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Application of a stress-equation model to flow over hills.

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APPLICATION OF A STRESS-EQUATION MODEL TO FLOW OVER HILLS

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Summary

Turbulent flow over topography can be divided into flow regions with different physical characteristics. In particular, the Reynolds stresses vary according to rapid distortion theory aloft, but are in equilibrium near to the ground. A turbulence closure which takes account of these features is used in a numerical model to obtain the mean flow field and stress fields for flow over topography.

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Introduction

An understanding of boundary-layer flow over topography has many applications in meteorology. Changes in wind speed and turbulence stucture as the flow passes over a hill can have a significant effect on local pollution dispersion, wind-loading on buildings and the viability of wind-energy systems. The drag induced by a hill or region of hills can also have an effect on larger scale atmospheric motions. Many studies have been undertaken for laminar flow over topography, but until recently there have been relatively few studies of turbulent flow, despite the fact that typically the boundary layer is turbulent.

One such early study was that carried out by Jackson and Hunt (1975). They solved for flow over a low hill (linearising the equations) and assumed that the equilibrium layer formula relating the shear stress to the mean velocity gradient was still valid. They found that, irrespective of the particular assumptions made in representing the stresses, the mean flow field could be split up into two regions; an inner region $z < l_{JH}$ where stress variations had a significant effect on the mean flow over the hill, and an outer region $z > l_{JH}$ where the velocity perturbation Δu was effectively inviscid and not influenced by the Reynolds stresses (Figure 1) .

Sykes (1980) carried out a similar analysis, but used a different turbulence model, presenting explicit equations for the Reynolds stresses (Launder, Reece and Rodi, 1975). Sykes found that the stress equations themselves gave rise to a second (unrelated) layer structure for stress variations. Within a region $z < l_s$, the turbulence is in equilibrium - in the outer region $z > l_s$ advection of the stresses is important and equilibrium no longer holds. It is here that equilibrium models become inaccurate. If we consider flow over a low hill of length L and height $\delta^{1/2}L$ where $\delta \ll 1$, the stresses are in equilibrium within a region $z < l_s \sim \delta L$, whilst the velocity perturbations are affected by the stresses in $z < l_{JH} \sim \delta^2 L$. As $l_{JH} < l_s$, the particular choice for turbulence closure makes little difference to the velocity perturbation for flow over a low hill, even though the closure used may model the turbulence rather badly in the region $z > l_s$. This is why mixing length models predict the mean flow (though not necessarily the turbulence) quite well for flow over low hills.

Over a 'real' hill, where non-linear effects are important, the flow may separate, forming a highly turbulent wake region, bounded by a detached shear layer. The choice of turbulence closure is now important as stress gradients in the free shear layer affect the mean flow and the size of the separation bubble. This region is inaccurately modelled by the equilibrium model of Jackson and Hunt and by mixing length models (Taylor, 1977).

Britter, Hunt and Richards (1981) have shown by comparison between theory and experiment that for flow over a two-dimensional hill the stress changes are governed by rapid distortion theory in the outer region ($z > l_s$). Field measurements for flow over the hill 'Blashaval' taken by the Meteorological Office are also consistent with rapid distortion theory in this outer region. To predict more accurately the mean flow for steep hills where there may be flow separation, and to obtain the stress fields, a turbulence closure is required which gives equilibrium effects near the ground, corresponds to rapid distortion theory aloft and can model detached shear layers successfully.

The numerical model

As a first step in this direction, we consider the closure proposed by Launder, Reece and Rodi (1975), which has been tested in a variety of flows. The equations of motion and the Reynolds stress equations to be solved are then

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\tau_{11} + \frac{\partial}{\partial z}\tau_{13} + Vf$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + W\frac{\partial V}{\partial z} = \frac{\partial}{\partial x}\tau_{12} + \frac{\partial}{\partial z}\tau_{23} + (u_g - U)f$$

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + W\frac{\partial W}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\tau_{13} + \frac{\partial}{\partial z}\tau_{33}$$

where u_g is the geostrophic wind

Stresses are given by

$$\frac{\partial}{\partial t}\tau_{ij} + U_k\frac{\partial \tau_{ij}}{\partial x_k} = -\left(\tau_{jk}\frac{\partial U_i}{\partial x_k} + \tau_{ik}\frac{\partial U_j}{\partial x_k}\right) + {}^2/_3\,\epsilon\,\delta_{ij}$$

$$-\,c_1\frac{\epsilon}{k}\left(\tau_{ij} + \tfrac{2}{3}k\,\delta_{ij}\right) + A_{ij} \quad + \text{ diffusion terms.}$$

where
$$A_{ij} = -\left(\frac{c_2+8}{11}\right)\left(P_{ij} - {}^2/_3\,P\,\delta_{ij}\right) + \left(\frac{30c_2-2}{55}\right)k\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$
$$-\left(\frac{8c_2-2}{11}\right)\left(D_{ij} - {}^2/_3\,P\,\delta_{ij}\right)$$

$$P_{ij} = -\left(\tau_{ik}\frac{\partial U_j}{\partial x_k} + \tau_{jk}\frac{\partial U_i}{\partial x_k}\right)$$

$$D_{ij} = -\left(\tau_{ik}\frac{\partial U_k}{\partial x_j} + \tau_{jk}\frac{\partial U_k}{\partial x_i}\right)$$

and
$$\frac{\partial \epsilon}{\partial t} + U_k\frac{\partial \epsilon}{\partial x_k} = -\,c_\epsilon\frac{\partial}{\partial x_k}\left(\frac{k}{\epsilon}\tau_{kj}\frac{\partial \epsilon}{\partial x_j}\right) + c_{\epsilon 1}\frac{\epsilon}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j} - \frac{\epsilon^2}{k}$$

$$k = -\tfrac{1}{2}\left(\tau_{11} + \tau_{22} + \tau_{33}\right)$$

c_1, c_2, c_ϵ, $c_{\epsilon 1}$, $c_{\epsilon 2}$ are numerical constants.

U_k is the mean flow field,

$\tau_{ij} = -\overline{u_i u_j}$ are the Reynolds Stresses

ε is dissipation

The model described is that for two-dimensional turbulent fluid flow over an irregular lower boundary. A co-ordinate transformation, based on that of Gal-Chen and Somerville (1975) and later developed by Clark (1977) is used to map the domain with bottom topography in (x,z) space into a rectangular region in (ζ, ξ) space (figure 2). If h(x) specifies the topography, and D is the depth of the domain, the transformation is given by

$$\zeta = x$$

$$\xi = D(z-h)/(D-h)$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\zeta} + (\xi/D - 1).dh/dx\,\frac{\partial\phi}{\partial\xi}\,/(1 - h/D)$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial\xi}\,/(1 - h/D)$$

Boundary conditions are given by

(i) No-slip lower boundary

(ii) $\frac{\partial \underset{\sim}{u}}{\partial\xi} = 0$ at upper boundary

(iii) Specified equilibrium stress ratios for the normal stresses at the lower boundary

(iv) $\frac{\partial \underset{\approx}{\tau}}{\partial\xi} = 0$ at upper boundary

(v) Law of the wall for the shear stresses

(vi) Periodic or inflow-outflow lateral boundary conditions

Results

At the time of writing, the model has been run successfully for flow over low topography, for both periodic terrain and isolated hills. We look briefly at the case of periodic terrain, a sinusoidally varying surface (as used by Sykes (1980)). As the ground is approached, the stress perturbations are such that the stress-energy ratios tend smoothly to their equilibrium limits (the lower boundary condition). Although Sykes was using a similar closure and the same form of boundary condition , he was unable to obtain a smooth match. (A fault in the numerics rather than the equations themselves) (Figures 3,4). In this region , the magnitude of the shear stress perturbation corresponds with that obtained from mixing length models, the maximum increase in stress occurring just forward of the crest. Moving vertically upwards from the summit, the (positive) perturbation in the shear stress decreases as we leave the equilibrium region, and reaches a (negative) minimum just above the height where Δu maximises and the mean shear is consequently decreased from its upstream value. Over the trough(x=0.5L), the perturbation for such a low sinusoidal series of hills is of the same magnitude but opposite phase, with a decrease in shear stress from the upstream value at the surface, and an increase higher up. This increase in shear stress in the 'matching region', where the stresses are not in equilibrium and rapid distortion theory does not fully apply, is of similar magnitude to the decrease above the crest. Whilst mixing length theory qualitatively agrees, the decrease in shear stress above the crest is much larger than the increase above the trough and larger than that predicted using the second-order closure model, because the shear stress is tied too strongly to the mean shear. Experiments for flow over water waves (Hsu, Hsu and Street ,1981) show that the stress profiles over crest and trough are of the same magnitude but 180° out of phase, as predicted by the second order closure model. (Figure 5)

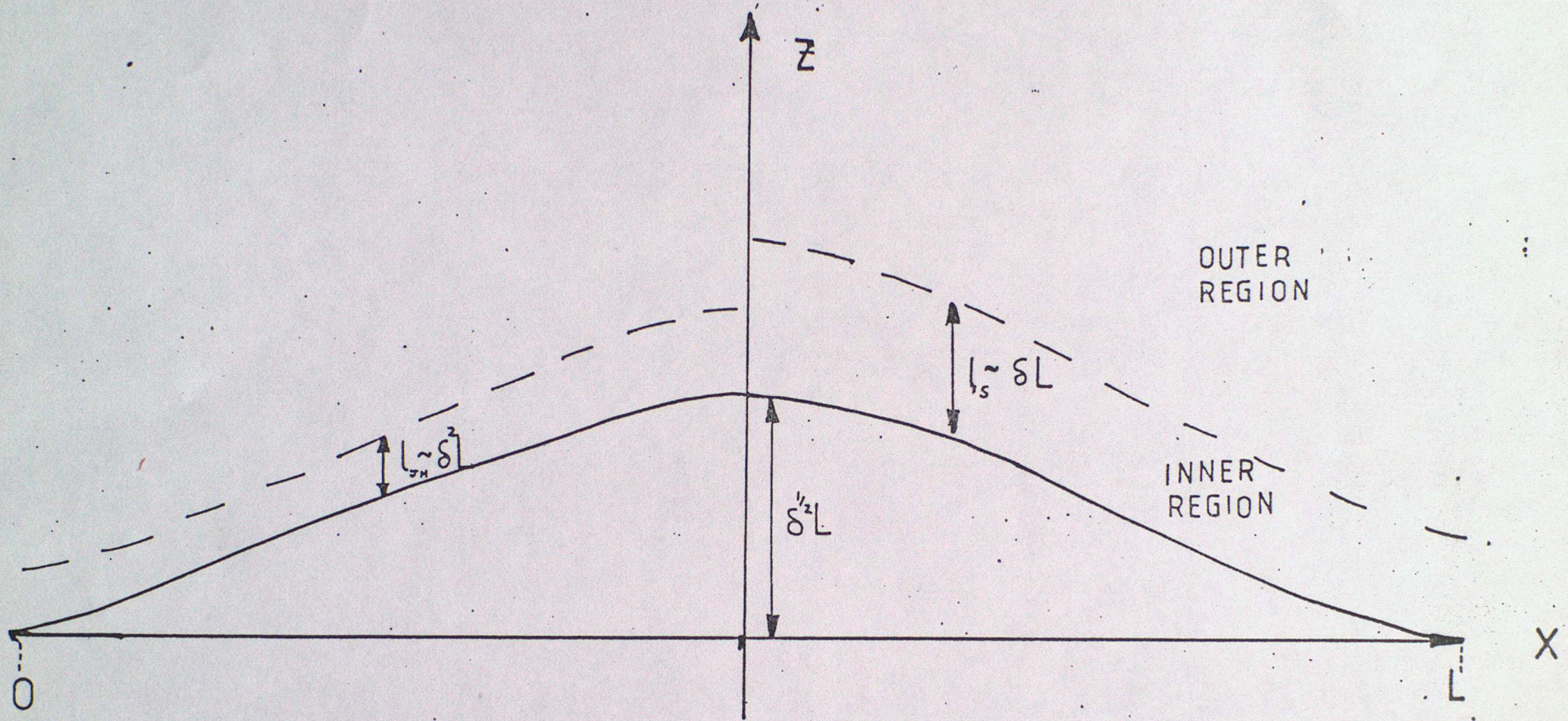
Further aloft, the stress perturbations tend qualitatively to the form predicted by a linear analysis (similar to that of Sykes) for this particular closure. Agreement of, for example $\overline{u^2}$, with the predictions of isotropic rapid distortion theory (Britter, Hunt and Richards 1981) is good. The numerically predicted values of ΔU were used in obtaining Figure 6. Allowance for the effect of anisotropy in the upstream turbulence improves the agreement.

Conclusions

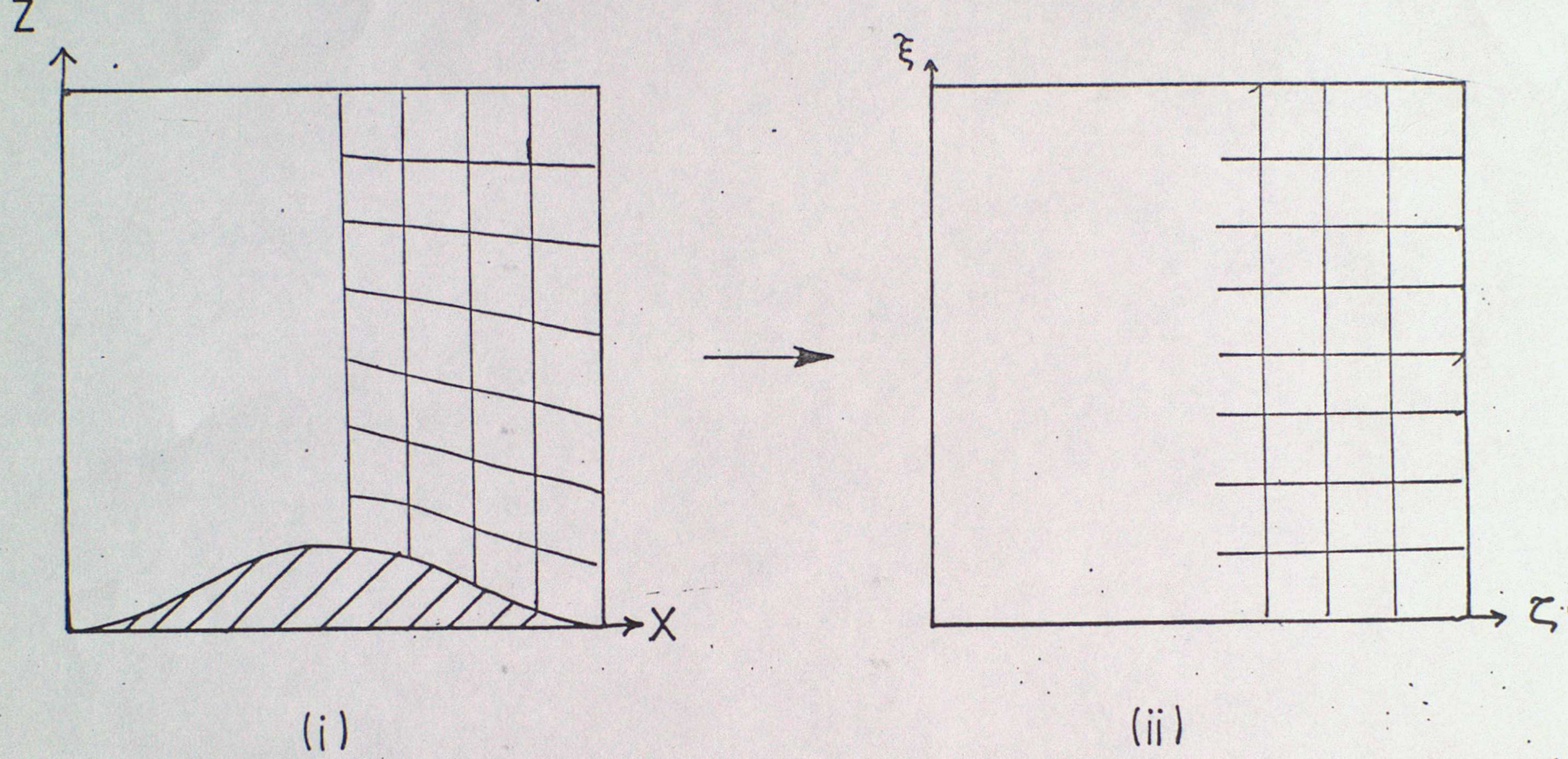
Comparison between theories using different turbulence models and experiments has shown the need for a sophisticated turbulence closure to model all the physical processes present in flow over a hill. These include equilibrium effects, rapid disortion theory and free shear layers. As a step in this direction, a second-order closure model has been employed numerically to solve for the mean flow and Reynolds stresses. For the case of low topography, such a model matches the stress fields satisfactorily between the limits of rapid disortion and equilibrium flows. Work is currently progressing on the effects of steep hills and the influence of separation.

References

Britter, Hunt and Richards	Quart.J.Roy.Met.Soc	1981	<u>107</u>	p91
Clark	J.Comp.Phys	1977	<u>24</u>	p186
Gal-Chen and Somerville	J.Comp.Phys	1975	<u>17</u>	p276
Hsu, Hsu and Street	J.Fluid.Mech	1981	<u>105</u>	p67
Jackson and Hunt	Quart.J.Roy.Met.Soc	1975	<u>101</u>	p929
Launder, Reece and Rodi	J.Fluid.Mech	1975	<u>68</u>	p137
Sykes	J.Fluid Mech	1980	<u>101</u>	p647
Taylor	Boundary Layer Met.	1977	<u>12</u>	p34



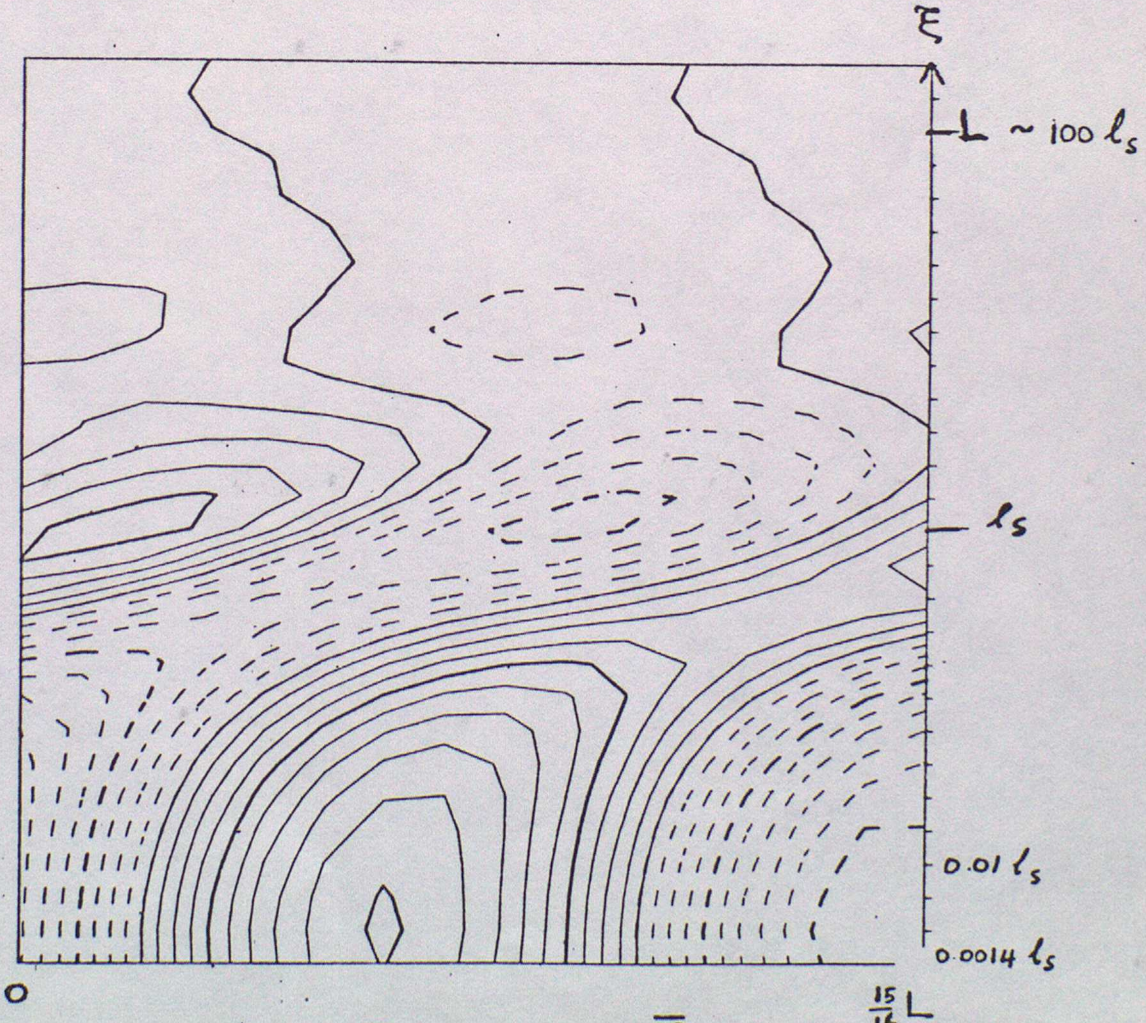
<u>Figure 1</u> Inner and outer regions for the flow over a hill. To the left of the centreline are shown the regions derived from the momentum equations (Jackson & Hunt) - to the right those from the stress equations (Sykes).



<u>Figure 2</u>   The solution domain in (i) (x,z) space

(ii) (ζ,ξ) space

Figure 3 Contour plot of perturbation from the horizontally averaged mean value for $\overline{u^2}$ from the numerical model for flow over a periodic sinusoidal hill of length L. The crest of the hill is at the left lateral boundary and flow is from left to right. Dashed lines indicate a positive perturbation in $\overline{u^2}$, solid lines a negative perturbation. Note that the constant stress lower boundary condition requires contours to be vertical at the lower boundary. $\overline{u_0^2}$ is the upstream value of $\overline{u^2}$ ($= -\tau_{11}$, as defined on page 5.) The vertical scale (in ξ)is logarithmic.



CONTOUR INTERVAL 0.022 $\overline{u_0^2}$ $\overline{u^2}$ PERTURBATION

FOR A HILL WITH $\delta^{1/2} \simeq 0.02$

Figure 4 Contour plot of perturbation in $\overline{u^2}$ from Sykes. Solid contours indicate a positive perturbation, dashes indicate negative perturbations. The crest is at the lateral boundary. In relating the magnitude of the perturbation to the mean value, the hill was assumed to have the same height as in figure 3, i.e. leading to a δ such that $\delta^{1/2} = 0.02$. This corresponds to a hill of length 2000m. and total height 40m.

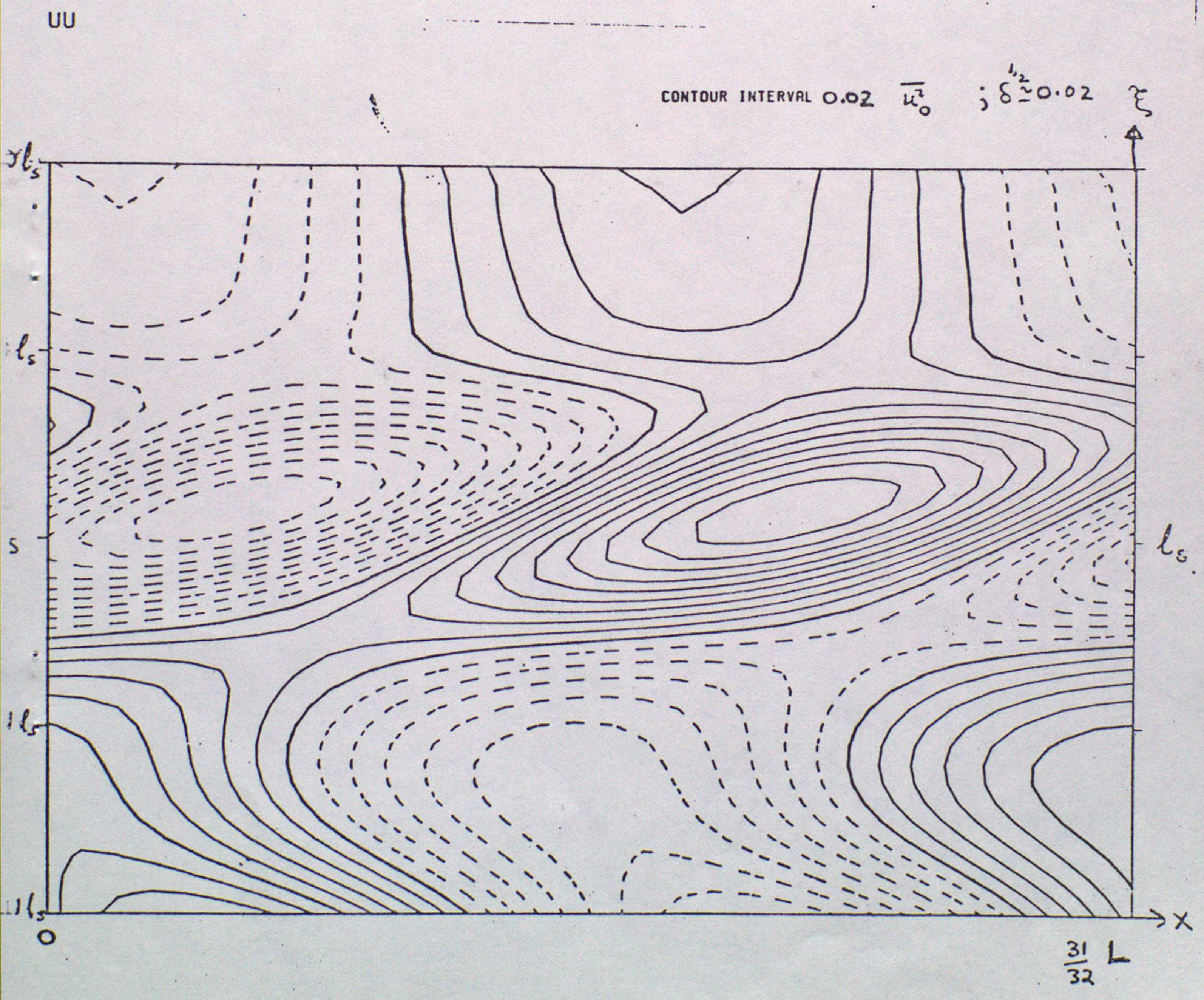
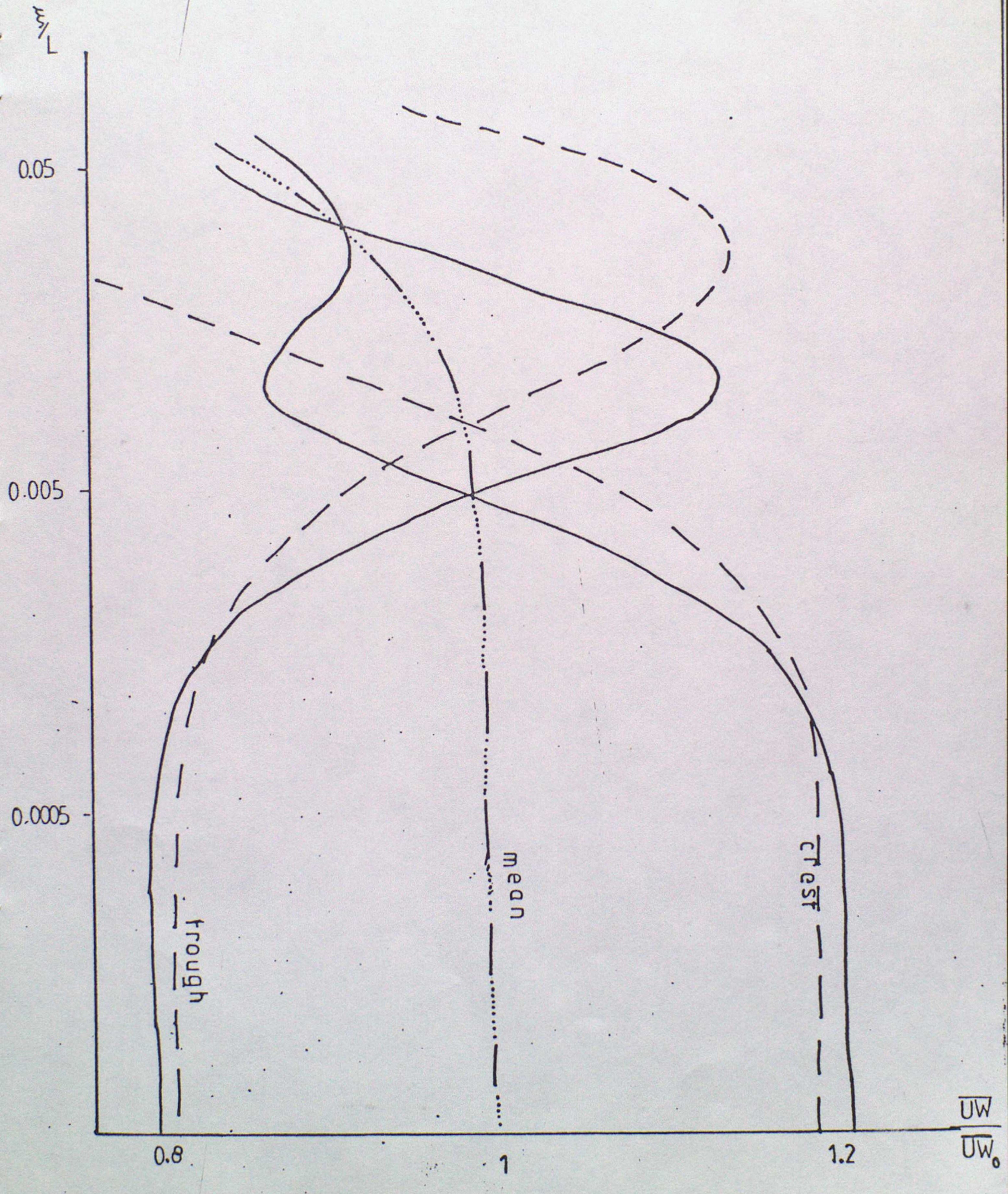


Figure 5   Profiles of shear stress at the crest and trough for the second-order model ——— and a mixing length model — — —



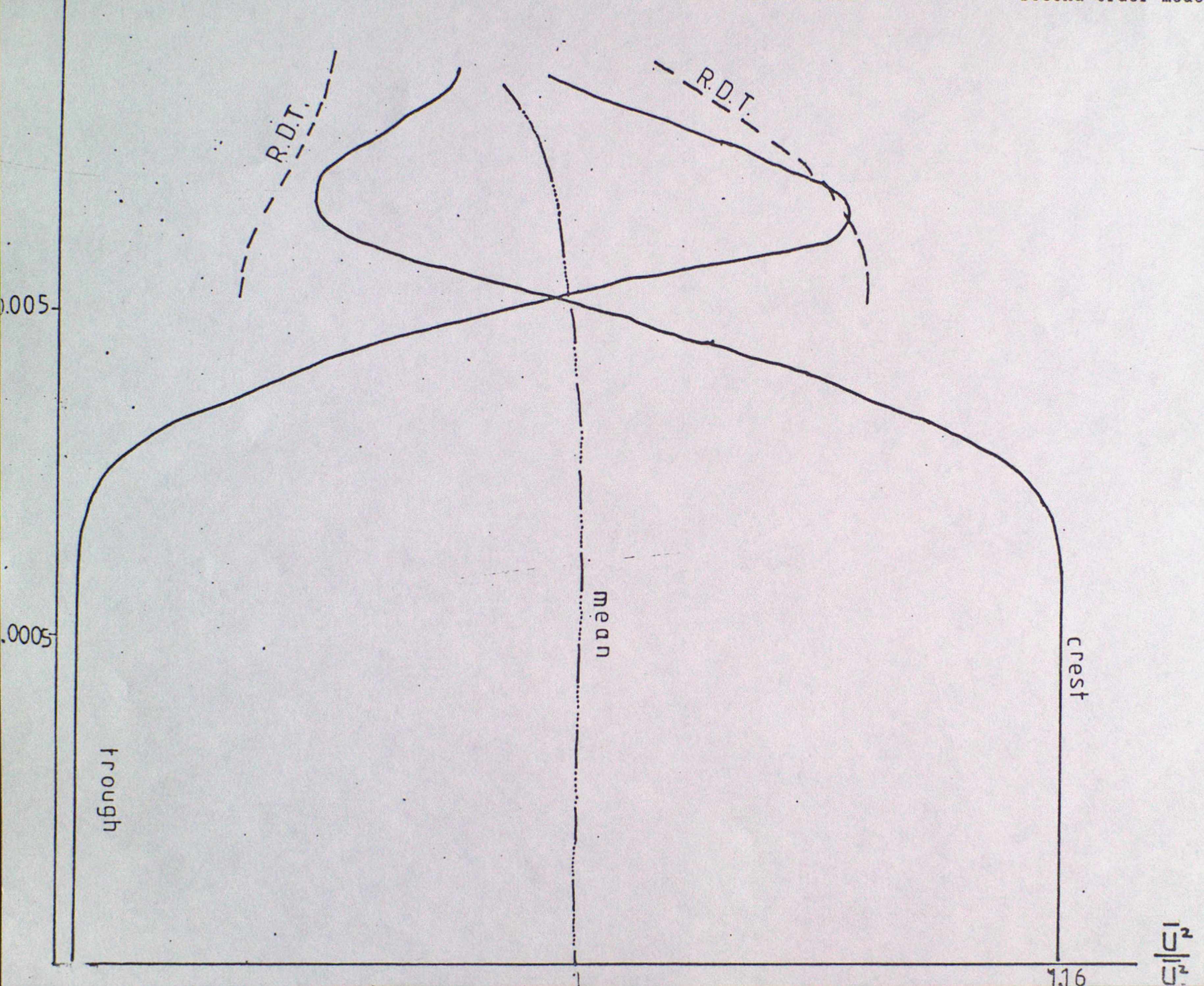


Figure 6 Profiles of $\overline{u^2}$ at the crest and trough . — RDT — Rapid distortion result : ——— second-order model.