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No 42

A NUMERICAL MODEL OF CONDITIONAL SYMMETRIC INSTABILITIY

by

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1. INTRODUCTION

Most mid latitude rainfall is associated with depressions. Understanding the structure of the associated frontal regions has produced great improvements in forecasting, from the early work of Bjerknes (1919) to the more recent studies of Browning (e.g. Browning (1974)) and Hobbs. A particularly clear review of the latter's group's work is given in Matejka et al (1980).

Precipitation is typically organised into several bands orientated parallel to the front. Many suggestions have been made to explain this phenomenon, and are discussed in Nash (1982a), but no one theory is generally accepted.

Bennetts and Hoskins (1979) proposed Conditional Symmetric Instability (CSI) as the explanation of the organisation. This is an attractive theory as the atmosphere in a frontal region is often preferentially unstable to such a motion. The instability produces a mesoscale roll parallel to the front, which moves with the local fluid velocity and which evolves slowly over many hours. The motion modifies the environment to produce convective activity in certain regions.

The theory of baroclinic symmetric instability in a dry atmosphere is discussed in Section II. Particular attention is drawn to both the nature of the motion in a bounded region and to the energy exchange with the environment. The conclusions of the dry analysis are used to motivate the description of CSI although no analytic solutions exist, for a wet atmosphere.

The numerical model, originally developed by Bennetts and Hoskins (1979), has been extended to study CSI and the results are described in Nash (1982a) and Nash (1982b). The formulation of the model is presented in Section III, together with a discussion of its characteristics. Approximate radiation boundary conditions are described in the form of damping layers that allow the modelling of CSI in an atmosphere with infinite horizontal extent. The finite difference energy equations are given in an Appendix.

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II. Analytic Theory (i) Basic equations.

Consider a fluid rotating with angular velocity 2f in sheared motion parallel to the y axis of the Cartesian reference frame Oxyz such that the velocity is given by

$$v = V(x,z) \tag{2.1}$$

and $\partial V/\partial x, \partial V/\partial z$ are constant.

The fluid is stabily stratified with a temperature distribution

$$\theta = \Theta(x,z) \tag{2.2}$$

in thermal wind balance with the velocity V, and $\partial\Theta/\partial z$ is also constant.

The flow is governed by the primitive equations

$$\frac{Du}{Dt} - fv + \partial p/\partial x = 0 \tag{2.3}$$

$$\frac{Dv}{Dt} + fu + \partial p/\partial y = 0 \tag{2.4}$$

$$\frac{Dw}{Dt} - \sigma + \partial p/\partial z = 0 \tag{2.5}$$

$$\frac{D\sigma}{Dt} = \mathcal{H} \tag{2.6}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.7}$$

where (u, v, w) are the Cartesian velocity components, $\sigma = \frac{g}{\bar{\theta}_0} . \theta$ (θ is potential temperature and $\bar{\theta}_0$ a constant reference value), $\mathcal{H}$ is the diabatic heating and the dynamic pressure $p = P/\bar{\rho}_0$ (P is the mechanical pressure and $\bar{\rho}_0$ a constant reference value of density). The Boussinesq approximation has been applied to the momentum equations in that variations of density are ignored (and $\bar{\rho}_0$ used) everywhere except where it leads to a buoyancy force (i.e. $\sigma = \frac{g}{\bar{\theta}_0}\theta = -g\rho/\bar{\rho}_0$). The motion is assumed two-dimensional so that all $\partial/\partial y$ terms are zero. In the initial state u = w = 0 and the statement of thermal wind balance is

$$f \frac{\partial V}{\partial z} = \frac{g}{\theta_0} \cdot \frac{\partial \Theta}{\partial x} \tag{2.8}$$

The equations will now be scaled to reveal the dimensional dependence of the flow.

Define the parameters

$$S^2 = f \frac{\partial V}{\partial z} = \frac{g}{\theta_0} \frac{\partial \Theta}{\partial x} = \frac{\partial \Sigma}{\partial x} \tag{2.9}$$

$$N^2 = \frac{g}{\theta_0} \cdot \frac{\partial \Theta}{\partial z} = \frac{\partial \Sigma}{\partial z} \tag{2.10}$$

$$F^2 = f\left(f + \partial V/\partial x\right) \tag{2.11}$$

Let L, H be length scales in the horizontal and vertical respectively and u_0 be a scaling velocity in the x direction. Then linearising the equations (2.3 - (2.7) about the basic state (V, Σ) gives

$$\frac{\partial u}{\partial t} \qquad - fv + \frac{\partial p}{\partial x} = 0 \tag{2.12}$$

$$f\frac{\partial v}{\partial t} + F^2 u + S^2 w = 0 \tag{2.13}$$

$$\frac{\partial w}{\partial t} \qquad - \sigma + \frac{\partial p}{\partial z} = 0 \tag{2.14}$$

$$\frac{\partial \sigma}{\partial t} + S^2 u + N^2 w = \mathcal{H} \tag{2.15}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.16}$$

where u, v, w, p and 6 are all peturbations about the basic state variables.

The non dimensional (primed) variables are defined by $x = Lx'$, $z = Hz'$, $u = u_0 u'$, $v = \frac{Fu_0}{f} . v'$, $w = \frac{H}{L} u_0 w'$, $p = LFu_0 p'$, $\sigma = \frac{L}{H} F u_0 \sigma'$ and substituting in (2.12) - (2.16) gives

$$\frac{\partial u'}{\partial t'} \qquad -v' + \frac{\partial p'}{\partial x'} = 0 \tag{2.17}$$

$$\frac{\partial v'}{\partial t'} + u' + Ri^{-1} w' \qquad = 0 \tag{2.18}$$

$$\left(\frac{H}{L}\right)^2 \frac{\partial w'}{\partial t'} \qquad -6' + \frac{\partial p'}{\partial z'} = 0 \tag{2.19}$$

$$Ri . \frac{\partial 6'}{\partial t} + u' + w' \qquad = \frac{\mathcal{H}}{S^2 u_0} \tag{2.20}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial w'}{\partial z'} = 0 \tag{2.21}$$

where $Ri = \frac{F^2 N^2}{S^4} \cdot \frac{H}{L}$ is now chosen *equal to S^2/N^2 to give a balance of the advective terms in (2.20). If the hydrostatic approximation is made the term $\left(\frac{H}{L}\right)^2 \partial w'/\partial t'$ disappears in (2.19), and the equations are the same as used by Stone (1966) to describe symmetric instability in the case $\mathcal{H} = 0$.

In the following analysis, $\mathcal{H}$ will be given by

$$\mathcal{H} = A w \tag{2.22}$$

where $A = \mathcal{A}(x, z)$. H (w), H is the Heaviside step function and so $\mathcal{H}$ is a nonlinear term in general.

Then (2.20) can be rewritten

$$Ri . \frac{\partial 6'}{\partial t'} + u' + w' = \left(\frac{A}{N^2}\right) w' \tag{2.23}$$

Thus in an unbounded atmosphere or an unbounded channel with lids at $z = 0$ and $z = H$ the flow can only depend on Ri, $\frac{S^2}{N^2}$ and $\frac{\mathcal{A}}{N^2}$. In the adiabatic, hydrostatic limit only Ri determines the motion.

(ii) Infinite atmosphere solutions.

It is simple to derive normal mode solutions in an infinite atmosphere, following Hoskins (1974), Elliasen (1962), Ooyama (1966). The adiabatic equations ((2.17) - (2.21) with $\mathcal{H} = 0$), are combined into a single equation for w where $w = w'(z') e^{i(k'x' + \omega' t')}$ is assumed.

* H is chosen from the flow geometry, e.g. between lids H would be the distance of separation, then $L = N^2 H / S^2$.

Then

$$(1-\omega'^2)Ri\frac{d^2w'}{dz'^2} - 2ik'\frac{dw'}{dz'} - \left(1-\left(\frac{S^2}{N^2}\right)^2\omega'^2 Ri\right)k'^2 w' = 0 \qquad (2.24)$$

If $w' = w_0 e^{im'z'}$ then

$$\omega'^2 = \left\{1-\left(\frac{k'}{m'}\right)\left(2-\frac{k'}{m'}\right)Ri^{-1}\right\} \Big/ \left\{1+\left(\frac{S^2}{N^2}\right)^2\left(\frac{k'}{m'}\right)^2\right\} \qquad (2.25)$$

The hydrostatic approximation will now be made, i.e. $S^2/N^2 \cdot k'^2/m'^2 \ll 1$, and then the denominator becomes unity. ω'^2 varies as a quadratic function of the aspect ratio ($\frac{k'}{m'}$) which is order one from the scaling. The minimum value of ω'^2, which corresponds to the most unstable mode, is at $\frac{k'}{m'} = 1$, motion parallel to the isentropic surfaces. This mode has growth rate given by

$$\omega'^2_{min} = -\gamma'^2_{max} = 1 - Ri^{-1} \qquad (2.26)$$

$\omega'^2 < 0$ corresponds to instability and $\omega'^2 > 0$ to neutral propagation with horizontal velocity ω'/k'. For unstable modes to exist

$$Ri^{-1} > 1 \qquad (2.27)$$

and these correspond to aspect ratios such that

$$1-\lambda < k'/m' < 1+\lambda \qquad (2.28)$$

where $1-\lambda^2 = Ri$

The most unstable mode was described by Bennetts and Hoskins (1979). In a non-hydrostatic atmosphere the most unstable mode has particle displacements at a small angle above the θ surfaces, (Hoskins 1978). It is important to notice that (2.25) has no explicit wavenumber dependence, only the aspect ratio appears. Thus there is no preferred length scale. McIntyre (1970) showed that the addition of laminar incosity favoured the longest scales and the most unstable mode has infinite horizontal scale in an unbounded atmosphere.

(iii) Pseudo-energetics.

It is informative to consider the energetics of the instability. The equations (2.17) - (2.21) lead to the equation

$$\frac{\partial}{\partial t'}\left\{ \frac{u'^2 + v'^2 + \left(\frac{S^2}{N^2}\right)^2 w'^2}{2} \right\} + \frac{\partial}{\partial x'}(u'p') + \frac{\partial}{\partial z'}(w'p') + Ri^{-1}\, v'w' - \sigma' w' = 0 \qquad (2.29)$$

This equation is not equivalent to the energy equation which will be revealed below. A growing disturbance must have

$$\frac{\partial}{\partial t'} \int \frac{u'^2 + v'^2 + \left(\frac{S^2}{N^2}\right) w'^2}{2}\, dx\, dz > 0$$

and so

$$\int (Ri^{-1}\, v'w' - \sigma' w')\, dx\, dz < 0$$

since the other terms reduce to surface integrals which can be assumed zero in most cases.

$Ri^{-1} v'w'$ represents a local exchange between the zonal shear and the peturbation. It is analogous to the mechanical work term of the kinetic energy equation. It is the dominant term for the most rapidly growing modes of symmetric instability.

$\sigma' w'$ represents the interaction of the disturbance with the thermodynamic state of the environment. It is analogous to a potential energy conversion. Let $\overline{\qquad}$ denote an average taken over a horizontal wavelength

$$\overline{\phi} = \frac{k}{2\pi} \int_{x_0}^{x_0 + 2\pi/k} \phi(x)\, dx$$

for any ϕ, independent of x_0.

(2.29) integrated over a representative wavelength gives

$$\frac{\partial \overline{K'}}{\partial t'} = -Ri^{-1}\, \overline{v'w'} + \overline{\sigma' w'} \qquad (2.30)$$

where

$$K' = \frac{u'^2 + v'^2 + \left(\frac{S^2}{N^2}\right)^2 w'^2}{2}$$

is a measure of the amplitude of the disturbance.

If $\phi_1 = \mathrm{Re}\, \tilde{\phi}_1 e^{ikx}$, $\phi_2 = \mathrm{Re}\, \tilde{\phi}_2 e^{ikx}$ then

$$\overline{\phi_1 \phi_2} = \frac{1}{2} \mathrm{Re}\,(\phi_1 \phi_2^*)$$

and the correlations in (2.30) can be simply written down in terms of the normal mode solutions.

The full field solutions to (2.17) - (2.21) for $\mathcal{H} = 0$ are

$$\begin{aligned}
u' &= -m'\omega'\phi' \\
v' &= i(Ri^{-1}k' - m')\phi' \\
w' &= k'\omega'\phi' \\
\sigma' &= Ri^{-1}.\, i(k' - m')\phi' \\
p' &= [Ri^{-1}k' + (\omega'^2 - 1)m']\phi'/k' \qquad (2.31)
\end{aligned}$$

where real parts are understood, $\phi' = \phi_0' \exp i(k'x' + m'z' + \omega' t')$ and ω is given by the dispersion relation (2.25).

Then

$$-Ri^{-1}\overline{v'w'} = -\frac{Ri^{-1}}{2} \mathrm{Re}\left[i(Ri^{-1}k - m)\phi.\, k\omega^*\phi^*\right]$$

$$= Ri^{-2}\left(\frac{k}{m} - Ri\right)|\phi|^2 \frac{km}{2} \mathrm{Im}(\omega^*) \qquad (2.32)$$

and

$$\overline{\sigma' w'} = \frac{\mathrm{Re}}{2}\left[Ri^{-1}.\, i(k-m)\phi\, k\omega^*\phi^*\right]$$

$$= -Ri^{-1}\left(\frac{k}{m} - 1\right)|\phi|^2 \frac{km}{2} \mathrm{Im}(\omega^*) \qquad (2.33)$$

or writing $A = km\, \mathrm{Im}(\omega^*)\, |\phi|^2/2$

$$-Ri^{-1}\overline{v'w'} = \left(\frac{k}{m} - Ri\right)Ri^{-2}A \qquad \overline{\sigma'w'} = -\left(\frac{k}{m} - 1\right)Ri^{-1}.\, A$$

Writing $\frac{k}{m} = \lambda$ $\left(= -\frac{u}{w}\right)$, the phase line slope or aspect ratio of the particle displacements, (2.32), (2.33) become

$$-Ri^{-1}\overline{v'w'} = \left(\frac{\lambda - Ri}{Ri^2}\right)A \quad = MK \qquad (2.34)$$

$$\overline{\sigma'w'} = \left(\frac{1-\lambda}{Ri}\right)A \quad = PK \qquad (2.35)$$

where MK and PK represent net contributions to the scale of the disturbance from the dynamical correlations and the transport of heat.

Fig (2.1) shows the signs of these quantities in relation to the phase slope relative to the absolute vorticity vector $\underline{\zeta_a}$, which has dimensionless slope Ri , and the isotherms which have unit slope. Also marked are the phase lines corresponding to neutral stability. All the unstable modes lie in between.

The diagram shows clearly that MK $>$ 0 if the particle motion is steeper than the absolute vorticity and negative if less. PK $>$ 0 if the phase slope lies below the isothemal surfaces and PK $<$ 0 if above. Both quantities tend to zero as the $\omega = 0$ lines are approached.

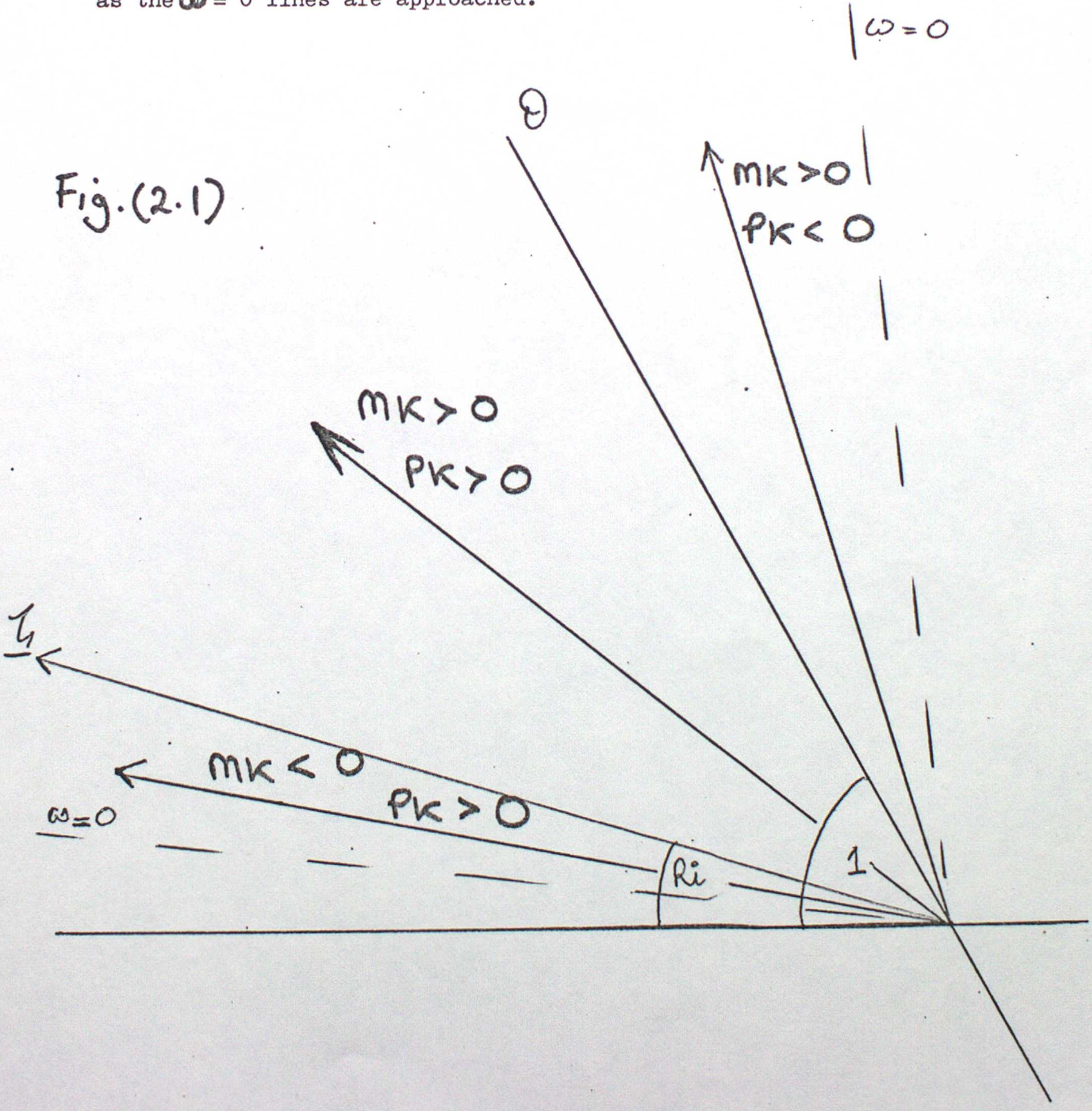
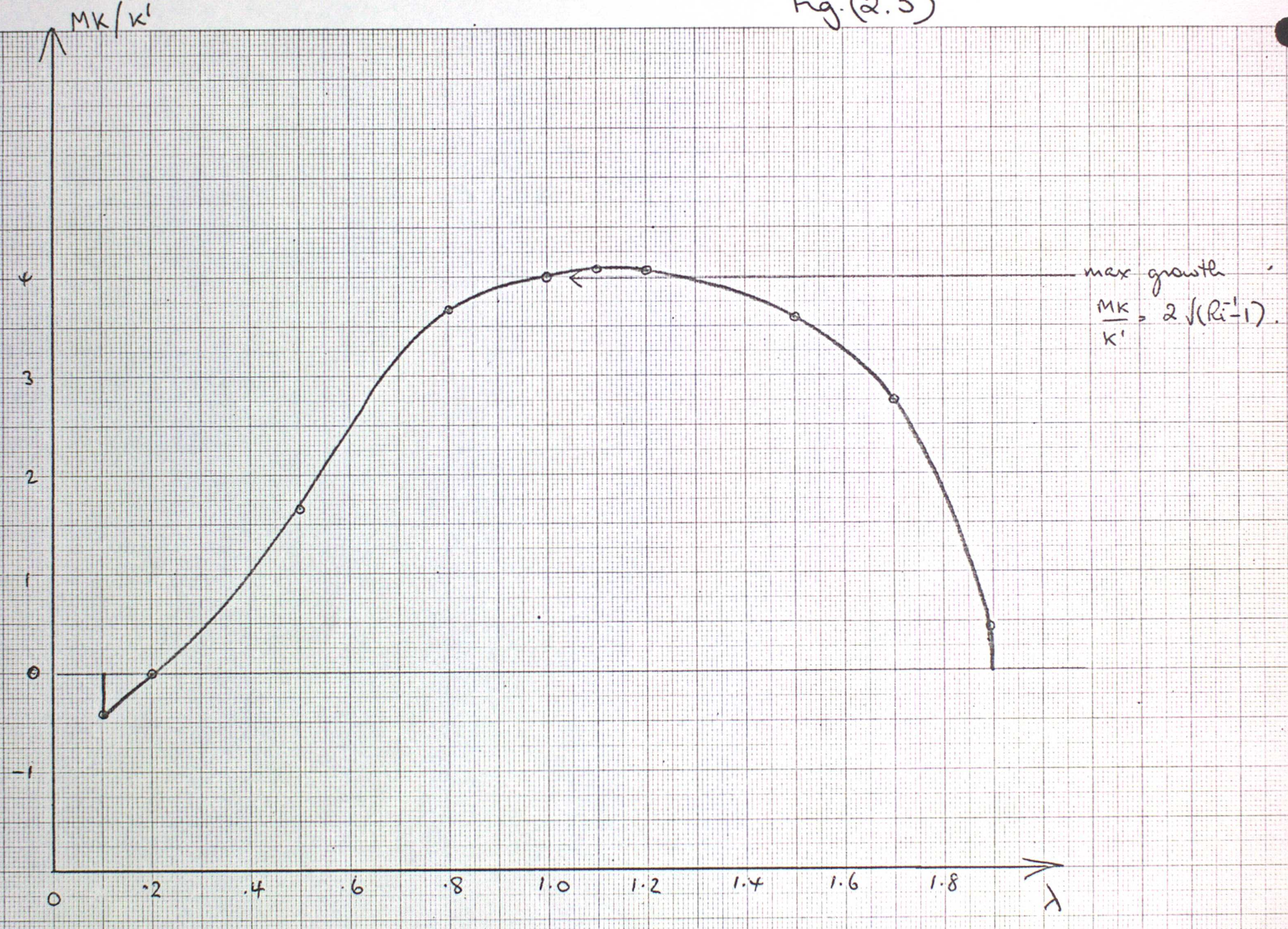


Fig. (2.1)

Fig. (2.3)



$Ri = 0.2 \quad S/N = 0$

The relative importance of the terms for λ lying between these lines is given by

$$\frac{PK}{MK} = -Ri\left(\frac{\lambda-1}{\lambda-Ri}\right) \qquad (2.36)$$

The $\omega = 0$ points are given by $\lambda = 1 \pm (1-Ri)^{1/2}, \ (Ri < 1)$.

It is apparent that the PK contribution is dominant near $\lambda = Ri$. The PK contribution is a minimum when $\lambda = 1$, motion on isentropic surfaces. Thus the most unstable mode of symmetric instability grows almost exclusively from the mechanical term.

In general the efficiency of this conversion depends on the shear in the zonal wind V. For a given aspect ratio λ lying in the unstable region, (2.34) shows that the rate of growth due to MK is given by

$$\frac{MK}{K'} = \frac{2(\lambda-Ri)|\omega|}{\lambda(1-Ri)} = \frac{2(\lambda-Ri)}{\lambda(1-Ri)}\left\{\frac{Ri^{-1}\lambda(2-\lambda)-1}{1+\frac{S^4}{N^4}\lambda^2}\right\}^{\frac{1}{2}} \qquad (2.37)$$

The value of this quantity gives a measure of the efficiency with which energy is **fed** into the peturbation from the zonal wind. Fig (2.3) shows the form of (2.37) for the case $Ri = 0.2$ and the hydrostatic limit. The most unstable mode has $|\omega| = 2$ at $\lambda = 1$. The efficiency of energy transfer increases rapidly with λ to a maximum near the most unstable mode. For $\lambda > 1$ the modes are attenuated by working against the stable stratification and this produces a weaker correlation between v' and w', reducing the efficiency of the mechanical conversion as well.

Further since

$$Ri = \frac{F^2/S^2}{S^2/N^2} = \phi_\zeta/\phi_d \qquad (2.38)$$

and

$$\max\left(\frac{MK}{\overline{k'}}\right) = 2\sqrt{(Ri^{-1}-1)} \qquad (2.39)$$

where ϕ_{ζ} and ϕ_d are the slope angles of the absolute vorticity vector and isentropes respectively, growth is large if $\phi_d >> \phi_{\zeta}$. ϕ_{ζ} is small in baroclinic zones, i.e. large $\partial\theta/\partial x$, or on the warm side of a jet, i.e. $\partial v/\partial x < 0$. ϕ_d is large near enhanced baroclinicity or lower stability, i.e. reduced N^2. In a frontal situation where there is often a jet parallel to the front, instability will be encouraged. Also the release of latent heat in a moist atmosphere will reduce the effective stability and this leads naturally to the concept of Conditional Symmetric Instability.

However before considering this, the nature of symmetric instability in an infinitely long channel bounded by horizontal lids will be explored. This reveals roll peturbations and justifies the use of a rigid upper boundary as an approximation to an upper level inversion.

(iv) Channel solutions.

Consider a fluid confined between rigid lids at $z = -H$ and $z = H_{\infty}$, on which $w = 0$, and unbounded in x . The fluid is composed of two layers separated by $z = 0$ (an approximation to a material surface) each of which is governed by (2.17) - (2.21). The upper layer has Brunt-Vaisala frequency N_2^2 and the lower symmetrically unstable layer, N_1^2. Both layers have the same F^2 and S^2.

The length scale L is chosen for both layers such that $\frac{H}{L} = \frac{S^2}{N_1^2}$ and the height scale in the lower layer is chosen as H. The height scale of the upper layer H_2 is chosen as $\frac{H_2}{L} = \frac{S^2}{N_2^2}$. Then this scaling ensures that the dispersion relation (2.25) holds in each layer, given the local scaling.

The interface matching conditions, with subscript to denote the solution region, are

$$w_1(0) = w_2(0)$$

$$\frac{dw_1}{dz}(0) = \frac{dw_2}{dz}(0) \qquad (2.39)$$

in dimensional form, where the second condition follows from continuity of pressure and temperature at $z = 0$.

If the dimensionless vertical velocities are written as $W_i'(z)\, e^{i(k_i' x + \omega_i' t')}$, $i = 1, 2$, then (2.39) becomes in dimensionless form,

$$W_2'(0) = (N_2^2/N_1^2)\, W_1'(0)$$
$$\frac{dW_1'}{dz_1'}(0) = dW_2'/dz_2'(0) \qquad (2.40)$$

(2.40) shows immediately that

$$k_1' = k_2' = k' \;, \quad \omega_1' = \omega_2' = \omega \qquad (2.41)$$

The general solution to (2.24) in each layer is

$$W_1' = A_1 e^{im_{1+}' z'} + B_1 e^{im_{1-}' z'}$$
$$W_2' = A_2 e^{im_{2+}' z'} + B_2 e^{im_{2-}' z'} \qquad (2.42)$$

where

$$m_{i\pm}' = \frac{k'}{Ri_i\,(1-\omega'^2)} \left\{ 1 \pm \sqrt{(1 - Ri_i(1-\omega'^2))} \right\} \qquad (2.43)$$

(the hydrostatic approximation has been made).

Now let $H_\infty = 0$ so that there is only one layer. Then $W_1' = W'$ in (2.42) and the boundary conditions

$$W'(0) = 0 \Rightarrow A_1 + B_1 = 0$$
$$W'(-1) = \Rightarrow A_1 + B_1\, e^{-i(m_{1-}' - m_{1+}')} = 0 \qquad (2.44)$$

That is

$$B_1 = -A_1$$
$$m_{1+}' - m_{1-}' = 2n\pi \qquad (2.45)$$

where n is an integer other than zero. (If n = 0 no non-trivial solution exists).

Then Appendix 1 shows that ω is real and such that

$$\omega'^2 \geq 1 - Ri^{-1} \tag{2.46}$$

W_1' can now be written as

$$w_1' = A_1^* e^{i(m'_{1+} + m'_{1-})z'} \sin\left(\frac{m'_{1+} - m'_{1-}}{2}\right) z' \tag{2.47}$$

where $A_1^* = 2iA_1$.

Then writing $m'_{1+} + m'_{1-} = 2a'$ and using (2.45), the full solution is

$$w' = A_1^* \sin n\pi z' . \, e^{i(k'x' + a'z' + \omega' t')} \tag{2.48}$$

These solutions first found by Elliäsen (1949) are a series of closed cells contained within regular parallelograms with edges parallel to z' = constant and $k'x' + a'z'$ = constant lines. The characteristic slope of the cells is given by

$$\frac{k'}{a'} = Ri\,(1 - \omega'^2) \tag{2.49}$$

Since $\omega^2 > (1 - Ri^{-1})$, $\frac{k'}{a'} < 1$: the cell boundaries slope at an angle less than the isentropic surfaces. (The mode with $\omega^2 = 1 - Ri^{-1}$ corresponds to n = 0 for which no solution exists).

The neutral mode has slope Ri and the growth increases with the slope, tending towards but never attaining the limit at $k'/a' = 1$.

ω is related to the vertical wavenumber n from (2.43) and (2.45) by

$$n\pi = \frac{k'\sqrt{(1 - Ri\,(1 - \omega'^2))}}{Ri\,(1 - \omega'^2)} \tag{2.50}$$

As the cell slope increases and $\omega^2 \rightarrow 1 - Ri^{-1}$, then,

$$\frac{n\pi}{k'} \rightarrow 0 \tag{2.51}$$

Thus the most unstable mode for fixed n occurs in the limit $k' \rightarrow 0$, and so in the atmosphere there would be a large number of very small cells, not of mesoscale proportions.

This unfortunate result was approached by Emanuel (1979) who showed that in a viscous fluid, over a range of viscosities, that the most unstable mode has $\frac{n\pi}{k'} \sim 1$ and has the structure of the inviscid modes outside the lid boundary layers. This is the justification for considering inviscid solutions in interpreting symmetric instability in the atmosphere.

These modes can be understood in terms of a superposition of the infinite atmosphere solutions. The energy transfers are analogous and the importance of a high aspect ratio for the cell growth is clearly evident from (2.49).

If a cell has slope $\lambda_c = k'/\alpha'$, then the mode is a superposition of two infinite atmosphere modes with aspect ratios

$$\lambda = 1 \pm \sqrt{(1 - \lambda_c)} \tag{2.52}$$

Now consider the two layer problem with $\frac{H_\infty}{H} \to \infty$, and $Ri_2 > 1 > Ri_1$ so that the upper layer solution is

$$w_2' = A_2 e^{im_2' z} \tag{2.53}$$

where m_2' is the appropriate value from (2.43) which gives evanescence as $z' \to \infty$. Appendix I shows that for general ω^2 there is one and only one such root of (2.43).

Then substituting into the boundary conditions gives

$$\left[1 - e^{i(m_{1-}' - m_{1+}')}\right]\left(\frac{N_2^2}{N_1^2}\right) = \left(\frac{A_2}{A_1}\right) \tag{2.54}$$

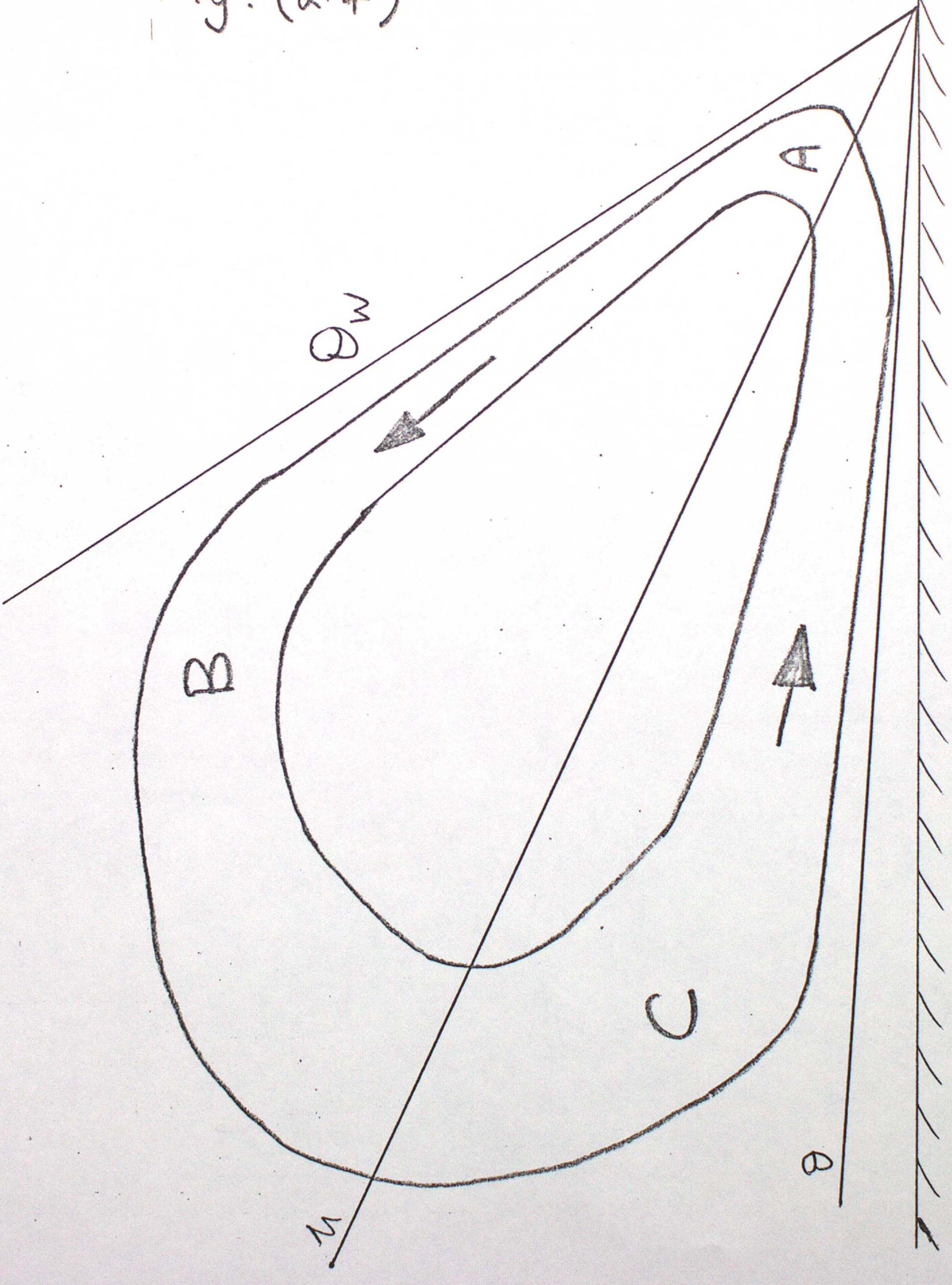
$$m_{1+}' - m_{1-}' e^{i(m_{1-}' - m_{1+}')} = m_2\left(\frac{A_2}{A_1}\right) \tag{2.55}$$

and $B_1 = -A_1 e^{i(m_{1-}' - m_{1+}')}$

From (2.43) it follows that

$$m_{1+}' - m_{1-}' = \frac{2k'\sqrt{(1 - Ri_1(1-\omega'^2))}}{Ri_1\,(1-\omega'^2)} \tag{2.56}$$

Fig. (2.4)



and

$$m_1' + m_{1-}' = \frac{2k'}{Ri_1(1-\omega'^2)} \qquad (2.57)$$

Thus $\frac{A_2}{A_1}, m_{1+}', m_{1-}'$ and ω'^2 are determined by (2.54) - (2.51). The solution is difficult to express in closed form. Assume $Ri_2|1-\omega'^2| >> 1$, i.e. the lower layer is symmetrically unstable lying underneath a very stable upper inversion and ω^2 is not close to 1.

The leading order solution in an expansion in $Ri_2^{-1/2}$ is given in Appendix II and shows that the single layer solution is a first approximation to the inversion capped problem and has an error as small as $0(Ri_2^{-1})$.

Appeal to this result justifies the upper boundary condition used in a substantial portion of the numerical work on the development of Conditional Symmetric Instability. This modified form of symmetric instability will now be described.

(v) Conditional Symmetric Instability.

The conclusion of the energy discussion above was that the dominant energy conversion is from the zonal kinetic energy into the perturbation. This is maximised by particle motions as steep as possible but not so steep as to induce retarding buoyancy forces. In a dry atmosphere this means motions shallower than the isentropic, θ, surfaces. However θ surfaces, even in frontal regions, are often shallower than the absolute vorticity vector and then no growth can result (except possibly baroclinic travelling disturbances along the front). In intense fronts or near strong jets instability can be realised, $Ri < 1$, provided there is a horizontal temperature gradient of 4-5 K $(100 \text{ Km})^{-1}$ or a zonal velocity shear $\partial v/\partial x \sim 10 \text{ ms}^{-1} (100 \text{ Km})^{-1}$.

Bennetts and Hoskins (1979) suggested that in a saturated atmosphere, these requirements might be relaxed. The release of latent heat on ascent allows a neutrally buoyant motion with a steeper angle of ascent than would be possible in a dry atmosphere, making it more likely that a peturbation can extract energy from the zonal wind on ascent. On descent the atmosphere is assumed dry and so the downdraought branch of the roll will be unsaturated on the mesoscale. Then the particle motions on descent lie close to θ surfaces. This return branch might be a sink of energy due to its shallow slope. For a viable instability the extraction of energy from the environment on the updraught must be sufficient to dominate this sink.

Fig (2.4) indicates the circulation envisaged. The updraught, parallel to the surface, θ_ω = constant, where θ_ω is the wet bulb potential temperature, is an intense flow while the return branches are broader and slower.

The updraught AB is a strong source of eddy kinetic energy which is maximised by the angle of slope and the intensity of the flow. BC is an energy sink as warm air is forced down against the atmospheric stability. This loss is minimised by the flow being very slow, and correspondingly broad. The downdraught, CA, is close to thermodynamically neutral but loses energy to the zonal wind since the motion is below the vorticity vector. This sink is also minimised by a broadening of the flow.

The numerical results of the work performed in Met O 15 and those of Bennetts and Hoskins (1979) show that the instability is as described. On the memsoscale the particle velocities are small, and the time for a particle to traverse the whole circulation is much longer than characteristic growth times. In particular moisture advection is not important and the flow is equivalent to one in which only the updraught region is saturated.

Bennetts and Hoskins (1979) offered an heuristic argument to predict the dimensional growth rate γ of a Conditional Symmetric Instability roll.

$$\gamma^2 = \frac{-1}{1+\alpha}\left\{\alpha\frac{q_W}{N_W^2} + \frac{q}{N^2}\right\} \qquad (2.58)$$

where α is an empirical constant relating to the geometry of the flow and is usually about 2,

$$q_w/N_w^2 = F^2 - S^4/N_w^2 \qquad (2.59)$$

$$q/N^2 = F^2 - S^4/N^2 \qquad (2.60)$$

where N_w^2 is the moist static stability, ($\frac{g}{\theta} \frac{\partial \theta_w}{\partial z}$).

q/N^2 and q_w/N_w^2 can be regarded as the components of absolute vorticity perpendicular to the θ and θ_ω surfaces respectively, multiplied by a buoyancy parameter.

That is

$$q_w/N_w^2 = \left(\frac{fg}{\theta_0}\right) \underline{\zeta} \cdot \nabla \theta_\omega \qquad (2.61)$$

$$q/N^2 = \left(fg/\theta_0\right) \underline{\zeta} \cdot \nabla \theta \qquad (2.62)$$

where $\underline{\zeta}$ is the absolute vorticity.

Emanuel (to appear) has proved a more rigorous result relating the geometry of a roll to the growth rate, which agrees very closely with the result above. Analytic solutions in the saturated atmosphere cannot be found and recourse to numerical results must be made.

The model that has been developed for this is now described in the next section.

3. <u>Numerical model</u> (i) **Formulation.**

A much more flexible tool is a finite difference model of the fluid equations. The Bennetts and Hoskins (1979) model was adopted and further developed.

The model equations are those for a compressible atmosphere with diffusion and diabatic heating Q, written with respect to coordinates (x, z) where $z = \left[1-\left(\frac{p}{p_0}\right)^{\kappa}\right]\frac{H_s}{\kappa}$, $\left(H_s = \frac{R\theta_0}{g}, \kappa = R/C_p\right)$, p is the pressure, and zero suffices refer to representative surface values. z is almost identical to physical height h in the troposphere, e.g. at p = 300 mb, z = 8000 m and h = 9200 m.

Defining r as the pseudo-density

$$r\,\Delta x\,\Delta z = \rho\,\Delta x\,\Delta h \tag{3.1}$$

and making the hydrostatic approximation leads to

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(uu) - \frac{1}{r}\frac{\partial}{\partial z}(r\omega u) + fv - \frac{\partial \phi}{\partial x} + \nu_x\frac{\partial^2 u}{\partial x^2} + \nu_z\frac{\partial^2 u}{\partial z^2} \tag{3.2}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial x}(uv) - \frac{1}{r}\frac{\partial}{\partial z}(r\omega v) - fu \qquad + \nu_x\frac{\partial^2 v}{\partial x^2} + \nu_z\frac{\partial^2 v}{\partial z^2} \tag{3.3}$$

$$\frac{\partial \phi}{\partial z} = \left(\frac{g}{\theta_0}\right)\theta \tag{3.4}$$

$$\frac{\partial \theta}{\partial t} = -\frac{\partial}{\partial x}(u\theta) - \frac{1}{r}\frac{\partial}{\partial z}(r\omega\theta) \qquad + Q \quad + \nu_x\frac{\partial^2 \theta}{\partial x^2} + \nu_z\frac{\partial^2 \theta}{\partial z^2} \tag{3.5}$$

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial z}(r\omega) = 0 \tag{3.6}$$

The flow is assumed two-dimensional so that all $\partial/\partial y$ terms have disappeared. ϕ is the geopotential,

$$r = \rho_0\left(p/p_0\right)^{1/\gamma} \quad \text{and} \quad r\omega = -\frac{1}{g}\frac{dp}{dt} \tag{3.7}$$

defines the "vertical velocity" ω.

If $r \equiv 1$ the equations **describe a Boussinesq fluid** with pressure ϕ and this corresponds to the limit $\frac{H}{H_s} \rightarrow 0$ where H is the height scale of the motion.

The domain of the model is a rectangle bounded by horizontal $\omega = 0$ boundaries at $z = 0$ and $z = H$ and lateral boundaries at $x = 0$ and $x = L$. The peturbation of potential temperature, θ', ($\theta' = \theta - \theta^0$ where θ^0 is the initial temperature) and all other variables are assumed periodic at $x = 0, L$.

The grid arrangement is shown in Fig (3.1). The grid is staggered with u, v, θ and ϕ held at the main points and $r\omega$ held at intermediate points.

The finite difference equations follow Williams (1967), are written in flux form, centred in space and time. Using the notation of Shuman (1962) ; for example

$$(\overline{\psi}^x)(x_0, z_0) = \frac{\psi(x_0 + \frac{\Delta x}{2}, z_0) + \psi(x_0 - \frac{\Delta x}{2}, z_0)}{2} \tag{3.8}$$

$$(\delta_x \psi)(x_0, z_0) = \frac{\psi(x_0 + \frac{\Delta x}{2}, z_0) - \psi(x_0 - \frac{\Delta x}{2}, z_0)}{\Delta x} \tag{3.9}$$

the finite difference equations can be written

$$\delta_t \overline{u}^t = -ADVU + fv - \delta_x \overline{\phi}^x + \nu_x \delta_{xx} u^{t-1} + \nu_z \delta_{zz} u^{t-1} \tag{3.10}$$

$$\delta_t \overline{v}^t = -ADVV - fu \qquad + \nu_x \delta_{xx} v^{t-1} + \nu_z \delta_{zz} v^{t-1} \tag{3.11}$$

$$\delta_z \phi = \left(\frac{g}{\theta_0}\right) \overline{\theta}^z \tag{3.12}$$

$$\delta_t \overline{\theta}^t = -ADV\theta + Q \qquad + \nu_x \delta_{xx} \hat{\theta}^{t-1} + \nu_z \delta_{zz} \hat{\theta}^{t-1} \tag{3.13}$$

$$\delta_x \overline{u}^x + \frac{1}{r} \delta_z (r\omega) = 0 \tag{3.14}$$

where all variables on the right hand side are taken at the central time level except for the diffusion terms which are taken at the previous step to ensure stability. $\hat{\theta}$ is the perturbation of θ from the initial temperature θ^0.

An occasional first order step is made to suppress the computational mode associated with the leapfrog time integration.

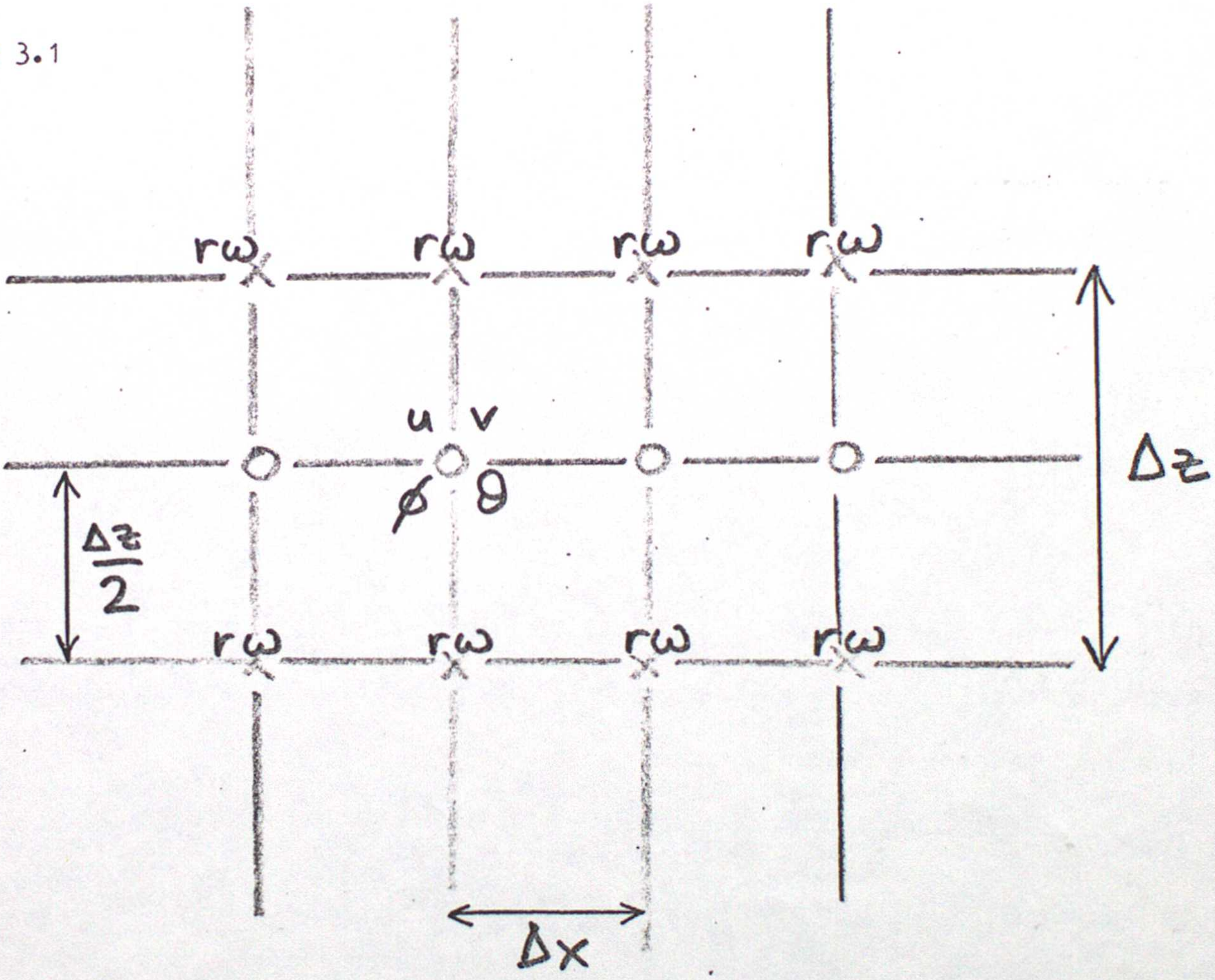
The advective terms are written as

$$\mathrm{ADV}\,\xi = \delta_x\left(\overline{u}^x\,\overline{\xi}^x\right) + \frac{1}{r}\,\delta_z\left(r\omega\,.\,\overline{\xi}^z\right) \tag{3.15}$$

where ξ can be u, v or θ.

In this form ξ is conserved exactly by this term and ξ^2 is conserved with an error of the order of $D = \delta_x \overline{u}^x + \frac{1}{r}\delta_z(r\omega)$ which is always very small.

Fig 3.1



The upper and lower boundaries run through $r\omega$ points.

The lateral boundaries lie between u points.

The boundary conditions are

$$r\omega = 0$$
$$\delta_{zz} u = \delta_{zz} v = \delta_{zz}\hat{\theta} = 0 \tag{3.16}$$

at $z = 0, H$. The second derivative conditions were chosen in order to close the system without inducing an Ekman boundary layer.

The periodic lateral boundary conditions are formulated as follows. The flow is regarded as a perturbation about a simple baroclinic state, given by

$$\theta^{\circ} = \left(\frac{\partial\theta}{\partial x}\right)^{\circ}\cdot x + \theta^{z}(z, x) \qquad u^{\circ} = (r\omega)^{\circ} = 0$$

$$v^{\circ} = \left(\frac{g}{f\theta_0}\right)\cdot\left(\frac{\partial\theta}{\partial x}\right)^{\circ}\cdot z \tag{3.17}$$

$$\phi^{\circ} = \left(\frac{g}{\theta_0}\right)\cdot\left(\frac{\partial\theta}{\partial x}\right)^{\circ}\cdot xz$$

where θ^{z} is any function such that $\theta^{z}(x=0, z) = \theta^{z}(x=L, z)$ Then the perturbation fields $u' = u - u_0, v' = v - v_0$ etc are assumed to have period L.

For example, in order to calculate θ at $x = -\frac{\Delta x}{2}$ the first point outside the domain,

$$\theta\left(-\frac{\Delta x}{2}\right) = \theta\left(L - \frac{\Delta x}{2}\right) - \left(\frac{\partial\theta}{\partial x}\right)^{\circ}\cdot L \tag{3.18}$$

v° is calculated explicitly from ϕ° in turn calculated from θ° using the same finite differencing as the model in order that truncation errors do not lead to spurious forcing on the boundaries.

The final boundary condition is $\int_0^H ru\,dz = 0$, i.e.

$$\sum_{z_j = \Delta z/2}^{H - \Delta z/2} r(z_j)\, u(z_j)\, \Delta z = 0. \quad \text{for all } (x, t) \tag{3.19}$$

This constraint implies there is no inertial oscillation of the geostrophic wind, and can be deduced from the equations and the choice of a balanced initial state.

The heating function Q is defined as

$$Q = gr\omega A \quad \text{if } \omega > 0 \tag{3.20}$$
$$= 0 \quad \text{otherwise}$$

where $A = -\frac{\partial\theta}{\partial\phi}\Big|_{\theta_\omega = \text{constant}}$ is the rate of increase of potential temperature as a saturated parcel ascends in the atmosphere from absolute pressure ϕ.

From the equation

$$\frac{d\theta}{dt} = gr\omega A . I(\omega) \tag{3.21}$$

where $I(\omega) = 1$ if $\omega > 0$, zero otherwise it follows immediately that

$$\theta^* = \theta - \int_0^z grA \, dz \tag{3.22}$$

is conserved on ascent and is equivalent to the wet bulb potential temperature θ_ω, provided the parcel remains saturated. On descent parcels are assumed unsaturated but sufficiently moist that any subsequent upward motion results in immediate saturation. θ^* will now be written as θ_ω.

The model definition of θ_ω is

$$\theta_\omega = \theta - \Delta \tag{3.23}$$

where $\delta_z \Delta = grA$ defined at $r\omega$ points and $\Delta(z = \frac{\Delta z}{2}) = (grA)_{z=0} \cdot \frac{\Delta z}{2}$.

Then requiring the finite difference analogue of $d\theta_w/dt = 0$ on ascent leads to the heating function

$$Q = \frac{1}{r} \cdot \overline{r\omega . grA}^{z} + O(D) \tag{3.24}$$

where $O(D)$ means a term of the order of $D = \delta_x \bar{u}^x + \frac{1}{r}\delta_z(r\omega)$.

Hence the heating function is chosen as

$$\begin{aligned} Q &= \frac{1}{r} \cdot \overline{r\omega . grA}^{z} && \text{if } \omega > 0 \\ &= 0 && \text{if } \omega < 0 \end{aligned} \tag{3.25}$$

in the model.

The calculation of ϕ is made at every time step in two **stages, prior to** incrementing the u field. The first step is to integrate the hydrostatic equation (3.12) assuming $\phi = 0$ at $z = 0$ to give ϕ^*,

$$\phi^*_{z_{N+\frac{1}{2}}} = \sum_{j=1}^{N} \alpha\overline{\theta}^z\Big|_{z_j} \cdot \Delta z + \alpha\theta\Big|_{\Delta z/2} \cdot \frac{\Delta z}{2} \tag{3.26}$$

where $z_j = j\,\Delta z$ the $r\omega$ grid points of the model.

Then the final pressure is calculated as

$$\phi = \phi^* + \phi'(x) \tag{3.27}$$

where ϕ' is calculated from a balance equation, derived as follows.

If the u momentum equation (3.10) is written as

$$\delta_t \overline{u}^t = S_u - \delta_x \overline{\phi}^x \tag{3.28}$$

where S_u represents all the terms on the right hand side that do not involve the pressure, then on using (3.27) and integrating over z from 0 to H gives the balance equation

$$\sum_{z_j=\frac{\Delta z}{2}}^{H-\Delta z/2} r(z_j) \cdot \delta_t \overline{u}^t \cdot \Delta z = \sum r\left(S_u - \delta_x \overline{\phi^*}^x\right)\Delta z - \left(\sum r\,\Delta z\right) \cdot \delta_x \overline{\phi'}^x \tag{3.29}$$

The left hand side is the discrete analogue of (3.19) and thus $\delta_x \overline{\phi'}^x$ is determined. Only ϕ^* and $\delta_x \overline{\phi}^x$ are held in the model as ϕ itself is never needed.

A stream function ψ is also calculated diagnostically such that

$$\overline{ru}^x = \delta_z \psi \qquad r\omega = -\delta_x \psi \tag{3.30}$$

by using a direct method (*) to solve the Poisson equation

$$(\delta_{xx} + \delta_{zz})\psi = \delta_z(\overline{ru}^x) - \delta_x(r\omega) \tag{3.31}$$

with boundary conditions $\psi = 0$ at $z = 0, H$ and
$\psi(x=0) = \psi(x=L)$.

(*) Ref; Met 0 2b Technical Note 33 C Temperton (PSOLV1)

The model is initialised in two steps. First a background state is assumed of the form

$$\theta = \theta^{\circ}(x, z) \tag{3.32}$$

A zonal velocity field V is calculated in thermal wind balance assuming $V = 0$ at $Z = 0$ The discrete equationsare used to ensure no spurious forcing in the u momentum equation (3.10) due to round off error.

Then $(\partial\theta/\partial x)^{\circ}$ is identified with $[\theta^{\circ}(L, z) - \theta^{\circ}(0, z)]/L$ and must be constant with height. This parameter is used in the boundary formulation, e.g. equation (3.18(.

The distribution of moisture is specified by defining rA_g as a function of height.

The second step is to add a dynamical perturbation inside the domain whose evolution is then studied. The circulation chosen is specified by

$$u = u^{\circ}(x, z) \tag{3.33}$$

where $\sum_{z_j = \Delta z/2}^{H - \Delta z/2} r u \Delta z = 0$

$r\omega$ is calculated from the continuity equation (3.14) to ensure mass conservation in the disturbance.

A single first order step then initiates the integration.

(ii) Rayleigh damping.

A mixture of propagating and stationary components are excited by the perturbation. The travelling disturbances have gravity wave phase speeds and amplitude comparable to the velocities of the initial circulation. Although damped by the diffusion they can survive for a significant time. Due to the periodicity the waves will then travel around the domain and will interact with the stationary components. If strong growth has not occurred before this happens then the interaction distorts the development of these components.

In order to model the flow in a rather longer or infinite box simple damping boundary conditions can be used which effectively absorb the propagating waves when they reach the computational boundaries. Rayleigh damping layers for $L-d<x<L$ and $0<x<d$ were used to damp the perturbation fields of u, v and θ. The treatment of each is the same so that describing the implementation in the v momentum equation (3.11) is sufficient.

Writing (3.11) as

$$\delta_t \bar{v}^t = S_v \tag{3.34}$$

where S_v is the entire right hand side, then the equation is modified to

$$\delta_t \bar{v}^t = S_v - \frac{1}{\tau}(v - v^{\circ}) \tag{3.35}$$

where v° is the v field at the initialisation time $t=0$ and τ is a relaxation time given by

$$\begin{aligned} \tau &= 0 && \text{outside the damping layers} \\ &= \tau(x) && \text{inside the damping layers} \end{aligned}$$

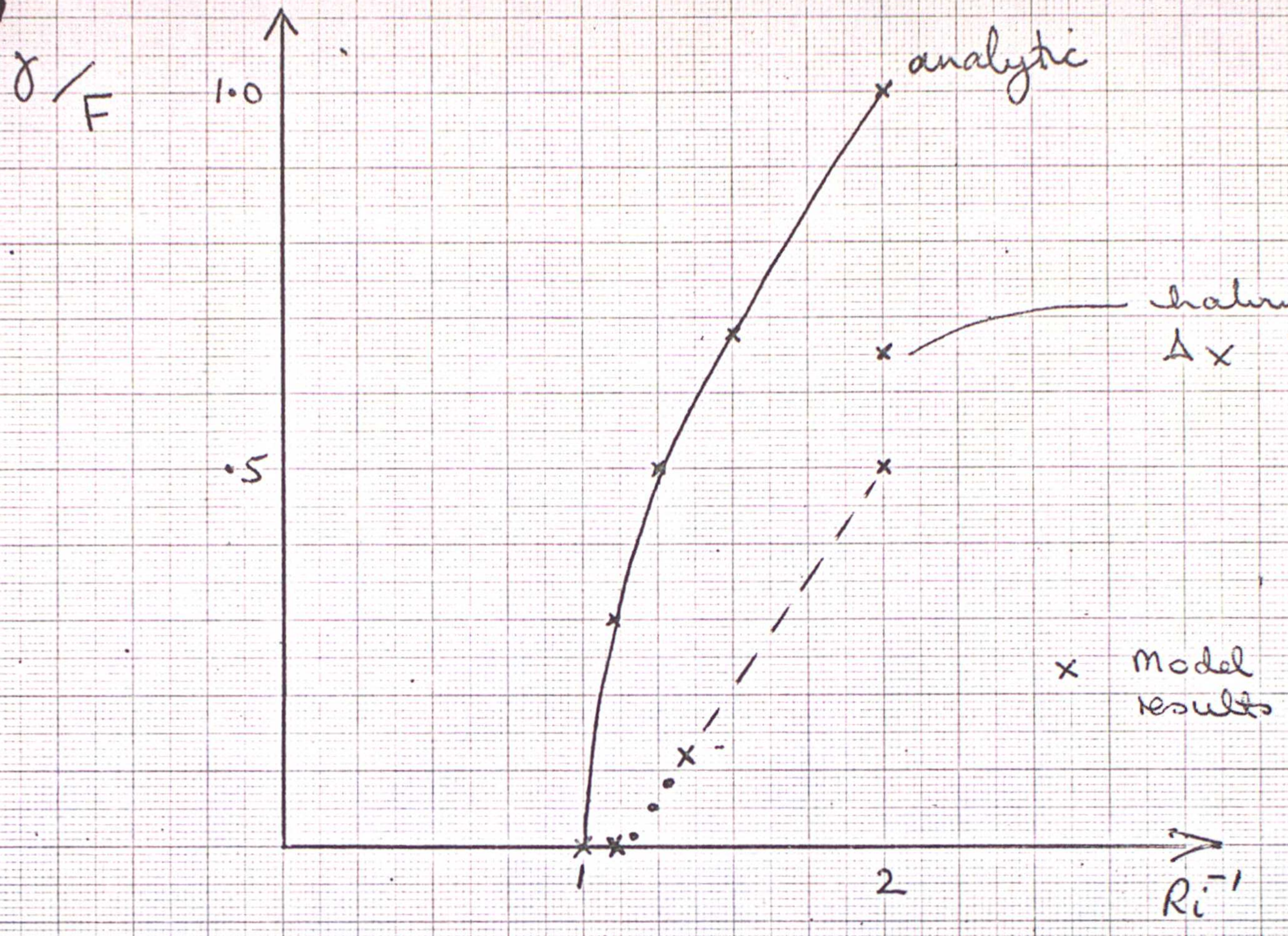
$\tau(x)$ increases slowly to a constant value in order to minimise reflection of waves back into the interior of the domain.

No evidence of reflections has been seen with the model. The calculated reflection coefficient shows that significant reflection will occur if $\Delta\tau$ is comparable with the angular frequency of the incident wave, where $\Delta\tau$ is the change in τ from one grid point to the next.

Care was taken to ensure the layers were not too near the motion of interest. Comparison with very long domain integrations showed that even for an unstable cell reaching the Rayleigh regions the growth was little affected, but was inhibited if the regions were much larger, as the slowly growing circulation cannot penetrate the layers.

(iii) Dissipation.

It is important to consider the dissipative characteristics of the model. Diffusion is represented explicitly by the laminar terms in ν_x and ν_z (eqns (3.10) - (3.13)). These terms model anisotrophic diffusion in a Boussinesq fluid. In the compressible equations they are only an approximation to the equivalent property. This can lead to spurious sources and sinks in the flow



$$u = u_0(x,z)\, e^{\gamma t} \qquad F = f\left(f + \frac{\partial V_0}{\partial x}\right)$$

Fig.(3.2) DIMENSIONLESS GROWTH RATE AGAINST Ri^{-1} FOR MOST UNSTABLE MODE; ANALYTIC AND NUMERICAL RESULTS.

but these are always small when the model is used close to the Boussinesq limit.

The advective differencing and leapfrog time integration is formally non-dissipative but a full analysis of the numerical scheme is very complex. The model was used with no explicit diffusion and the model proved stable. The fields developed some roughness only when the nonlinear terms became significant.

Then the Richardson number was varied in a dry integration to find the neutral point of symmetric instability. Analytic theory shows $Ri^{-1} = 1$ is the inviscid value. Fig (3.2) shows the variation of symmetric growth rate from the same arbitrary initial disturbance for various Ri.

The growth rate is scaled with the parameter F (defined in II). The analytic curve is based on the most unstable inviscid mode which has vanishing horizontal length scale. The model results suggest a neutral point of $Ri^{-1} = 1.1$ The model growth rates are significantly different from the analytic prediction away from this point. At $Ri^{-1} = 2$ the growth rate is different by a factor of 2.

The most unstable mode in the model will not be the analytic one due to the finite resolution. Reducing the grid spacing by a factor of 2 smaller wavelength modes were resolved, the growth rate increased by 30% and the cell slope doubled. Further improvements in resolution would be expected to bring the growth rate nearer the analytic value.

Usually the model is used to study the viscous problem with typical diffusion coefficients $\nu_x = 200\,m^2s^{-1}$, $\nu_z = 1\,m^2s^{-1}$. Then for $\Delta x = 1000\,m$ and $\Delta z = 100\,m$ the flow is generally well resolved as the results are not sensitive to further reductions in grid spacing.

The stability criteria that must be satisfied are the following conventional ones

$$\frac{u\Delta t}{\Delta x}, \frac{w\Delta t}{\Delta z}, \frac{\Delta x^2}{4\nu_x}, \frac{\Delta z^2}{4\nu_z} < 1 .$$

The diffusive terms are handled with a forward time step to give stability. The scheme is vulnerable to time splitting and so occasionally, say every 150 steps, a first order step is made for all the fields. The inexact conservation of kinetic emergy means there is the possibility of a nonlinear error growth but this has not been seen.

THE CONCLUSIONS FOLLOW THE APPENDICES.

Appendix I

Proof that for complex , there is just one evanescent mode in region 2 of the two layer analysis of Section II.

Given (2.43) in region 2

$$m'_{2\pm} = \frac{k'}{Ri_2(1-\omega'^2)}\left\{1 \pm \sqrt{(1 - Ri_2(1-\omega'^2))}\right\} \quad \text{AI.1}$$

where $\omega^2 \in \mathbb{C}$, write $z = 1 - Ri_2(1-\omega'^2) \in \mathbb{C}$

Now

$$m_{2\pm} = \frac{k}{1-z}\left(1 \pm z^{1/2}\right) \quad \text{AI.2}$$

dropping redundant suffices and primes.

Write

$$m = \frac{k}{|1-z|^2} R \quad \text{AI.3}$$

where $R = (1-z^*)(1 \pm z^{1/2})$, * denoting complex conjugate.

Then if $z = re^{i\theta}$, $|\theta| < \pi$,

$$\begin{aligned} \text{Im}(R) &= r \sin\theta \pm r^{1/2} \sin\frac{\theta}{2}(1+r) \\ &= \sin\frac{\theta}{2}\left\{x^2 \frac{\sin\theta}{\sin\theta/2} \pm x(1+x^2)\right\} \qquad |\theta| \leq \pi \\ &= \pm x \sin\theta/2 \left\{x^2 \pm 2x\cos\theta/2 + 1\right\} \end{aligned} \quad \text{AI.4}$$

where $x^2 = r$,

and it is easily seen that $x^2 \pm 2x\cos\theta/2 + 1 > 0$ for all $x \in \mathbb{R}$.

Thus for any z such that $x \sin\frac{\theta}{2} \neq 0$, there is one root of (AI.2) with positive imaginary part and one with negative. Hence there is one and only one evanescent mode for any ω^2 provided $1 - Ri_2(1-\omega^2)$ is not a positive real number or zero.

Further from (AI.2) if m_+, m_- are the two roots ,

$$m_+ - m_- = \frac{2kz^{1/2}}{1-z} = \frac{2k}{|1-z|^2} z^{1/2} (1-z^*) \tag{AI.5}$$

and making the substitution $z = x^2 e^{i\theta}$, $|\theta| < \pi$, as above

$$\mathrm{Im}\,(m_+ - m_-) = \frac{2k}{|1-z|^2} \cdot x(1+x^2)\sin\frac{\theta}{2} \tag{AI.6}$$

Thus the difference of m_+ and m_- is real if and only if z is real and positive (θ = 0), or zero.

Appendix II. Asymptotic solution of (2.54) - (2.57) + (2.43)

From (2.43),

$$m_2' = \frac{k'}{Ri_2(1-\omega'^2)}\left\{1 \pm \sqrt{(1 - Ri_2(1-\omega'^2))}\right\} \qquad \text{(II.1)}$$

where the sign is chosen such that Im (m_2') > 0 and $\sqrt{Z}$ is defined such that $0 \leq \arg\sqrt{Z} < \pi$.

If $Ri_2 \gg 1$ and $Ri_2|1-\omega^2| \gg 1$ too, then

$$m_2 = ik(1-\omega^2)^{-1/2}\varepsilon + (1-\tfrac{1}{2}i)k(1-\omega^2)^{-1}\varepsilon^2 + O\left(k(1-\omega^2)^{-3/2}\varepsilon^3\right) \qquad \text{(II.2)}$$

where $\varepsilon = Ri_2^{-1/2}$.

Now assuming m_{1+}, m_{1-} remain bounded as $Ri_2 \to \infty$ then (2.55) shows

$$\left|m_2\frac{A_2}{A_1}\right|^2 < |m_{1+}|^2 + |m_{1-}|^2 < \infty$$

$$\Rightarrow \frac{A_2}{A_1} = \frac{K_0}{k(1-\omega^2)^{-1/2}\varepsilon} + \frac{K_1}{k} + O(\varepsilon) \qquad \text{(II.3)}$$

then (2.55) gives

$$m_{1+} - m_{1-}e^{i(m_{1-}-m_{1+})} = iK_0 + [iK_1 + (1-\tfrac{1}{2}i)K_0](1-\omega^2)^{-1/2}\varepsilon + O(\varepsilon^2) \qquad \text{(II.4)}$$

while (2.54) gives

$$1 - e^{i(m_{1-}-m_{1+})} = \frac{Ri_1 K_0 \varepsilon}{k(1-\omega^2)^{-1/2}} + O(\varepsilon^2) \qquad \text{(II.5)}$$

Now write $m_{1+} - m_{1-} = M_0 + M_1\varepsilon + O(\varepsilon^2)$, then (II.5) becomes

$$1 - e^{iM_0}(1 - iM_1\varepsilon + O(\varepsilon^2)) = \frac{Ri_1 K_0 \varepsilon}{k(1-\omega^2)^{-1/2}} + O(\varepsilon^2) \qquad \text{(II.6)}$$

$$\Rightarrow 1 - e^{iM_0} = 0 \qquad \text{(II.7)}$$

$$\text{and} \quad iM_1 e^{iM_0} = \frac{Ri_1 K_0}{k(1-\omega^2)^{-1/2}} \qquad \text{(II.8)}$$

$$\Longrightarrow \qquad M_0 = 2n\pi \qquad \text{(II.9)}$$

$$M_1 = \frac{-i\,Ri_1 K_0}{k(1-\omega^2)^{-1/2}} \qquad \text{(II.10)}$$

Hence the leading order solution has the same structure in the lower layer as the single layer solution considered in Section 2.

The first order correction can also be evaluated from the expansion terms above, but is beyond the scope of this appendix.

Appendix III. Dimensional analogues of some Symmetric Instability results

The dimensional analogue of the structure equation (2.24) is

$$\frac{d^2w}{dz^2} - \frac{2S^2 ik}{F^2-\omega^2}\frac{dw}{dz} - \left(\frac{N^2-\omega^2}{F^2-\omega^2}\right)k^2 w = 0 \tag{III.1}$$

and the dispersion relation (2.25)

$$\omega^2 = \frac{F^2 - 2S^2 k/m + N^2 k^2/m^2}{1+(k/m)^2} \tag{III.2}$$

so that the most unstable mode's growth rate (2.26) is given by γ such that

$$\omega^2_{min} = -\gamma^2 = F^2 - S^4/N^2 = q/N^2 \tag{III.3}$$

where $q \propto$ Ertel potential vorticity of the basic flow and is a conserved quantity in the adiabatic frictionless equations ((2.3) - (2.7); = 0)).

The energy equation is

$$\frac{\partial}{\partial t}\left(\frac{u^2+v^2+w^2}{2}\right) = -\frac{\partial}{\partial x}(up) - \frac{\partial}{\partial z}(wp)$$

$$-uv\frac{\partial V}{\partial x} - vw\frac{\partial V}{\partial z} + w\sigma \tag{III.4}$$

III. ENERGY AND MOMENTUM BUDGETS IN THE NUMERICAL MODEL

The analytic equations on which the model is based (3.2) - (3.6) obey the following kinetic energy equation

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{u^2+v^2}{2}\right) = & -\frac{\partial}{\partial x}\left(u\left[\frac{u^2+v^2}{2}\right]\right) - \frac{1}{r}\frac{\partial}{\partial z}\left(r\omega.\left[\frac{u^2+v^2}{2}\right]\right) + \frac{g}{\theta_0}\omega\theta \\
& -\frac{\partial}{\partial x}(u\phi) - \frac{1}{r}\frac{\partial}{\partial z}(r\omega.\phi) \\
& +\frac{\partial}{\partial x}\left(\nu_x\frac{\partial}{\partial x}\frac{u^2+v^2}{2}\right) + \frac{1}{r}\frac{\partial}{\partial z}\left(r\nu_z\frac{\partial}{\partial z}\frac{u^2+v^2}{2}\right) \\
& - \varepsilon
\end{aligned}
\qquad (\text{III}.1)
$$

where
$$
\begin{aligned}
\varepsilon = & \ \nu_x\left(\frac{\partial u}{\partial x}\right)^2 + \nu_z\left(\frac{\partial u}{\partial z}\right)^2 + \nu_x\left(\frac{\partial v}{\partial x}\right)^2 + \nu_z\left(\frac{\partial v}{\partial z}\right)^2 \\
& + \frac{\nu_z}{r}\frac{\partial r}{\partial z}.\frac{\partial}{\partial z}\frac{u^2+v^2}{2}
\end{aligned}
\qquad (\text{III}.2)
$$

The terms are grouped onto lines so that their meaning can be summarised. The first line is the advection of kinetic energy and the last term the buoyancy production term. The next line are the pressure work terms which only give a surface contribution. The third line is a viscous diffusion of energy across the bounding surface of the fluid domain. ε is the dissipation but is not strictly positive definite unless $\frac{\partial r}{\partial z} = 0$; this is the spurious source/sink alluded to in the main text due to the incorrect form of diffusion in the transformed coordinate system. The term is however small in practice.

The potential energy equation is

$$\frac{\partial}{\partial t}\left(-\frac{g}{\bar{\theta}_0}\cdot z\theta\right) = -\frac{\partial}{\partial x}\left(u\cdot -\frac{gz\theta}{\bar{\theta}_0}\right) - \frac{1}{r}\frac{\partial}{\partial z}\left(r\omega\cdot -\frac{gz\theta}{\bar{\theta}_0}\right)$$

$$-\frac{g\omega\theta}{\bar{\theta}_0} - \frac{gzQ}{\bar{\theta}_0}$$

$$+\frac{\partial}{\partial x}\left(\nu_x\frac{\partial}{\partial x}\left[-\frac{gz\theta}{\bar{\theta}_0}\right]\right) + \frac{\partial}{\partial z}\left(\nu_z\cdot\frac{\partial}{\partial z}\left[-\frac{gz\theta}{\bar{\theta}_0}\right]\right) + \frac{2g\nu_z}{\bar{\theta}_0}\cdot\frac{\partial\theta}{\partial z} \qquad \text{(III.3)}$$

The first line terms are advection of potential energy, the second line has loss by buoyant production of kinetic energy and generation by diabatic sources. The third line is diffusion and is a surface effect but the last term has no significance and is due to the unphysical form of diffusion. It is a surface effect and generally small but unlike its kinetic counterpart does not affect the dynamics.

The finite difference analogues of (III.1) and (III.2) are derived analogously, using the identity

$$\tilde{p}\,\delta_y(\phi\,\bar{p}^y) = \delta_y\left(\phi\cdot\frac{\overline{pp}^y}{2}\right) + \frac{p^2}{2}\,\delta_y\,\bar{\phi}^y$$

$$+\Delta t^2\cdot\delta_{tt}p\cdot\delta_y(\phi\,\bar{p}^y) \qquad \text{(III.4)}$$

where p can be u or v, ϕ is any function and

$$2\tilde{p} = p(t+\Delta t) + p(t-\Delta t)$$

$$\widetilde{pp}^y = p\left(y+\frac{\Delta y}{2}\right)\cdot p\left(y-\frac{\Delta y}{2}\right)$$

Now

$$\delta_t \overline{\frac{u^2+v^2}{2}}^t = \tilde{u}\,\delta_t \overline{u}^t + \tilde{v}\,\delta_t \overline{v}^t \qquad \text{(III.5)}$$

so that applying this combination to (3.10) and (3.11) gives

$$\begin{aligned}\delta_t \overline{\frac{u^2+v^2}{2}}^t = &-ADVKE - DIVPV + CPK \qquad \text{(III.6)}\\ &+DISS\\ &+\{CKI + NLADV + COR\}\end{aligned}$$

where

$$ADVKE = \delta_x\left(\overline{u}^x \, \frac{\widetilde{u}\overline{u}^x + \widetilde{v}\overline{v}^x}{2}\right) + \frac{1}{r}\,\delta_z\left(r\omega .\ \frac{\widetilde{u}\overline{u}^z + \widetilde{v}\overline{v}^z}{2}\right)$$

the analogue of advection;

$$DIVPV = \delta_x\left(\overline{\tilde{u}}^x . \ \overline{\phi}^x\right) + \frac{1}{r}\,\delta_z\left(\widetilde{r\omega} . \ \overline{\phi}^z\right)$$

the "pressure work";

$$CPK = \frac{1}{r} . \ \overline{\widetilde{r\omega}\, \frac{g}{\theta_0}\, \overline{\theta}^z}^z$$

the buoyant production;

$$\begin{aligned}DISS = &\ \delta_x\left[\nu_x\left(\overline{\tilde{u}}^x \delta_x u + \overline{\tilde{v}}^x \delta_x v\right]\right.\\ &+ \frac{1}{r}\,\delta_z\left[\nu_z\left(\overline{r\tilde{u}}^z \delta_z u + \overline{r\tilde{v}}^z \delta_z v\right)\right] - E\end{aligned}$$

$$E = \nu_x \overline{(\delta_x u\, \delta_x \tilde{u})}^x + \nu_x \overline{(\delta_x v\, \delta_x \tilde{v})}^x$$

$$+ \frac{1}{r}\nu_z \overline{(\overline{r}^z \delta_z u\, \delta_z \tilde{u})}^z + \frac{1}{r}\nu_z \overline{(\overline{r}^z \delta_z v\, \delta_z \tilde{v})}^z + \frac{\nu_z}{r} \delta_z r . \overline{\tilde{u}}^z \delta_z u$$

the viscous diffusion and dissipation; and the purely numerical terms

$$CKI = \overline{\overline{\phi}^x \delta_x \tilde{u}}^x + \frac{1}{r}\phi\, \delta_z(r\omega)$$

$$NLADV = -\left(\frac{u^2+v^2}{2}\right)\left[\delta_x \overline{u}^x + \frac{1}{r}\delta_z r\omega\right] - \Delta t^2\left\{\delta_{tt}u .\ ADVU + \delta_{tt}v .\ ADVV\right\}$$

$$COR = f(\tilde{u}v - \tilde{v}u) = \Delta t^2 f\left(v\,\delta_{tt}u - u\,\delta_{tt}v\right)$$

where the last two have been written correct to $O(\Delta t^3)$.

These last three terms are all small. The dissipation has the analogue of the unphysical term in (III.2) as its last term, this may lead to a source of energy but is small. The positive definite terms are only positive to $O(\Delta t)$ since for example

$$\nu_x \overline{\delta_x u . \delta_x \tilde{u}}^x = \nu_x \overline{(\delta_x u)^2}^x + \Delta t^2 \nu_x \overline{\delta_x u\, \delta_{xtt} u}^x + O(\Delta t^4)$$

but the second term is much smaller than the first so no net source of energy is possible.

The potential energy equation is found by considering $\delta_t \overline{z\theta}^t$, then defining $I = -gz\theta/\bar{\theta}$.

$$\delta_t \overline{I}^t = -ADVIE - CPK - DIFF\theta + [NLADI] \qquad \text{(III.7)}$$

where

$$ADVIE = \delta_x\left(\overline{u}^x . \overline{I}^x\right) + \frac{1}{r}\delta_z\left(r\omega . \overline{I}^z\right)$$

the advection of I;

$$CPK = \frac{1}{r} . \overline{r\omega \frac{g}{\bar{\theta}_0}\overline{\theta}^z}^z$$

the conversion between kinetic energy and potential energy;

$$DIFF\theta = \frac{-gz}{\bar{\theta}_0}\left\{\nu_x \delta_{xx}\theta + \nu_z \delta_{zz}\theta\right\}$$

and the numerical term

$$NLADI = \left(\frac{g\Delta z^2}{\bar{\theta}_0}\right) . \frac{1}{r}\delta_z\left(r\omega . \delta_z\theta\right)$$

The diffusion term can be written analogously to the form in (III.3) as

$$DIFF\theta = \delta_x\left(\nu_x \delta_x I\right) + \delta_z\left(\nu_z \delta_z I\right) + \frac{2g}{\bar{\theta}_0}\nu_z \delta_z \overline{\theta}^z \qquad \text{(III.8)}$$

4. CONCLUSIONS

Dry symmetric instability theory (SI) has been reviewed and presented to emphasise the energy exchanges and aspect ratios of the motion. SI is a motion that draws its energy predominantly from the kinetic energy of a sheared jet along a front and to do this particles ascend at an angle steep compared to the absolute vorticity vector. For this to be thermodynamically consistent the potential isentropes must slope at close to this angle.

The presence of an upper inversion was shown to only slightly modify the analytically simpler solution of the rigid upper lid. The work of Emmanuel (1978) is crucial for the physical interpretation of the bounded instability when viscosity forces the motion to adopt finite scales which are theemselves independent of the value of viscosity.

Moisture was shown to allow thermodynamically consistent motion with a steeper angle of ascent and this will promote an enhanced extraction of energy from the environment. A physical description of Conditional Symmetric Instability (CSI) then followed.

A two dimensional numerical model was described in which a simple moisture parameterisation allows the study of CSI. Results from this model are given in Nash (1982a, 1982b).

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