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AN ABSORBING UPPER BOUNDARY CONDITION FOR ATMOSPHERIC MODELS

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(This work was undertaken by Dr Davies while he was employed as a Vacation Consultant to Met O 11 during August 1978).

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A local energy-absorbing, upper boundary condition is developed for atmospheric models. The behaviour of the free modes of the bounded-atmosphere are compared to those of an infinite atmosphere for the simple case of an isothermal, quiescent, basic state. It is shown that, if the bounded-atmosphere is sufficiently deep, the free oscillations of the two systems differ in only relatively minor respects. Moreover, unlike the system with an upper rigid-lid approximation, no spurious free oscillations occur for the bounded-atmosphere system, and hence the possibility of spurious resonant response to forcing is excluded a priori.

The boundary condition is also shown to allow substantial upward propagation of wave energy out of the system for waves in a broad frequency and vertical wavenumber band.

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1. INTRODUCTION

The upper atmosphere acts both as an absorber and reflector of energy propagated from lower altitudes, and the formulation of atmospheric numerical models should be in accord with this fact. Thus an upper boundary condition applied at some finite height should allow for the possible transmission of energy through that level. This energy would correspond to that which in the real atmosphere would be absorbed at altitudes above the top of the model. Again, a model of the entire depth of the atmosphere must be sufficiently detailed to absorb and reflect upward propagating wave energy without inducing spurious reflection at any level.

An upper boundary condition that requires the vertical velocity (w) or a pseudo-vertical velocity $(\omega$ or $\dot{\sigma})$ to be set to zero at some finite height, pressure, or pseudo-pressure level will effect a perfect reflection of wave energy at that level. Again, due to truncation effects, setting ω or $\dot{\sigma}$ to zero at the model's level of zero pressure will also induce reflection. It is highly desirable to obtain an understanding of the possible effect of these 'rigid-lid' type of upper boundary conditions since virtually all GCMs and numerical weather prediction models currently employ a condition of this genre.

Several repercussions arise from applying a rigid-lid type upper boundary condition (u.b.c.) that are in addition to, or related to, the in general spurious reflection of wave energy. The free modes of oscillation of the model will differ somewhat from the corresponding atmospheric modes, and the model will also sustain certain spurious modes (Lindzen et al, 1968). Thus for a linear model we would infer that the projection of the initial data onto the model's free and forced modes would be misrepresented. For example the partition between the forced and free mode components might be in error (Kirkwood and Derome, 1977; Desmarais and Derome, 1978) with the possibility of resonant response to the aforementioned spurious modes (Lindzen et al, 1968; Hayashi, 1976). Again initial data representing a non-horizontal propagating mode would be misrepresented in terms of both 'realistic' and spurious horizontally propagating modes. These misrepresentations would give rise to forecast errors during the subsequent evolution of the flow. The 'realistic' free modes of oscillation will themselves have phase speeds that are incorrect and will also contribute to the forecast error. These possibly serious shortcomings emphasize clearly that the unjustifiable rigid-lid approximation to the upper boundary is open to suspicion unless the vertical resolution of the numerical model is adequate. Recent theoretical studies with linear models by Nakamura (1976) and by Derome and his collaborators tend to fuel this suspicion. One attractive possibility to alleviate the effect of the rigid-lid is to artificially remove upward propagating wave energy before it can be reflected at the top by including a region with Rayleigh friction (and/or Newtonian cooling) or a layer of high viscosity beneath the rigid-lid. (In the upper atmosphere Newtonian cooling does indeed contribute substantially to the removal of wave energy.) This procedure has been employed in simple gravity wave models (Houghton and Jones, 1969; Klemp and Lilly, 1978) in an inertia-gravity wave model (Eliassen and Rekustad, 1971) and in a linear planetary wave model (Tokioka and Arakawa, unpublished). However care must be taken in using this method since sharp gradients in the friction coefficient can induce spurious reflection (Klemp and Lilly, 1978). The study of the latter authors demonstrates that it is desirable to have of the order of eight model layers in the artificial damping zone. Computational considerations would suggest that the technique would be unacceptable for all but the simpler numerical models.

The foregoing considerations provide the motivation for seeking an u.b.c. that allows for the radiation of wave energy. A suitable radiation condition can be derived for various linear wave problems (Eliassen and Palm, 1961; Charney and Drazin, 1961), but its application in a time dependent problem requires knowledge of the time history of the flow structure across the entire domain (Beland and Warn, 1975; Bennett, 1976). It is therefore too complex for implementation in most numerical models, and moreover its validity for the non-linear atmospheric situation is in doubt (Beland, 1976). Again the validity of the pseudo-radiation condition preposed by Orlanski; (1976) is not clear for the particular problem under consideration.

In the succeeding sections we derive on the basis of simple physical reasoning a 'local' absorbing boundary condition for linear perturbations of the atmosphere about an isothermal basic state of no motion. A rigorous analysis of the free modes of oscillation of the system and an assessment of the reflectivity of the u.b.c. illustrate both its potential and limitations. The proposed condition is also examined in the context of the mathematical framework recently provided by Engquist and Majda (1977) for deriving local absorbing boundary conditions that approximate the theoretical nonlocal radiation boundary condition.

Finally some comments are made regarding the implementation of the proposed u.b.c. in a numerical model.

2. GENERAL CONSIDERATIONS

We consider the linear equations representing small perturbations of an hydrostatic, compressible, and inviscid atmosphere about a basic isothermal and quiescent state. The perturbations satisfy the following pseudo-energy constraint,

$$\frac{\partial}{\partial t} \iiint_V \tfrac{1}{2}\left\{ \bar{\rho}(u'^2+v'^2) + (p')^2/(c^2\bar{\rho}) + (g\bar{\rho}/S)(\theta'/\theta)^2 \right\} dV$$

$$= -\iiint_V \left\{ \nabla_h \cdot (p'\underset{\sim}{V_h'}) + \frac{\partial}{\partial z}(p'w') \right\} dV , \qquad (1)$$

where the overbarred and primed symbols refer respectively to the basic state and perturbed variables, and remaining notation is conventional. It follows immediately from Eq (1) that, if the domain is self contained in the horizontal, the norm of the perturbation variables will decrease with time if conditions at the upper and lower boundaries $(z=0, z_T)$ are such that,

$w'=0$ at $z=0$ corresponding to a rigid surface,

and $p'w'>0$ at $z=z_T$ corresponding to upward energy transfer.

Thus an upper boundary condition of the form $w'=f(p')$ such that $\{p'f(p')\}>0$ for all locations and for all time on $z=z_T$ will provide an energy absorbing boundary condition. If our system was unbounded in the vertical then the free modes would be either evanescent or wave-like in character in the vertical. The application of an absorbing upper boundary condition implies that the free modes of the bounded system will be either wave-like or if a mixed type in their vertical structure. The analysis in the next section illustrates and helps to clarify this somewhat unusual feature.

3. FREE OSCILLATIONS OF THE BOUNDED ATMOSPHERE

If our upper boundary condition is suitable then the governing equations of our system can be separated in the usual manner into horizontal and vertical structure equations. The latter equation takes the form,

$$\chi_{z^*z^*} + \lambda^2\chi = 0 , \qquad (2)$$

where $\lambda^2 = (\kappa H/h - \tfrac{1}{4})$, (3)

and $\chi = (p'/\bar{\rho})\, e^{\frac{1}{2}z^*}$ with $z^* = z/H$ Here H is the atmospheric scale height and h is the separation constant (the so-called equivalent depth). On the sphere the solutions of the horizontal structure equation take the form of Hough functions. For simplicity we shall assume a mid-latitude β-plane system in which case the trigonometric wave solutions of the form $\exp\{i(kx + ly - \sigma t)\}$ satisfy the frequency equation,

$$\tilde{\sigma}^3 + [1 + F(m^2+n^2)]\tilde{\sigma} - F\tilde{\beta}m = 0 , \tag{4}$$

with $F = 2gh/(fa)^2$, $\tilde{\beta} = \beta a/(\sqrt{2}f)$, where a is the earth's radius. Here $\tilde{\sigma}$, m, n represent the dimensionless frequency, zonal and meridional wavenumbers defined such that $\tilde{\sigma} = \sigma/f$, $m = ka/\sqrt{2}$, $n = la/2$ where σ, k and l refer to the corresponding dimensional variables. Our problem then is to determine λ as an eigenvalue of Eq (2) subject to the boundary conditions,

$$W' = 0 \qquad \text{i.e.} \quad \left\{\frac{\partial}{\partial z} + \left(\tfrac{1}{2} - \kappa\right)\right\}\chi = 0 \qquad \text{at} \quad z = 0$$

$$\text{and} \quad W' = f(p) \qquad \text{i.e.} \quad \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial z} + \left(\tfrac{1}{2} - \kappa\right)\right\}\chi = -\Gamma^*\chi \qquad \text{at} \quad z = z_T ,$$

where Γ^* is an, as yet, undefined operator. Then to obtain estimates of the eigenfrequencies from Eq (4) we allow for the fact, alluded to earlier, that λ and hence h and F may be complex. To derive these estimates we note, from Eq (3), that h and λ are related as follows,

$$h_r \approx \kappa H/(\tfrac{1}{4} - \lambda_i^2) \quad \text{and} \quad h_i \approx -(\kappa H)2\lambda_r\lambda_i/(\tfrac{1}{4} - \lambda_i^2) \text{ if } (\lambda_r/\lambda_i) \ll 1 \tag{5}$$

It then follows that $(F_i/F_r) \ll 1$ and hence to a good approximation the roots of Eq (4) are given by the formulae,

$$\tilde{\sigma}_r = -\tilde{\beta}F_r m/E , \quad \sigma_i = -\tilde{\beta}F_i m/E^2 \qquad \text{for the Rossby wave,} \tag{6}$$

$$\text{and} \quad \tilde{\sigma}_r = \pm E^{\frac{1}{2}} , \quad \sigma_i = \pm\tfrac{1}{2}F_i(m^2+2n^2)/E^{\frac{1}{2}} \qquad \text{for the inertia-gravity waves} \tag{7}$$

where $E = [1 + F_r(m^2 + 2n^2)]$

After some algebraic manipulation it can be shown that the imaginary and real parts of the vertical wavenumber are related by the following two relationships,

$$\tanh 2\lambda_i z_r^* = 2\lambda_i L/(L^2 + \lambda_i^2) , \tag{8}$$

$$\tan 2\lambda_r z_r^* = -\{2\Gamma/(\Gamma^2 + \sigma_r^2)\}[\lambda_i\sigma_r - \lambda_r L\Gamma/(L^2 - \lambda_i^2)], \tag{9}$$

where $L = (\tfrac{1}{2} - \kappa)$ provided the value of $L\Gamma$ is such that

$$L\Gamma \gg \{(2\lambda_r\lambda_i\sigma_r), \sigma_r[L^2 - \lambda_i^2]\}, \quad \text{and} \quad (\lambda_i/\lambda_r)^2 \ll 1 .$$

Here Γ is the spectral coefficient of the operator Γ^*.
We shall verify 'a posteriori' that these inequalities are valid for the solutions discussed hereafter. It can be readily deduced from Eqs (8) and (9) that the waves are purely evanescient (i.e. $\lambda_r \equiv 0$) only in the limit of $Z_T \longrightarrow \infty$
For finite Z_T the vertical wavenumber (λ) is complex, but we can determine the form of the solutions by specifying Z_T and Γ and solving sequentially for λ_i (Eq 8), h_r (Eq 5), σ_r (Eqs 6,7), λ_r (Eq 9), h_i (Eq 5) and σ_i (Eqs 6,7).

In Fig 1 the variation of $|\lambda_i|$ is shown as a function of the depth of the model atmosphere. The dashed line refers to the comparable value for the unbounded atmosphere subject to a radiation condition. It is evident that a very large error in the value of λ_i is incurred if the model top is lower than $Z_T^* = 6$ (≈ 48 Km) The variation of λ_i with Z_T produces a related change in the value of the equivalent depth (h). The (Z_T^*, h) variation of the single-valued h of our model, with an absorbing u.b.c., is displayed in Fig 2, and compared with the variation of the multi-valued equivalent depths of models with a $W = 0$ u.b.c. and an $\omega = 0$ u.b.c. The single-value of h for a given Z_T implies that there are no spurious free modes, and it is seen that h is within 4 per cent of its correct value for $Z_T^* \geqslant 8.2$. An incorrect value for the equivalent depth implies that the frequency and phase speed of the waves will also be in error (Eqs 6,7) and some indication of the magnitude of this error for Rossby waves is given in Table 1. The percentage error in the frequency is seen to be 7 per cent, or less, for $Z_T^* = 6$, but it must be remembered that the generating formulae are themselves only approximations.

To conclude our study of the free modes we determine the values of the vertical wavenumber (λ_r) and the time-decay rate $(-\sigma_i)$. Table 2 is a plot of these variables against the zonal wavenumber (m) for $Z_T = 6$, and we have set $\Gamma = m$ for reasons that will become apparent later. We note that these solutions are in accord with the assumptions made in deriving Eqs 8 and 9. Moreover the implied vertical wavelength and e-folding decay time are so large as to effect little change in the structure of the free modes over the vertical domain and for reasonable time periods. The absorbing u.b.c. is thus seen to produce only modification of the free Rossby modes of the unbounded atmosphere, and if the bounded atmosphere is sufficiently deep these modifications are only minor.

4. THE REFLECTIVITY OF THE UPPER BOUNDARY CONDITION

We consider an upward-energy propagating wave, with a given frequency $(\tilde{\sigma})$ and vertical wavenumber (λ), of the form

$$\chi_A = A\, e^{i(mx + ny - \tilde{\sigma} t)}\, e^{-i\lambda z},$$

impinging upon a boundary at $z^* = z^*_T$ where the following condition must be satisfied,

$$\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial z} + \left(\frac{1}{2} - k\right)\right\}\chi = -\Gamma^* \chi \tag{10}$$

The reflectivity of the boundary will be measured by the ratio $R = |B/A|$ where B is the amplitude of a downward-energy propagating wave of the form

$$\chi_B = B e^{i(mx + ny - \tilde{\sigma} t)} e^{+i\lambda z}$$

letting $\chi = \chi_A + \chi_B$ and substituting into Eq (10), we obtain, after some manipulation, the following expression for R^2,

$$R^2 = \frac{\tilde{\sigma}^4 (\ell^2 + \lambda^2)^2 + 2\tilde{\sigma}^2 \Gamma^2 (\ell^2 + \lambda^2) + \Gamma^4}{\tilde{\sigma}^4 (\ell^2 + \lambda^2)^2 + 2\tilde{\sigma}^2 \Gamma^2 (\ell^2 + \lambda^2)\mu + \Gamma^4 \mu^2} , \tag{11}$$

where $\mu = 1 + (2\tilde{\sigma}\lambda/\Gamma)$

In particular if we assume that $(L/\lambda) \ll 1$ i.e. the vertical wavelength is much less than 300 km, then Eq (11) may be approximated by the relation,

$$R^{\cdot} = \frac{\tilde{\sigma}^4 \lambda^4 + \Gamma^2(\Gamma^2 - 2\tilde{\sigma}^2\lambda^2)}{\tilde{\sigma}^4 \lambda^4 + \mu \Gamma^2 (\mu \Gamma^2 + 2\tilde{\sigma}^2 \lambda^2)} \tag{12}$$

In this limit R is seen to become a function of $\delta = (\tilde{\sigma}\lambda/\Gamma)$ and this dependency is illustrated in Table 3. Almost total absorption occurs for $\delta = 1\ (\Gamma = \tilde{\sigma}\lambda)$ and the reflectivity is less than $1/3$ and $1/10$ respectively for δ in the ranges $(\cdot 5, 2)$ and $(\cdot 8, 1\cdot 2)$.

Thus the effectiveness of our absorbing boundary condition will depend upon our ability to design the operator Γ^* such that $\Gamma \approx \tilde{\sigma}\lambda$ for the range of atmospheric wave motions that propagate energy vertically through the level $z = z_T$ For the particular system considered herein we note that,

$$\lambda^2 = \left(\frac{KH}{h} - \frac{1}{4}\right)$$

and from Eq (4)

$$\lambda^2 = \frac{1\cdot 11}{10^2}\left\langle \frac{m^2 + \tilde{\beta} m/\tilde{\sigma}}{(\tilde{\sigma}^2 - 1)} \right\rangle - \frac{1}{4} \tag{13}$$

It follows that for geostrophic modes,

$$(\tilde{\sigma}\lambda)^2 \approx (1.11 . 10^{-2}\, \tilde{\beta}\, m \sigma)^{1/2}$$

If the main contribution to $\tilde{\sigma}$ is the purely advective effect of the mean flow, i.e. $\sigma \approx -Uk\ (\tilde{\sigma} \approx \cdot 24\ 10^{-2} U m)$, then we infer that,

$$|\tilde{\sigma}\lambda| \approx \cdot 48\ 10^{-2} (\tilde{\beta} U)^{1/2} m ,$$

i.e. $|\tilde{\sigma}\lambda| = \alpha m$, where $\alpha \sim \cdot 01$

Thus we require $\Gamma \approx \alpha m$, and the operator Γ^* must be chosen accordingly. One possible choice would be to set

$$\Gamma^* \chi = -\frac{1}{\left\{ \left| K_H \frac{\partial^2 \chi}{\partial x^2} \right|^{1/2} + C \right\}} K_H \frac{\partial^2 \chi}{\partial x^2}$$

where C would be given a very small, but non-zero, value to avoid a singularity in the response of the operator. In principle an estimate of the value of α could be inferred from theoretical studies of the vertical propagation of energy by planetary scale waves (e.g. Charney and Drazin, 1961; Holton, 1975), but in practice it would be probably preferred to select a value of α based upon experimentation with the particular numerical model in hand.

We are now in a position to compare our 'ad hoc' approach with the systematic framework proposed by Engquist and Majda (1977). The non-local boundary condition $\chi_{z^*} = -i\lambda\chi$ applied at $z^* = z^*_T$ would annihilate the reflected wave, and Engquist and Majda propose that an approximate local, absorbing condition can be derived by some suitable expansion of Eq (13) for λ. For the particular case considered by them on expansion about the case of normal incidence proved very effective. For the problem of the atmospheric u.b.c. for the planetary scale waves the limit of normal incidence $(m \to 0)$ is inappropriate at least at the lowest order of approximation, since it corresponds to non-propagation in the vertical. Our approach indicates that a natural expansion would be for $(\tilde{\sigma}\lambda)$ with Eq (10) providing the base formula. Moreover the energy estimate of Eq (1) demonstrates the well-posedness of our mixed initial boundary value problem.

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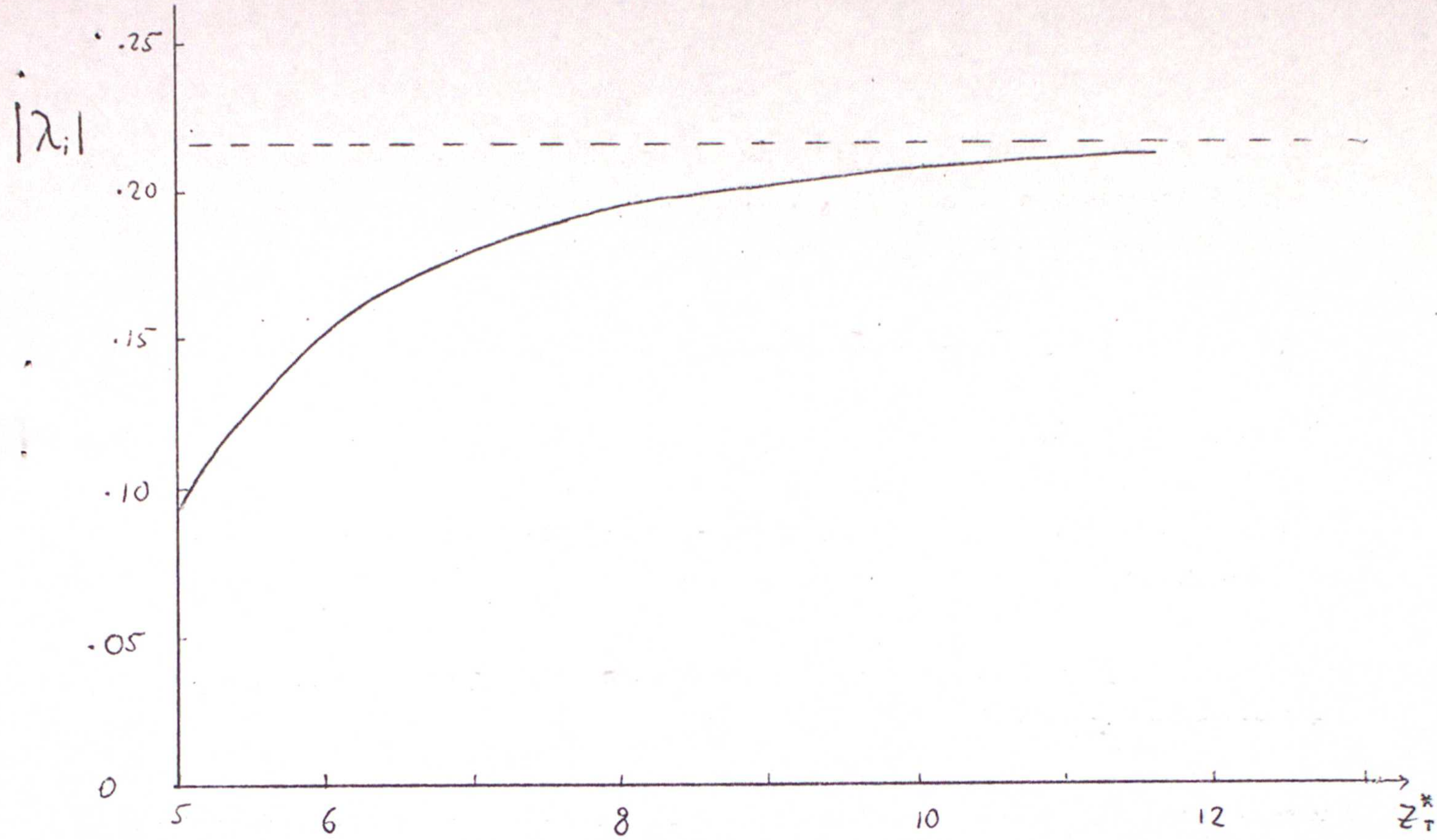
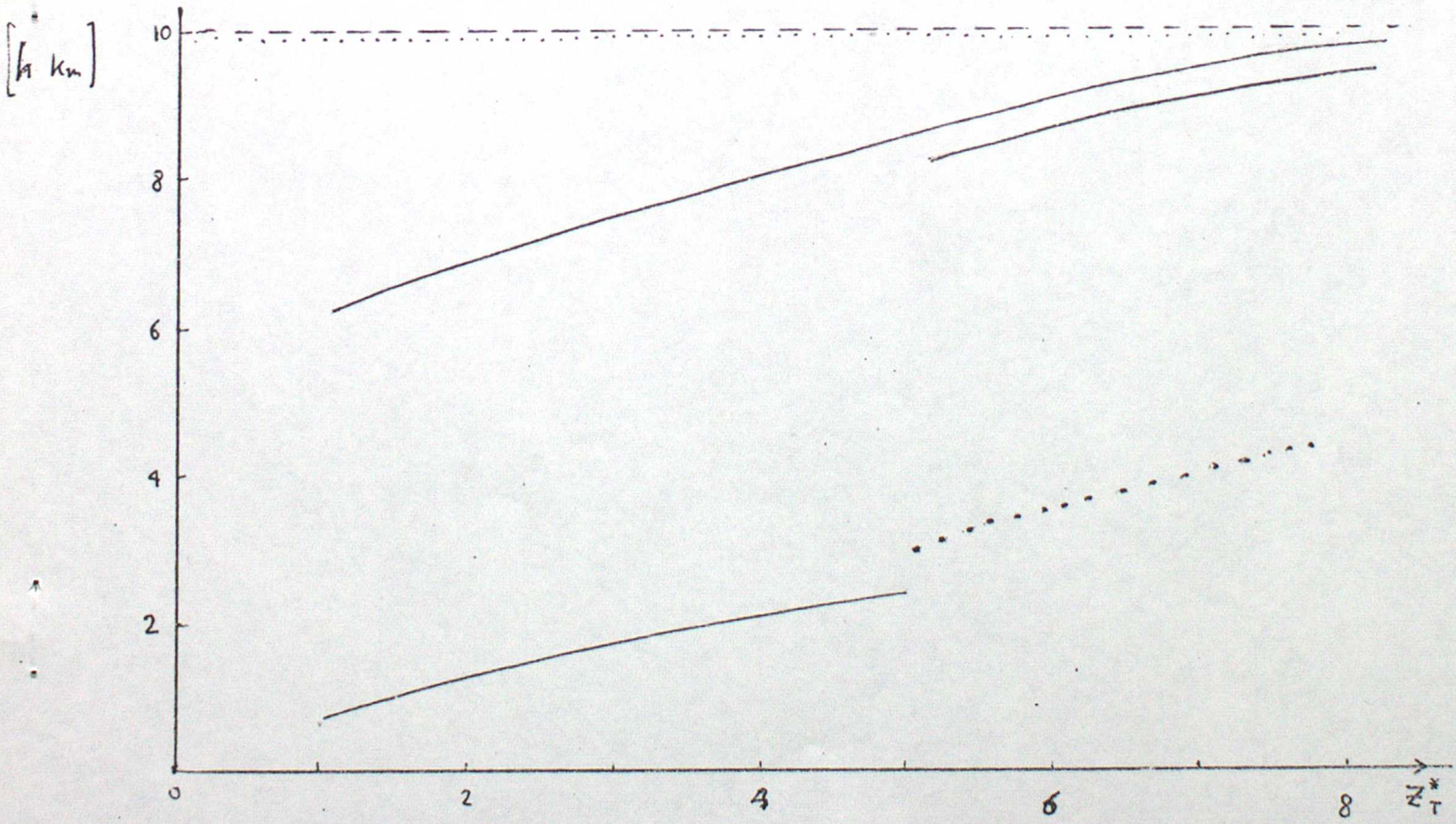


- - - - UNBOUNDED ATMOS. ——— ABSORBING UPPER BOUNDARY CONDITION

FIGURE 1



——— $\omega = 0$ u.b.c. . . . . . $W = 0$ u.b.c

FIGURE 2

TABLE 1   VALUES OF $\tilde{\sigma}(Z_T^*, m)$

$m \backslash Z_T^*$	∞	8	6	5
1	·0882	·0871	·0850	·0825
2	·1092	·1024	·1.047	·1047
4	·0866	·0863	·0857	·0851
8	·0507	·0506	·0505	·0504
10	·0414	·0413	·0413	·0413

TABLE 2

m	1	2	4	8	10
λ_r	·093	·053	·822	·0074	·0051
$\lvert\sigma_i\rvert$	·0048	·0022	·00029	$1{\cdot}7\ 10^{-5}$	$1{\cdot}4\ 10^{-6}$

TABLE 3

$\left(\frac{\tilde{\sigma}\lambda}{r}\right)$	0·1	0·5	0·8	1·0	1·2	2	4	10
R	·82	·33	·11	0·0	·091	·33	·6	·82