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Implicit finite difference methods for computing discontinuous atmospheric flows

by

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IMPLICIT FINITE DIFFERENCE METHODS

FOR COMPUTING DISCONTINUOUS ATMOSPHERIC FLOWS

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1. Introduction

An important mathematical model for representing discontinuous atmospheric flow is given by first calculating an equilibrium velocity field from the requirement that the horizontal pressure gradient and frictional forces balance the Coriolis acceleration due to the Earth's rotation, and that the vertical pressure gradient is hydrostatic. In the absence of friction this requires the horizontal velocity to be geostrophic. The momentum in the equation of motion is then approximated by its equilibrium value, but the trajectory is not approximated. In the absence of friction this leads to the well-known geostrophic momentum approximation, Hoskins (1982), Salmon (1985).

Cullen and Purser (1984) showed that the frictionless equations possess generalised solutions which may contain discontinuities.. A number of subsequent studies, e.g. Cullen et al. (1987), have shown that these solutions are useful simplified models of a number of atmospheric flows of direct relevance to weather forecasting, such as the deflection of the flow round large mountain barriers and the intensification of depressions due to convective mass transfer. In these cases the equations can be solved exactly for piecewise constant data by using a Lagrangian method.

In this paper we include the friction. A finite difference method is used to solve the equations, which still admit discontinuous solutions. The method has to be at least partly implicit, since the trajectory is determined implicitly. A predictor-corrector method is used. The velocity field from the previous timestep is used as a first guess, and the evolution equations solved. The residual in the balance of forces, which should be zero, is calculated and used to generate a correction to the velocity field. The structure of the method is similar to that of the pressure correction method for the incompressible Navier-Stokes equations.

2. Model problem

The equations are for flow in a two-dimensional atmospheric cross-section. The coordinates are x and σ where σ is pressure divided by surface pressure. The upper and lower boundaries are therefore $\sigma= 0,1$. The balance of forces in the x direction is given by:

$$\partial\phi/\partial x + C_p\sigma^{\kappa}\theta\partial\pi/\partial x - fv = F_u. \qquad (1)$$

where ϕ is the geopotential, C_p the specific heat of air at constant pressure, $\kappa=R/C_p$ where R is the gas constant, $\pi=(p_*/p_o)^{\kappa}$ where p_* is the surface pressure and p_o a reference pressure, f is the Coriolis parameter and F_u the frictional force. There is assumed to be no pressure gradient in the y direction, so that the balance in that direction is trivial. The hydrostatic relation is

$$\partial\phi/\partial\sigma = -RT/\sigma \qquad (2)$$

where T is the temperature. The evolution equations for the y momentum and potential temperature $\theta=T/(\pi\sigma^{\kappa})$ are

$$Dv/Dt - fu = F_v \qquad (3)$$

$$D\theta/Dt = H \qquad (4)$$

where F_v is the frictional force in the y direction and H is the heat source. The continuity equation is

$$\partial p_*/\partial t + \partial(p_*u)/\partial x + \partial(p_*\dot{\sigma})/\partial\sigma = 0 \qquad (5)$$

$\dot{\sigma}$ is the vertical velocity.

In the case without friction, the generalised solutions are constructed at each time as an incompressible rearrangement of θ and $(v+fx)$. There is a unique rearrangement which satisfies (1) and (2) and the dynamical stability condition

$$(\partial(v+fx)/\partial x)(\partial\theta/\partial\sigma) - (\partial\theta/\partial x)(\partial v/\partial\sigma) \geqslant 0 \qquad (6)$$

When forcing terms such as H in (4) are included, the solution evolves in time as a sequence of rearrangements. If the rearrangements are smooth, they can be represented as advection by a smooth velocity field $(u,\dot{\sigma})$. In general the rearrangement will not be smooth and may, for instance, require fluid to detach from the boundary. In the presence of friction, the structure of the solution is more complicated and the problem must be solved as an initial value problem. If the term H in (4) implies heating from below, the condition (6) will be weakened and considerable vertical rearrangement of the fluid will occur. If the vertical stability is completely destroyed by the heating, convective overturning will result. This cannot be represented as advection by a smooth velocity field and requires separate numerical treatment.

The frictional forces and heat fluxes are calculated as functions of the local Richardson number using the turbulence model employed in the U.K. operational weather forecasting models. In the problem illustrated here the cross-section is half land and half sea. The land is assumed to be rougher than the sea, and is heated and cooled on a diurnal cycle. The sea temperature is assumed constant. A sea breeze circulation is set up. During the day the hydrostatic pressure difference between land and sea implied by the heating is partly balanced by friction and there is only limited penetration of the sea-breeze inland. During the evening the air becomes more stable, the friction is reduced, and the sea-breeze accelerates. Later in the night the land becomes colder than the sea and a shallow land breeze develops.

3. <u>Numerical method</u>

A finite difference method is used. The variables are stored on a staggered grid, with u and v held at the same points, and θ and $\dot{\sigma}$ held together at points staggered apart in both x and σ. $\dot{\sigma}$ is held at the upper and lower boundaries, where it is set to zero. p_* is held instead of θ at the lower boundary points. The scheme is described in more detail in Cullen(1987).

A predictor-corrector method is used. In the predictor step the variables v,θ and p_* are stepped forward in time using equations (3) to (5) and the current values of the mass-weighted velocity field $(p_*u, p_*\dot{\sigma})$. An implicit single step method is used, with slight backward weighting (0.55). The heating and friction increments are then added, these are also calculated using an implicit method. The fields must then be adjusted to ensure satisfaction of (6). It has not yet been found practicable to use a two-dimensional adjustment, and therefore successive sweeps of the data are made in the x and σ directions to ensure that $\partial\theta/\partial\sigma<0$ and $\partial(v+fx)/\partial x>0$. These adjustments cover the cases where there is convective overturning not representable by a finite difference approximation to equations (3) to (5).

The correction step updates the mass-weighted velocity fields by calculating a streamfunction correction $\Delta\psi$ and a vertical mean momentum correction $\Delta\bar{U}$, with $\bar{U}=p_*\bar{u}$. The correction equations are derived by using (3) to (5) to convert the velocity corrections into corrections to v, θ and p_*, and then substituting into (1) and (2) to remove the residuals. This gives

$$\kappa\delta_x p_*^{-1}\delta_x\Delta\bar{U} - f^2 p_*^{-1}(\Delta\bar{U}-\delta_\sigma\psi)_* + C_{*j}(\Delta\bar{U}-\delta_\sigma\psi)_j = R_* \tag{7}$$

$$-\kappa C_p\sigma^{\kappa-1}\pi\delta_x(\overline{\delta_\sigma\theta}^{\sigma}\delta_x\psi) + f^2\delta_\sigma(p_*^{-1}\delta_\sigma\psi) + \delta_\sigma C_{ij}(\Delta\bar{U}-\delta_\sigma\psi)_j = R \tag{8}$$

In these equations, the suffix * denotes lower boundary values and δ a finite difference in the direction specified. The friction term F_u at level i is calculated in terms of values of u at model levels as the sum $C_{ij}u_j$, the details are given in Bell and Dickinson (1987). In deriving these equations, not all the terms in (3) to (5) were used. The selection was done to make (7) and (8) as elliptic as possible, and in particular to ensure that the one-dimensional conditions $\partial\theta/\partial\sigma<0$ and $\partial(v+fx)/\partial x>0$ are sufficient for ellipticity. This procedure is found to be needed to ensure stability where the solutions are discontinuous, and is analogous to the under-relaxation often used in the pressure correction method.

This scheme was used to integrate the equations on a 50×12 grid, with unequal vertical spacing giving higher resolution near the lower boundary. The horizontal grid-length was 4km and the timestep 1 hour. The correction step was performed three times for each predictor step, this was found to give adequate accuracy. The results are compared with a Lagrangian method developed by Chynoweth (1987) in which all effects can be included except the F_u term. By calculating this term *a posteriori* the accuracy of the finite difference calculation can be assessed. They are also compared with an explicit model where the Du/Dt term is included on the laft hand side of (1). This allows non-equilibrium wave motions to develop, and illustrates the difficulty of computing real motions which are close to non-smooth equilibrium states.

4. <u>Results</u>

The diagrams on the next page show:

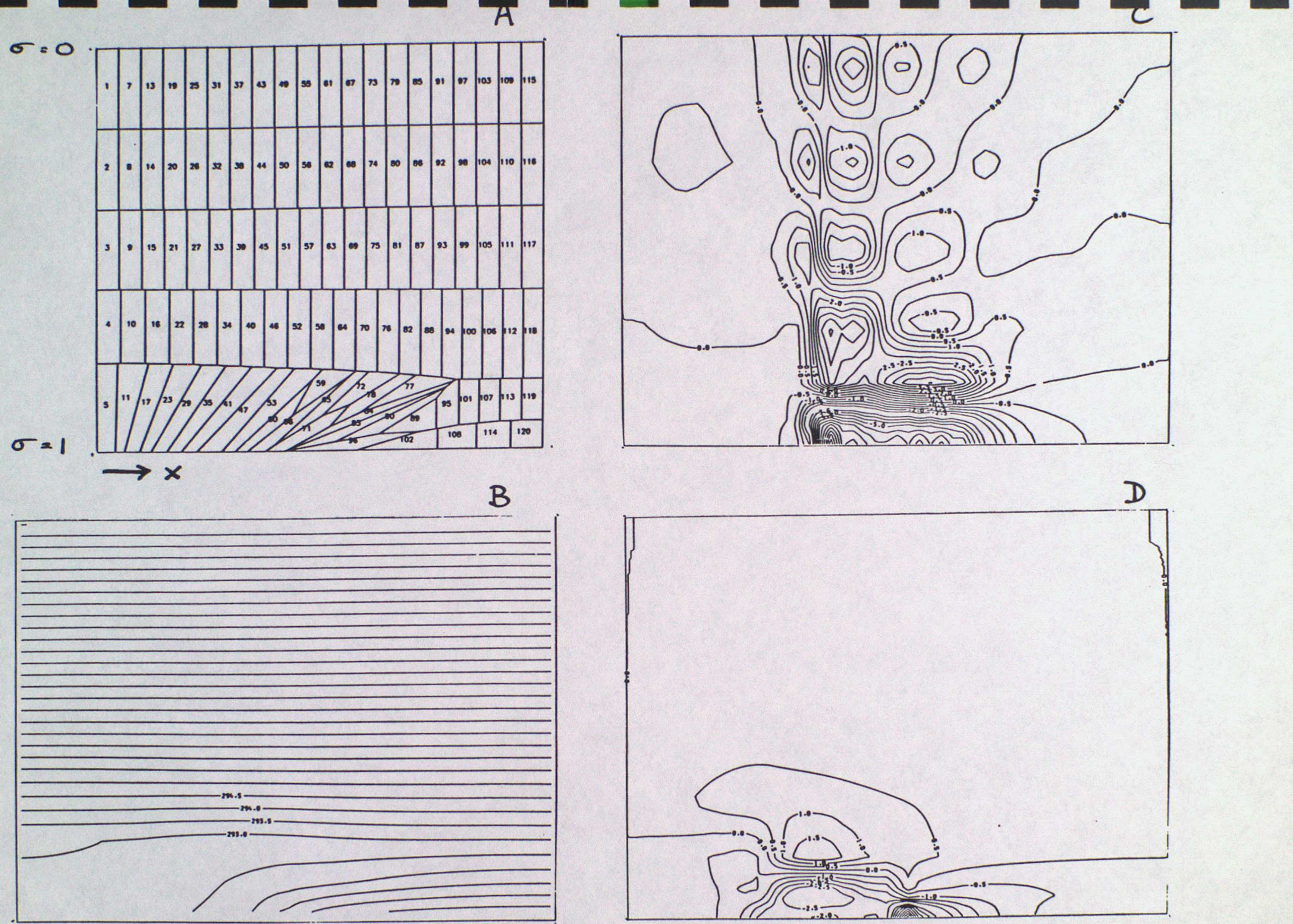
A: Lagrangian solution after 12 hours. The elements were originally in a regular rectangular array.

B: Finite difference θ distribution at the same time.

C: Finite difference u field.

D: The u field from the explicit model.

Both the Lagrangian and finite difference solutions show the low level sea-breeze circulation with a deeper and weaker return flow above. The air originally at the coast has moved 35km inland in the Lagrangian solution, element 96 has, however, moved 70km at the surface leading to the formation


A
B
C
D
σ = 0
σ = 1
→ x
294.5
294.0
293.5
293.0

of a sharp front. In the finite difference solution the greatest lateral displacement is about 40km at this stage, though after 18 hours, when the friction is weaker, it reaches about 80km. The velocity field shown in C indicates strong surface convergence near the coast, consistent with the strong front formed in the Lagrangian model. The temperature contrast shown in B is weaker because of the effect of the heating.

The solution of the apparently more general explicit equations is shown in D. The sea-breeze circulation is much stronger and sets up a wave train in the vertical. In this model the air tends to overshoot its equilibrium position and oscillate about it. These oscillations are of a lsrge horizontal scale and a small vertical scale. They can be damped by increasing the vertical diffusion, but this also tends to destroy the details of the main circulation. In reality small scale turbulent entrainment helps to damp such oscillations though some wave motions are excited. The wave response cannot be treated correctly within the context of an operational forecast model and the implicit computation of the equilibrium velocity field presented here provides a way of avoiding the problem.

References

R.S.Bell and A.Dickinson. *The Meteorological Office operational numerical weather prediction system*. Met. Office Scientific Paper no. 41, HMSO(1987).

S.Chynoweth. Ph.D thesis, Department of Maths., University of Reading(1987).

M.J.P.Cullen. *Implicit finite difference methods for modelling discontinuous atmospheric flows*. U.K.Met.Office, Met O 11 Tech. Note no. 259(1987), submitted for publication.

M.J.P.Cullen, J.Norbury, R.J.Purser and G.J.Shutts. *Modelling the quasi-equilibrium dynamics of the atmosphere*. Quart. J. Roy. Meteor. Soc., 113, 735-757(1987).

M.J.P.Cullen and R.J.Purser. *An extended Lagrangian model of semigeostrophic frontogenesis*. J. Atmos. Sci., 41, 1477-1497(1984).

B.J.Hoskins. *The mathematical theory of frontogenesis*. Ann. Rev. Fluid Mech. 14 ,131-151(1987).

R.Salmon. *New equations for nearly geostrophic flow*. J. Fluid Mech., 153, 461-477(1985).