1}{r_s\cos\phi} \left(\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi} (u\cos\phi) \right) \right)$$

- which is similar to $\tilde{\underline{Z}}$ (see (4.30)) but with $\tilde{w}$ terms included (cf.(3.11) of paper I).

This formulation incorporates the representation of the vertical acceleration term Dw/Dt used in Miller's (1974) p-coordinate model of nonhydrostatic convection. Solution would proceed by time integration for $\tilde{w}$ as well as for u,v and T, Φ being determined by solving the global 3D Poisson equation obtained by taking the divergence ($\tilde{\nabla}.$) of (5.2) and applying the continuity equation (4.20). A σ-coordinate version of this model may be envisaged (see Miller and White (1984)).

Nonhydrostatic, global models which retain the term Dw/Dt in the vertical component of the momentum equation have been discussed by Daley (1988), but his formulations allowed acoustic modes. Nonhydrostatic, Boussinesq fluid models using spherical radius (r) as vertical coordinate have been integrated numerically by Gilman (1975, 1977), Miller and Gall (1982) and others. The model proposed here is nearly isomorphic to these Boussinesq models, the only difference being that the mean density is a function of the vertical coordinate $r_s(= r_s(p))$ rather than being a constant: the continuity equation (4.20) can be written in terms of r_s and $\tilde{w}$ as

$$\tilde{\nabla}_p . \underline{v} + \frac{1}{\rho_s r_s^2} \frac{\partial}{\partial r_s} \left(\rho_s r_s^2 \tilde{w} \right) = 0 \qquad (5.3)$$

(cf.(2.9)).

(b) A "local temperature" modification

Following Miller (1974), a reference temperature profile $T_s(p)$ has been used in sections 3 and 4 to define an approximate vertical velocity (see Eqs (3.21) and (4.15)). It can be shown (White 1989) that Miller's nonhydrostatic convection model may be consistently extended to use the approximate vertical velocity $\hat{w} = - \omega RT/gp$. The appearance of the local temperature T (instead of $T_s(p)$) in this expression allows a more accurate representation and does not impair the energy and potential vorticity conservation properties of the model. Such an extension is possible for the global model put forward in section 4 (or for the modification discussed in section 5(a)) but its properties remain to be established. Instead of $r_s(p)$, the extended model would use $\hat{r}$ defined by

$$\hat{w} = \frac{D\hat{r}}{Dt} = - \frac{RT\omega}{gp} \qquad (5.4)$$

Clearly it would be necessary to integrate (5.4) numerically in order to obtain $\hat{r}$ as a function of time. Examination of the potential vorticity properties would be complicated by the fact that $\hat{r}$ (unlike $r_s(p)$) does not commute with the operators $\partial/\partial t$, $\partial/\partial\lambda$, $\partial/\partial\phi$ taken at constant pressure; hence the isomorphism exploited in section 4 could not be used. A further difficulty concerns the form of the continuity equation. The height-coordinate form (see (3.6) of paper I) may be transformed to p-coordinates as

$$\frac{\partial}{\partial t}(r^2) + \frac{1}{\cos\phi}\left[\frac{\partial}{\partial\lambda}(ur) + \frac{\partial}{\partial\phi}(vr\cos\phi)\right] + \frac{\partial}{\partial p}(\omega r^2) = 0 \qquad (5.5)$$

(assuming constant g and the applicability of hydrostatic balance to the mass budget). In (5.5) the temporal and horizontal derivatives are taken at constant p, so that the term $\partial r^2/\partial t$ is in general non-zero, and r cannot be taken through the $\partial/\partial\lambda$ and $\partial/\partial\phi$ derivatives. These disadvantages remain if r is replaced by $\hat{r}$, but they disappear if $r \to r_s(p)$ (in which case the simple form (4.20) is obtained). As it stands, (5.5) is not convenient for finding surface pressure tendencies. The σ-coordinate form is

$$\frac{\partial}{\partial t}(p_* r^2) + \frac{1}{\cos\phi}\left[\frac{\partial}{\partial\lambda}(p_* ur) + \frac{\partial}{\partial\phi}(p_* vr\cos\phi)\right] + p_* \frac{\partial}{\partial\sigma}(r^2\dot{\sigma}) = 0$$

from which $\partial p*/\partial t$ cannot be readily calculated because of the accompanying r^2 factor. (Note, however, that (4.34) may be recovered if $r = r_s(p)$.) These complications reflect the fate of the relationship between total atmospheric mass and mean surface pressure as the shallow atmosphere approximation is relaxed. In the shallow atmosphere case, the total mass M of the atmosphere (assuming hydrostatic balance) is (area mean surface pressure) x $4\pi a^2/g$. When r is allowed to vary, M = (area mean value of $\int_0^{p_*} r^2 dp$) x $4\pi a^2/g$; this is expressible in terms of the surface pressure field if $r = r_s(p)$, but not otherwise. Use of $r_s(p)$, as in section 4, represents a natural first step away from the shallow atmosphere approximation, and it may be that anything more precise would be computationally unwieldy.

(c) Variable gravity models

The shallow atmosphere approximation has been relaxed in this study for reasons of dynamical consistency, and not because it is considered to be a source of appreciable inaccuracy. Its relaxation nevertheless prompts a suspicion that the radial variation of gravity should be allowed for too. If the form $g \propto 1/r_s^2$ is used as a crude exploratory representation it is found that the p-coordinate form of the continuity equation is

$$\frac{1}{r_s\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right) + \frac{1}{r_s^4}\frac{\partial}{\partial p}\left(r_s^4\,\omega\right) = 0$$

(As usual, hydrostatic balance is assumed to be applicable to the mass budget.) The r_s^4 variation in the $\partial/\partial p$ term arises because the decrease of g with radial distance reinforces the accompanying increase of spherical surface area. Its appearance suggests that the usual shallow atmosphere, constant g representation of the HPEs may be quantitatively undesirable in stratospheric simulations quite apart from the importance of the $\cos\phi$ Coriolis terms. The possibility of including latitude variation of g - to take into account the varying centrifugal contribution - should also be investigated. Indeed, it is probably necessary to assume such a variation in order to take best advantage of the angular momentum principle as it governs the form invariance of the energy conservation law.

Table 1

Definitions and typical values of the nondimensional quantities

E, F, Ri^{-1} and Ro

Quantity	E	F	Ri^{-1}	Ro
Definition	$2\Omega U\cos\Phi/g$	U^2/gH	U^2/N^2H^2	U/fL
Typical value in tropics	10^{-4}	10^{-3}	10^{-2}-10^{-1}	1
Typical value in middle latitudes	10^{-4}	10^{-3}	10^{-2}	10^{-1}

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APPENDIX A

Sketch proof of Equation (4.9)

Upon use of the spherical polar expression of grad (see (3.13) of paper I) it follows that

$$\mathrm{curl}(\underline{k}\,\frac{Dw}{Dt}) = -\,\underline{k} \times \mathrm{grad}(\frac{Dw}{Dt}) = \frac{1}{r}\,(\underline{i}\,\frac{\partial}{\partial\phi} - \underline{j}\,\frac{1}{\cos\phi}\,\frac{\partial}{\partial\lambda})\,\frac{Dw}{Dt} \qquad \text{(A1)}$$

because the radial unit vector $\underline{k}$ is irrotational. (Fig. 1 of paper I illustrates the spherical polar coordinates λ, ϕ, r and the associated velocity components u, v, w and unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$).

It is readily shown that

$$\left.\begin{aligned}\frac{1}{r}\,(\underline{i}\,\frac{\partial}{\partial\phi} - \underline{j}\,\frac{1}{\cos\phi}\,\frac{\partial}{\partial\lambda})\,\frac{Dw}{Dt} &= \frac{D\underline{\xi}}{Dt} - \frac{1}{r}\,\frac{\partial w}{\partial\phi}\,\frac{D\underline{i}}{Dt} + \frac{1}{r\cos\phi}\,\frac{\partial w}{\partial\lambda}\,\frac{D\underline{j}}{Dt} \\ &\quad + \underline{i}\,\frac{w}{r^2}\,\frac{\partial w}{\partial\phi} + \underline{j}\,(v\tan\phi - w)\,\frac{1}{r^2\cos\phi}\,\frac{\partial w}{\partial\lambda}\end{aligned}\right\} \qquad \text{(A2)}$$

in which $\underline{\xi} \equiv \frac{1}{r}\,(\underline{i}\,\frac{\partial w}{\partial\phi} - \underline{j}\,\frac{1}{\cos\phi}\,\frac{\partial w}{\partial\lambda})$ (A3)

Now

$$\left.\begin{aligned}&\frac{\partial\underline{i}}{\partial\lambda} = \underline{j}\sin\phi - \underline{k}\cos\phi\,, &&\frac{\partial\underline{i}}{\partial\phi} = 0\,, &&\frac{\partial\underline{i}}{\partial r} = 0 \\ &\frac{\partial\underline{j}}{\partial\lambda} = -\,\underline{i}\sin\phi\,, &&\frac{\partial\underline{j}}{\partial\phi} = -\underline{k}\,, &&\frac{\partial\underline{j}}{\partial r} = 0 \\ (\text{and}\quad &\frac{\partial\underline{k}}{\partial\lambda} = \underline{i}\cos\phi\,, &&\frac{\partial\underline{k}}{\partial\phi} = \underline{j}\,, &&\frac{\partial\underline{k}}{\partial r} = 0\quad)\end{aligned}\right\} \qquad \text{(A4)}$$

$$\text{Hence } r \frac{D\underline{i}}{Dt} = \underline{j}u\tan\phi - \underline{k}u \tag{A5}$$

$$r \frac{D\underline{j}}{Dt} = - \underline{i}u\tan\phi - \underline{k}v \tag{A6}$$

Using (A3)-(A5), and (3.12) of paper I for div $\underline{u}$, (A2) becomes

$$\left.\begin{aligned} &\frac{1}{r}\left(\underline{i}\frac{\partial}{\partial\phi} - \underline{j}\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}\right)\frac{Dw}{Dt} = \frac{D\underline{\xi}}{Dt} + \underline{\xi}\text{ div }\underline{u} \\ &\qquad + \frac{1}{r^2\cos\phi}\frac{\partial w}{\partial\lambda}\left(\underline{i}\frac{\partial u}{\partial\phi} + \underline{j}\left(\frac{\partial v}{\partial\phi} + w\right) - \underline{k}v\right) \\ &+ \frac{1}{r^2}\frac{\partial w}{\partial\phi}\left[\underline{i}\left(v\tan\phi - w - \frac{1}{\cos\phi}\frac{\partial u}{\partial\lambda}\right) - \underline{j}\left(u\tan\phi + \frac{1}{\cos\phi}\frac{\partial v}{\partial\phi}\right) + \underline{k}u\right] \end{aligned}\right\} \tag{A7}$$

But

$$- (\underline{\xi}.\text{grad})\ \underline{u} = \frac{1}{r^2\cos\phi}\left(\frac{\partial w}{\partial\lambda}\frac{\partial}{\partial\phi} - \frac{\partial w}{\partial\phi}\frac{\partial}{\partial\lambda}\right)(u\underline{i} + v\underline{j} + w\underline{k})$$

the right-hand side of which expands to give the terms in $\frac{\partial w}{\partial\lambda}$ and $\frac{\partial w}{\partial\phi}$ on the right-hand side of (A7), upon use of (A4). Hence, from (A1) and (A7):

$$\text{curl }(\underline{k}\ Dw/Dt) = \frac{D\underline{\xi}}{Dt} + \underline{\xi}\text{ div }\underline{u} - (\underline{\xi}.\text{grad})\ \underline{u}$$

which is (4.9).