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A THERMODYNAMIC MODEL FOR THE DEVELOPMENT OF A CONVECTIVELY UNSTABLE BOUNDARY LAYER.

by

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SUMMARY

Two approaches to the simple parameterization of the development of a dry, convectively unstable boundary layer, capped by a stable layer are discussed.

In the purely thermal models, the important interfacial entrainment process is supposed controlled, to a first approximation, only by the intensity of the thermal bombardment of the interface and is characterised by postulating that the heat flux due to entrainment is directly proportional to the surface sensible eddy heat flux. Attention is drawn to the general results and limitations of this type of model.

Entrainment is essentially a dynamical process and a novel, simple model which gives the parameterization a combined dynamical and thermal basis is formulated. It is argued that the entrainment process is a function not only of the surface heat flux, as in the first approach, but also of the changes in the wind velocity and potential temperature which are observed at the immediate top of the developing convectively unstable boundary layer. Numerical integration enables the evolution to be described in terms of two constants which await observational determination. Some provisional results are presented.

1. INTRODUCTION

At the present time there is a great deal of interest being shown in the study of the non-stationary aspects of the atmospheric boundary layer. One reason for this is the desire to incorporate the boundary layer realistically into numerical forecasting and general circulation models. Another is the need to provide reliable estimates of the depth and character of the mixing layer for use in schemes dealing with the dispersion of concentrations of atmospheric pollutants where steady state theories have proved to be totally inadequate.

The stability of the mixing layer is of primary importance in short-range pollution studies; however its depth, h, becomes increasingly more important to ground level concentrations when the distance of travel is greater than about 10 h from the source. The inclusion of such non-steady features in a practical scheme for estimating the vertical dispersion of pollutants has been outlined by Smith (1972). Also, the ability to specify the diurnal cycle of the boundary layer's evolution becomes important when dealing with pollutants tracked for several days on a regional scale and such effects are discussed elsewhere in this Symposium (Pasquill, 1973).

Although intricate parameterizations and numerical models are being developed for the study of evolving boundary layers (Deardorff, 1972a, 1972b, 1973) there remains a need to provide relatively simple parameterizations for general use and a start would be to consider the development of the important dry, convectively unstable layer capped by a stable layer.

Observations of the boundary layer in convective situations show that in general it is a diurnally evolving system with discontinuities around sunrise and sunset. The discontinuities arise because we distinguish between the relatively shallow nocturnal inversion layer in which buoyancy and viscous effects suppress any mechanically generated turbulent motions and the more rapidly evolving daytime convectively unstable layer which is generally capped by a non-turbulent stable layer. Two of the main factors controlling the development of the convective layer are the flux of sensible eddy heat entering the boundary layer at the ground and also a turbulent mixing process which occurs at the interface between the well-mixed boundary layer air and the non-turbulent air in the capping stable layer. The process whereby stable air from above is mixed into the developing convectively unstable boundary layer is called entrainment.

The part played by the dynamics in the entrainment process has received little attention and the simplest parameterizations treat it from purely thermal considerations. However, recent observations (Readings et al, 1973) indicate that wind shear at the interface may be of fundamental importance not only to the entrainment of eddy momentum but also of eddy sensible (and latent) heat.

This contribution sets out,

(i) to draw attention to the general results and limitations of the simple thermal approach, and

(ii) to provide a combined, if grossly simplified, dynamical and thermal approach to the parameterization of the entrainment process and the development of the convectively unstable boundary layer.

2. SIMPLE THERMAL MODELS

The history of simple thermal models for parameterizing the development of the convectively unstable boundary layer capped by a stable layer can be traced through the papers of Ball (1960), Lilly (1968), Deardorff et al (1969), Tennekes (1972) and Carson (1973). We summarize here the method and results of the model discussed in depth by Carson (1973) and independently proposed by Betts (1973).

The potential temperature profile, as illustrated in Fig. 2, is defined by

$$\theta(z,t) = \begin{cases} \theta_c(t) & z<h \\ \theta_s(z,t) = \theta_o + \gamma(t).z & z>h \end{cases} \qquad (1)$$

where h is nominally the depth of the convectively unstable boundary layer, θ_o is the effective surface temperature obtained by extrapolating the stable lapse rate down to $z = 0$, and $\gamma(t)$ is the vertical gradient of θ in the capping stable layer.

Advection, radiation and evaporation are not considered here although in certain circumstances each or all of these processes can be important. In this case the simple heat balance equation is

$$\frac{\partial H}{\partial z} = -\rho c_p \frac{d\theta}{dt} = -\rho c_p \left[\frac{\partial \theta}{\partial t} + w(z)\frac{\partial \theta}{\partial z}\right] \quad , (2)$$

such that $\quad H(z, t) = 0 \qquad z > h \quad , \qquad (3)$

where $H(z, t)$ is the sensible eddy heat flux, $w(z)$ is the mean, synoptically-induced vertical velocity, ρ is the mean air density and c_p is the specific heat of air at constant pressure.

Certain features of the simple model, such as the linearity of $w(z)$

and $H(z,t)$ with height and the exponential increase of $\gamma(t)$ with time are derived from Eqs. (1)-(3), as indeed is the expression for the entrained sensible eddy heat flux

$$H(h,t) = -\rho c_p w_e(t) . \Delta\theta(t), \tag{4}$$

where

$$w_e(t) = \frac{dh}{dt} - w(h) \tag{5}$$

is the entrainment rate and

$$\Delta\theta(t) = \theta_s(h,t) - \theta_c(t) \tag{6}$$

is the step discontinuity in θ across the interface at $z = h$.

The system of equations is closed by parameterizing the entrainment process at $z = h$. In this simple model it is postulated that the degree of entrainment is controlled, to a first approximation, by the intensity of the thermal bombardment of the interface which, in turn, is directly proportional to the magnitude of the surface sensible heat flux. Hence the closure equation is

$$H(h,t) = -A . H(0,t) \qquad 0 \leqslant A \leqslant 1 . \tag{7}$$

Straightforward analysis produces an ordinary differential equation for the development of the convectively unstable layer,

$$h \frac{d(\gamma h)}{dt} = \frac{H(0,t) - 2H(h,t)}{\rho c_p} \tag{8}$$

which although not explicitly dependent on A, does depend on A being constant. Strictly, in keeping with the assumptions, Eq. (8) should be written

$$\frac{dh^2}{dt} + 2\tilde{\beta}h^2 = \frac{2(1+2A)\,H(0,t)}{\rho c_p \gamma(t)}, \tag{9}$$

where $$\tilde{\beta} = -\frac{w(z)}{z} = \text{constant}, \tag{10}$$

and $$\gamma(t) = \gamma(0)\, exp(\tilde{\beta} t). \tag{11}$$

Integration of Eq. (9), with $h(0) = 0$, gives

$$h^2(t) = \frac{2(1+2A)\, exp(-2\tilde{\beta}t)}{\rho c_p \gamma(0)} \int_0^t exp(\tilde{\beta}\tau)\, H(0,\tau)\, d\tau, \tag{12}$$

and the corresponding evolutionary expressions for $w_e(t)$, $\Delta\theta(t)$ and $\theta_c(t)$ are,

$$w_e(t) = \frac{(1+2A)\, H(0,t)}{\rho c_p \gamma(t)\, h(t)}, \tag{13}$$

$$\Delta\theta(t) = \frac{A\, \gamma(t)\, h(t)}{1+2A}, \tag{14}$$

and $$\theta_c(t) = \theta_0 + \left(\frac{1+A}{1+2A}\right) \gamma(t)\, h(t). \tag{15}$$

The value of A which characterises the degree of interfacial mixing realised in the atmosphere during the typical development of a convectively unstable boundary layer remains to be chosen.

The extreme value $A = 1$ derives from Ball (1960) who, in his consideration of the integrated local turbulent kinetic energy balance equation, assumed that the contribution from molecular dissipation could be neglected. The other extreme, $A = 0$, describes the situation where the boundary layer is growing without entraining heat across the interface, i.e. $\Delta\theta$ in Eq. (4) is zero whereas $w_e(t)$ remains finite. In such circumstances the interface is a passive one with no mixing across it and we shall use the term <u>encroachment</u> of the stable layer by the unstable layer to describe this process. Such a state is strictly never realised in the atmosphere but is closely approached in the laboratory studies of penetrative convection by Deardorff et al. (1969) and when weak thermal activity is eroding a strong inversion, such as a nocturnally established inversion (Carson, 1973).

Available evidence favours small values of A; 0.2 is suggested by Deardorff (private communication) and Tennekes (1972), and Betts (1973) quotes evidence for 0.25. It seems unlikely that A remains constant throughout the various phases of the boundary layer's evolution and Carson (1973) from his analysis of the O'Neill 1953-data has suggested that A varies quite significantly during the day, being very small soon after dawn and reaching a maximum value, as high as 0.5, during a period of a few hours following the time of maximum surface heating.

The uncertainty about A and its likely time dependence may limit the range of applicability of the simple thermal model. Further, entrainment is essentially a dynamical process and therefore it seems inadequate to propose a model which omits the dynamics of the interfacial region. We seek then a simple model which will give the parameterization a combined dynamical and thermal basis and at the same time avoid the restriction that $H(h,t)/H(0,t)$ be constant.

3. A SIMPLE THERMODYNAMIC MODEL

3.1 The Entrainment Process

Observations such as those of Readings et al. (1973) and Browning et al. (1973) show that in general there is not only a temperature change across the convoluted interface between the deepening convectively unstable boundary layer and the capping stable layer but also a finite shear in the wind velocity. The general situation in the vicinity of the interface is illustrated schematically in Fig. 1 and we envisage several mechanisms contributing to the mixing process whereby stable air is entrained into the developing boundary layer across such an interface.

The change in temperature across the interface serves to create a narrow layer or zone which with the accompanying change in wind speed is also a zone of marked vorticity. Bombardment of the interface by thermals causes three-dimensional domes to protrude into the stable layer thereby stretching the vortex sheet and further enhancing the local vorticity. The net effect is a torque which causes a wave-like overturning of the convective dome which enables a tongue of relatively warm air to undercut the dome's colder boundary layer air. At the same time, small-scale interfacial Kelvin-Helmholtz instabilites grow on the crest of the dome, where gradients have been intensified (Readings et al., 1973). These may then be advected into the tongue by the wind shear and there play an important role in enhancing the mixing of the tongue into the convectively unstable boundary layer.

It is therefore postulated that the entrainment process and hence A of Eq.(7) are governed not only by $H(0,t)$, which partly determines the strength of the thermals, but also by $\Delta\theta(t)$ and the magnitude of the wind shear, $\Delta V(t)$, which relate to the dynamical stability of the interface. On dimensional grounds, then, the simplest formulation for A is,

$$\frac{H(h,t)}{H(0,t)} \equiv -A = -k\left[\frac{\rho c_p \Delta V . \Delta\theta}{H(0,t)}\right]^{a}, \qquad (16)$$

where a, k are two 'constants' which must be determined from observations.

3.2 Dynamical Considerations

It is necessary in our dynamical formulation to include the parameters needed to determine the degree of entrainment as expressed in Eq. (16) and, as a first attempt, we construct a simple model based on the idealised wind profiles of Fig. 2 . A right-handed system of axes is chosen such that the x-axis is directed along the geostrophic wind $\underset{\sim}{V}_g$. For $z < h$, the mean wind components are assumed to be virtually constant with height and, at $z = h$, we include a step discontinuity $\Delta\underset{\sim}{V}$ in the wind velocity which defines the angles α and β , α being the turning of the wind in the boundary layer from the geostrophic direction. For $z > h$ the mean wind is assumed to be $\underset{\sim}{V}_g$.

Assuming quasi-stationarity and horizontal homogeneity for all the relevant variables, including the pressure gradient, the boundary layer momentum equations can be written

$$\left.\begin{aligned} \frac{\partial \tau_x}{\partial z} &= -f\rho V \sin\alpha = \text{constant} \\ \frac{\partial \tau_y}{\partial z} &= -f\rho (V_g - V\cos\alpha) = \text{constant} \end{aligned}\right\}, \qquad (17)$$

implying

$$\left.\begin{aligned} \tau_x(z) &= \tau_x(0) - f\rho V \sin\alpha .\, z \\ \tau_y(z) &= \tau_y(0) - f\rho (V_g - V\cos\alpha)\, z \end{aligned}\right\} \qquad (18)$$

and, in particular,

$$\left.\begin{aligned} \tau_x(0) - \tau_x(h) &= f\rho V \sin\alpha .\, h \\ \tau_y(0) - \tau_y(h) &= f\rho (V_g - V\cos\alpha)\, h \end{aligned}\right\} \qquad (19)$$

where $\underset{\sim}{\tau} = (\tau_x, \tau_y)$ is the shearing stress, $V = |\underset{\sim}{V}|$, $V_g = |\underset{\sim}{V}_g|$ and f is the coriolis parameter.

Entrainment of stable air through the interface not only implies a transfer of heat but also momentum and Deardorff (1973) has recently used this concept to explain the large values of eddy momentum flux in the upper regions of a developing convectively unstable boundary layer, obtained from measurements analysed by Angell (1972).

In a form analogous to that for the heat flux due to entrainment, Eq. (4), the shearing stress components at z = h are expressed as

$$\left.\begin{aligned} \tau_x(h) &= \rho \frac{dh}{dt} . \Delta u = \rho \frac{dh}{dt} . \Delta V \cos\beta \\ \text{and} \quad \tau_y(h) &= \rho \frac{dh}{dt} . \Delta v = -\rho \frac{dh}{dt} . \Delta V \sin\beta \end{aligned}\right\} \tag{20}$$

where $\underset{\sim}{\Delta V} = (\Delta u, \Delta v)$, $\Delta V = |\underset{\sim}{\Delta V}|$, $\underset{\sim}{\tau}(z) = 0$ for $z > h$ and the mean, synoptically-induced, vertical velocity is assumed to be zero.

In the surface layer the shearing stress is specified by means of a drag coefficient C_D in the usual way,

$$\left.\begin{aligned} \tau_x(0) &= \rho C_D V^2 \cos\alpha \\ \tau_y(0) &= \rho C_D V^2 \sin\alpha \end{aligned}\right\} . \tag{21}$$

Finally, from Fig. 2, we have the kinematical relationships,

$$V \sin\alpha = \Delta V \sin\beta \tag{22}$$

$$V \cos\alpha = V_g - \Delta V . \cos\beta \tag{23}$$

and $$V^2 = V_g^2 + (\Delta V)^2 - 2V_g . \Delta V \cos\beta . \tag{24}$$

3.3 Thermal Considerations

In keeping with the simple thermal model of Section 2, the adopted profile of potential temperature in the thermodynamic model is that illustrated in Fig. 2 and defined by Eq. (1) (only in this analysis γ is assumed constant with time). This profile and the heat balance equation in the absence of radiational effects, Eqs. (2) and (3), give us three further equations for the thermodynamic model. These are :

Heat input by entrainment :

$$H(h,t) = -\rho c_p . \frac{dh}{dt} . \Delta\theta . \tag{25}$$

Heat balance for the whole of the mixing layer :

$$\rho c_p h . \frac{d\theta_c}{dt} = H(0,t) - H(h,t) . \tag{26}$$

The magnitude of the temperature step across the interface :

$$\Delta\theta = \theta_0 + \gamma h - \theta_c . \tag{27}$$

A knowledge of the constant parameters $a, k, C_D, f, \rho c_p$ and the variables $H(0,t)$, γ, V_g and initial values of h, $\Delta\theta$ enable us to use the system of Eqs. (16) - (27) to determine the evolutionary profiles of $h, \Delta\theta, \theta_c, V, \Delta V, \alpha, \beta, \tau_x, \tau_y$ and $H(h)$.

4. PROVISIONAL RESULTS AND CONCLUSIONS

The two constants a and k await observational determination and so at this stage it is only possible to indicate the nature of the results which can be obtained with the simple thermodynamic model. The full details of the procedure for determining the evolutions of $h, \Delta\theta, \theta_c, V, \Delta V, \alpha, \beta, \tau_x, \tau_y$ and $H(h)$ are given in the APPENDIX.

The results of a test case with $a = \frac{1}{2}$ and $k = 0.1$ are presented in Figs. 3 and 4. Parameters kept constant throughout the integration are ;

$$C_D = 4 \times 10^{-3}, \quad \rho c_p = 10^3 \text{ joule m}^{-3} \, {}^{o}K^{-1}, \quad f = 10^{-4} \text{ sec}^{-1},$$

$$V_g = 10 \text{ m sec}^{-1} \quad \text{and} \quad \gamma = 6 \times 10^{-3} \, {}^{o}K \text{ m}^{-1}.$$

We adopted a sinusoidal surface heat flux defined by $H(0,t) = \hat{H} \sin(\Omega t)$, where Ω is the Earth's angular rotation and $\hat{H} = 300$ watt m^{-2} is the maximum value. The integration was started at $t = 0.4$ hr and the initial values of h and $\Delta\theta$ were obtained from Eqs. (12) and (14) of the thermal model with $\tilde{\beta} = 0$ and $A = 0.2$. The time step for the integration was 0.2 hr.

Fig. 3 (a) shows the development of the depth, h, of the convectively unstable boundary layer. Also shown is the corresponding development derived from the purely thermal model with $\tilde{\beta} = 0$, $A = 0.2$, all other parameters being as stipulated for the thermodynamic model. It is not possible to critically compare the two curves and the thermal model is given for information only. Fig. 3 (b) shows the variations with time of $\Delta V / V_g$, $\tau(0)/\rho$ and $\tau(h)/\rho$ where $\tau = |\underset{\sim}{\tau}|$ and again the trends do not appear to be contrary to expectation. Fig. 4 (a) shows the specified surface sensible heat flux $H(0,t)$ and the derived entrained heat flux $H(h,t)$.

Also given is $H(h,t) = 0.2\ H(0,t)$ from the thermal model and here we note the differences in shape of the two curves for $H(h,t)$. In particular $H(h,t)$ from the thermodynamic model is not distributed symmetrically about $t = 6$ hr as is the $H(h,t)$ from the simple thermal model i.e. A in the thermodynamic model is a function of time.

Figs. 4 (b) and 4 (c) show the evolutions of $\Delta\theta$ and $[\theta_c(t)-\theta_o]$ for both models and again they exhibit similar trends.

The nature of these provisional results for the trial values of a and k provide encouragement for following up these ideas. In particular we await further observational data such that we can determine values of a and k directly and compare the computed evolutions with the actuals.

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APPENDIX

DETAILS OF THE NUMERICAL INTEGRATION PROCEDURE

Substitution of Eqs. (20) - (23) into Eq. (19) gives

$$C_D V \left(V_g - \Delta V . \cos\beta \right) - \frac{dh}{dt} . \Delta V . \cos\beta = fh . \Delta V \sin\beta , \tag{A1}$$

and $$C_D V . \Delta V \sin\beta + \frac{dh}{dt} . \Delta V . \sin\beta = fh . \Delta V \cos\beta . \tag{A2}$$

If we now introduce the non-dimensional variables,

$$\nu = \frac{V}{V_g} , \tag{A3}$$

$$\epsilon = \frac{\Delta V}{V_g} , \tag{A4}$$

$$\eta = \frac{fh}{C_D V_g} , \tag{A5}$$

and $$\lambda = \frac{(dh/dt)}{C_D V_g} , \tag{A6}$$

then Eq. (A2) implies

$$\tan\beta = \frac{\eta}{\nu + \lambda} , \tag{A7}$$

which with Eq. (A1) gives

$$\nu^2 = \epsilon^2 \left[\eta^2 + (\nu + \lambda)^2 \right] . \tag{A8}$$

Further, Eqs. (24), (A7) and (A8) give

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$$\epsilon = \left[\frac{\nu(1-\nu^2)}{\nu + 2\lambda} \right]^{1/2} , \qquad (A9)$$

which when substituted in Eq. (A8) gives

$$\eta = \left[\frac{\nu^2(\nu+\lambda)^2 - \lambda^2}{1-\nu^2} \right]^{1/2} , \quad \text{valid for } \lambda \leqslant \frac{\nu^2}{1-\nu} , \qquad (A10)$$

i.e. $$\lambda = \frac{\nu^3 \pm \left[\nu^4 - \eta^2(1-\nu^2)^2\right]^{1/2}}{1-\nu^2} , \quad \text{valid for } \eta \leqslant \frac{\nu^2}{1-\nu^2} . \qquad (A11)$$

Equation (22) provides α from

$$\sin\alpha = \frac{\epsilon}{\nu} \cdot \sin\beta , \qquad (A12)$$

and Eqs. (16) and (25) give

$$\frac{k}{C_D} \left[\frac{H(0,t)}{\rho c_p \, \Delta\theta . V_g} \right]^{1-a} = \lambda \epsilon^{-a} , \qquad (A13)$$

which couples the dynamical and thermal properties of the system.

The dynamical equations (A9) - (A11) relate the variables $\nu, \epsilon, \eta, \lambda$ to each other as shown graphically for $a = \frac{1}{2}$ in Fig. 5 and, finally, the remaining thermal equations are from Eqs. (25) - (27),

$$\rho c_p h . \frac{d\theta_c}{dt} = H(0,t) + \rho c_p . \frac{dh}{dt} . \Delta\theta \qquad (A14)$$

and $$\Delta\theta = \theta_0 + \gamma h - \theta_c . \qquad (A15)$$

The steps in the integration procedure are :

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(i) Constant Parameters

Values are assumed for a, k, C_D, f and ρc_p which remain constant throughout the development.

(ii) Known Time Variations

The values of $H(0,t)$, γ and V_g are assumed known at all times. Strictly speaking, the analysis of Section 3 has assumed values of γ and V_g which remain constant throughout the integration.

(iii) The Entrainment Equation

At $t = t_o$, the start of the integration, $h(t_o)$ and $\Delta\theta(t_o)$ are given as initial conditions and at any other time $t_i > t_o$, $h(t_i)$ and $\Delta\theta(t_i)$ will have been estimated from the system of equations integrated over the previous time step from t_{i-1}. All the varaiables and parameters needed to evaluate $\eta(t_i)$ and the L.H.S. of Eq. (A13) are therefore known at any time t_i. The value of the L.H.S. of Eq. (A13) is the value of $[\lambda\epsilon^{-a}]_i$.

(iv) The Dynamical Equations

Knowing $\eta(t_i)$ and $[\lambda\epsilon^{-a}]_i$, Eqs. (A7) - (A12) are solved (possibly with the help of a graph similar to Fig. 5 corresponding to the chosen value of a) to give ν_i, λ_i, ϵ_i, α_i, β_i and integration over some small time step δt gives,

$$h(t_{i+1}) \doteqdot h(t_i) + [\delta h]_i \tag{A16}$$

where

$$[\delta h]_i = \lambda_i C_D V_g . \delta t . \tag{A17}$$

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(v) The Thermal Equations

Equation (A14) gives

$$\left[\delta\theta_c\right]_i = \left[\frac{H(0,t_i)}{\rho c_p} + \lambda_i C_D V_g \cdot \Delta\theta(t_i)\right] \frac{\delta t}{h(t_i)} , \qquad \text{(A18)}$$

and Eq. (A15) can be expressed as

$$\Delta\theta(t_{i+1}) \doteq \Delta\theta(t_i) + \gamma(t_i)\left[\delta h\right]_i - \left[\delta\theta_c\right]_i \, . \qquad \text{(A19)}$$

(vi) With the new values $h(t_{i+1})$ and $\Delta\theta\,(t_{i+1})$ we return to (i) and repeat the process for the next step in the integration procedure.

This integration procedure provides us with all the parameters and variables needed to determine the evolutions of, for example, h , $\Delta\theta$, θ_c , V, ΔV, α, β , τ_x , τ_y and $H(h)$.

FIGURE LEGENDS

Fig. 1. Schematic representation of the dynamical and thermal effects which control the entrainment process at the interface between a deepening convectively unstable boundary layer and a capping non-turbulent stable layer.

Fig. 2. Schematic representation of the idealised profiles of potential temperature, θ , and the components u, v of the horizontal wind velocity, $\underline{V}$, used in the simple thermodynamic model. Also illustrated is the nature of the wind velocity shear, $\Delta \underset{\sim}{V}$, across the interface at $z = h$.

Fig. 3(a). The full line shows the development of the depth, h , of the convectively unstable boundary layer obtained from the trial integration using the thermodynamic model. The broken line shows the corresponding development obtained from the simple thermal model with $A = 0.2$, $\tilde{\beta} = 0$, $\gamma(0) = 6 \times 10^{-3}$ °K m^{-1} and a sinusoidal $H(0,t)$ as specified in Fig. 4 (a).

3(b). The evolutions of $\Delta V/V_g$, $\tau(0)/\rho$ and $\tau(h)/\rho$ obtained from the trial integration using the thermodynamic model.

Fig. 4(a). The sinusoidal surface heat flux $H(0,t)$ adopted for both the thermodynamic and thermal models. $H(h,t)$ (full line) is the entrained heat flux derived from the thermodynamic model and $H(h,t) = 0.2\ H(0,t)$ (broken line) is the entrained heat flux in the simple thermal model with $A = 0.2$.

4(b). $\Delta\theta$, the step in the potential temperature across the interface at $z = h$, for the trial thermodynamic model (full line) and the simple thermal model with $A = 0.2$ (broken line).

4(c). The warming of the convectively unstable boundary layer throughout the day expressed as $[\theta_c(t) - \theta_0]$, for the trial thermodynamic model (full line) and the simple thermal model with $A = 0.2$ (broken line).

Fig. 5. Contours of the dynamical parameters $\lambda = (dh/dt)/C_D V_g$ (full lines) and $\nu = V/V_g$ (broken lines) as functions of $\eta = fh/C_D V_g$ and $\frac{k}{C_D}\left[\frac{H(0,t)}{\rho c_p \Delta\theta . V_g}\right] = \lambda \epsilon^{-1/2}$, as determined by the thermodynamic model with $a = \frac{1}{2}$.

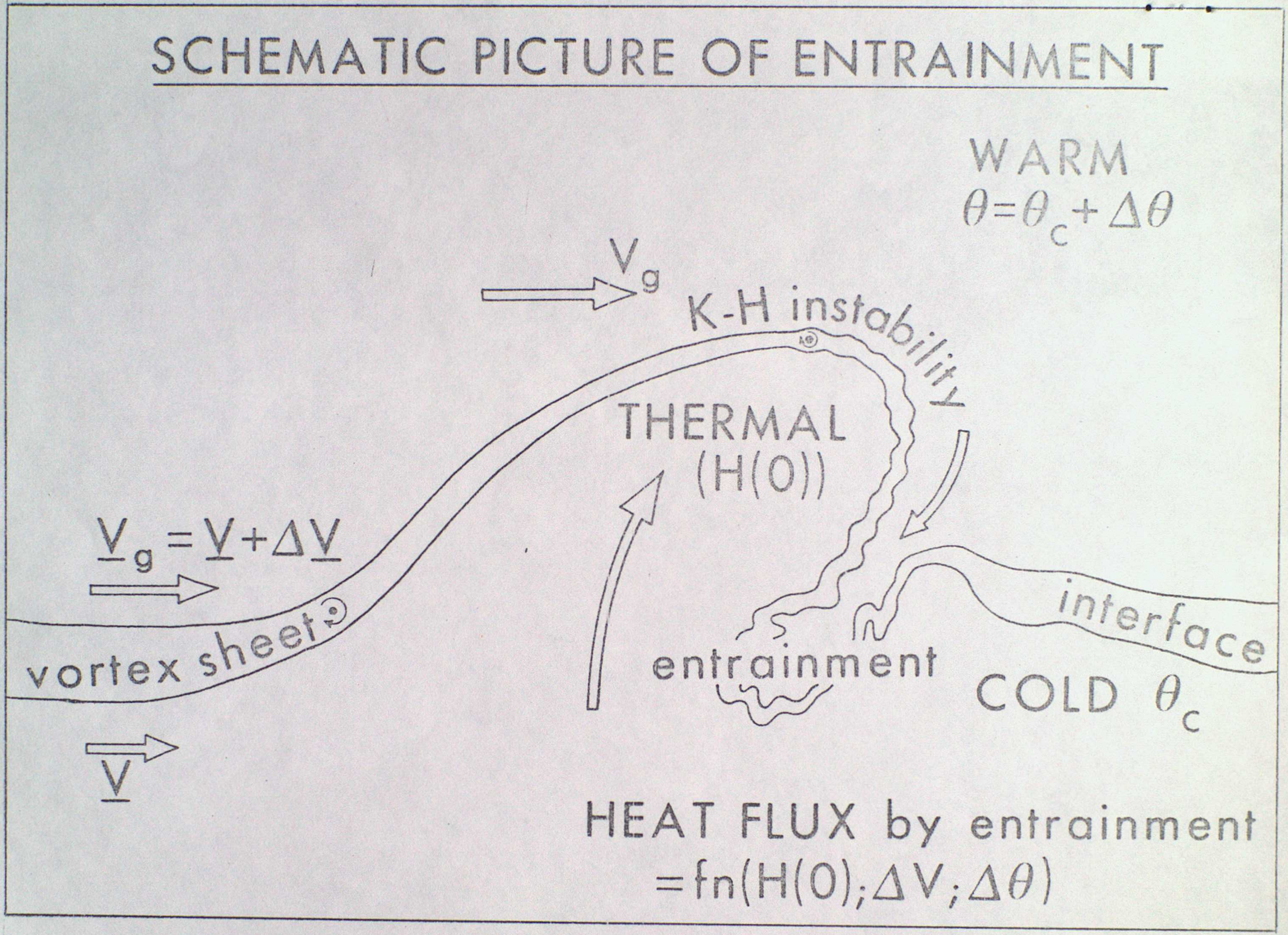

SCHEMATIC PICTURE OF ENTRAINMENT
WARM
$\theta=\theta_c+\Delta\theta$
V_g
K-H instability
THERMAL
(H(0))
$\underline{V}_g=\underline{V}+\Delta\underline{V}$
vortex sheet
entrainment
interface
COLD θ_c
$\underline{V}$
HEAT FLUX by entrainment
$=fn(H(0);\Delta V;\Delta\theta)$

Fig.1

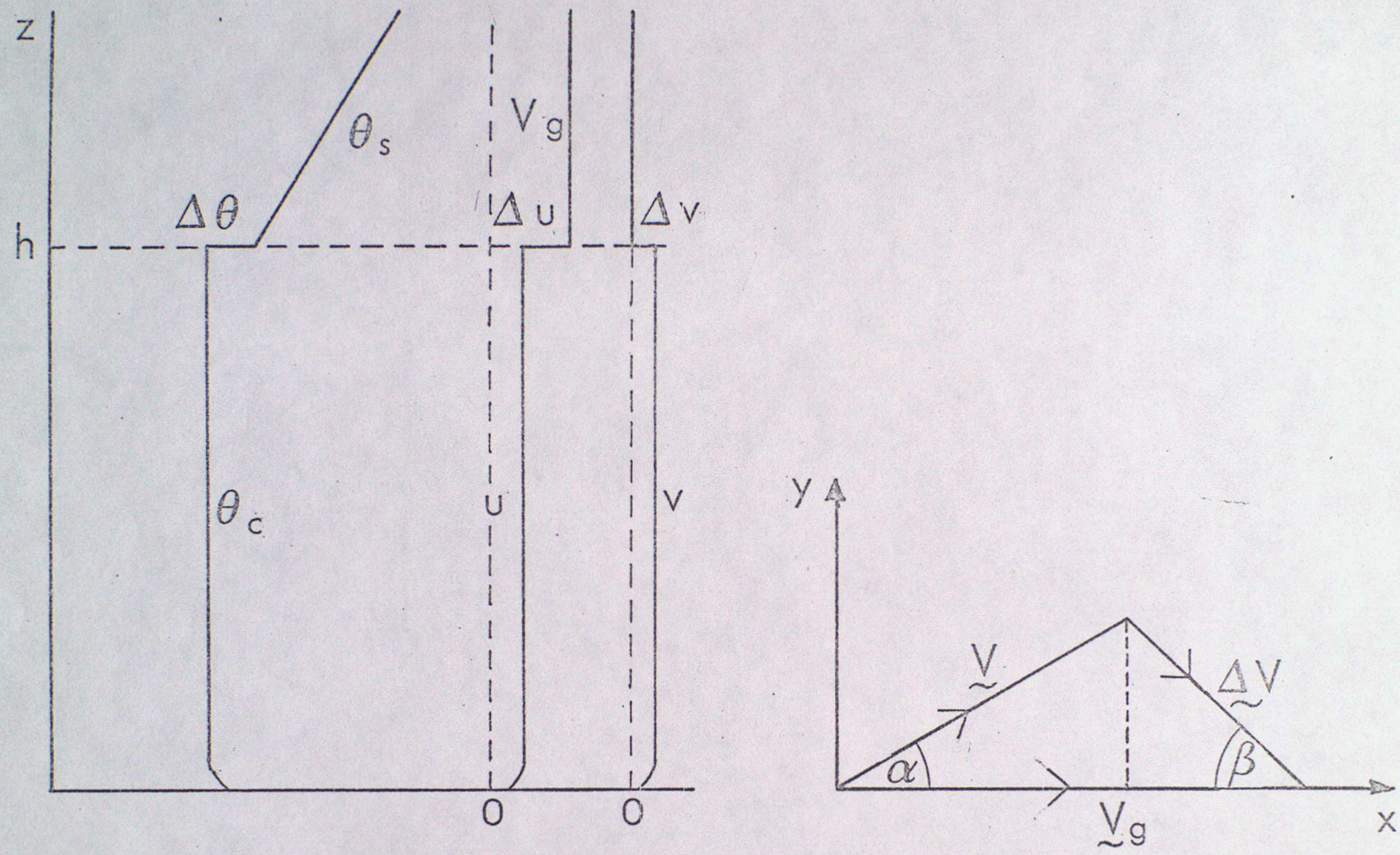
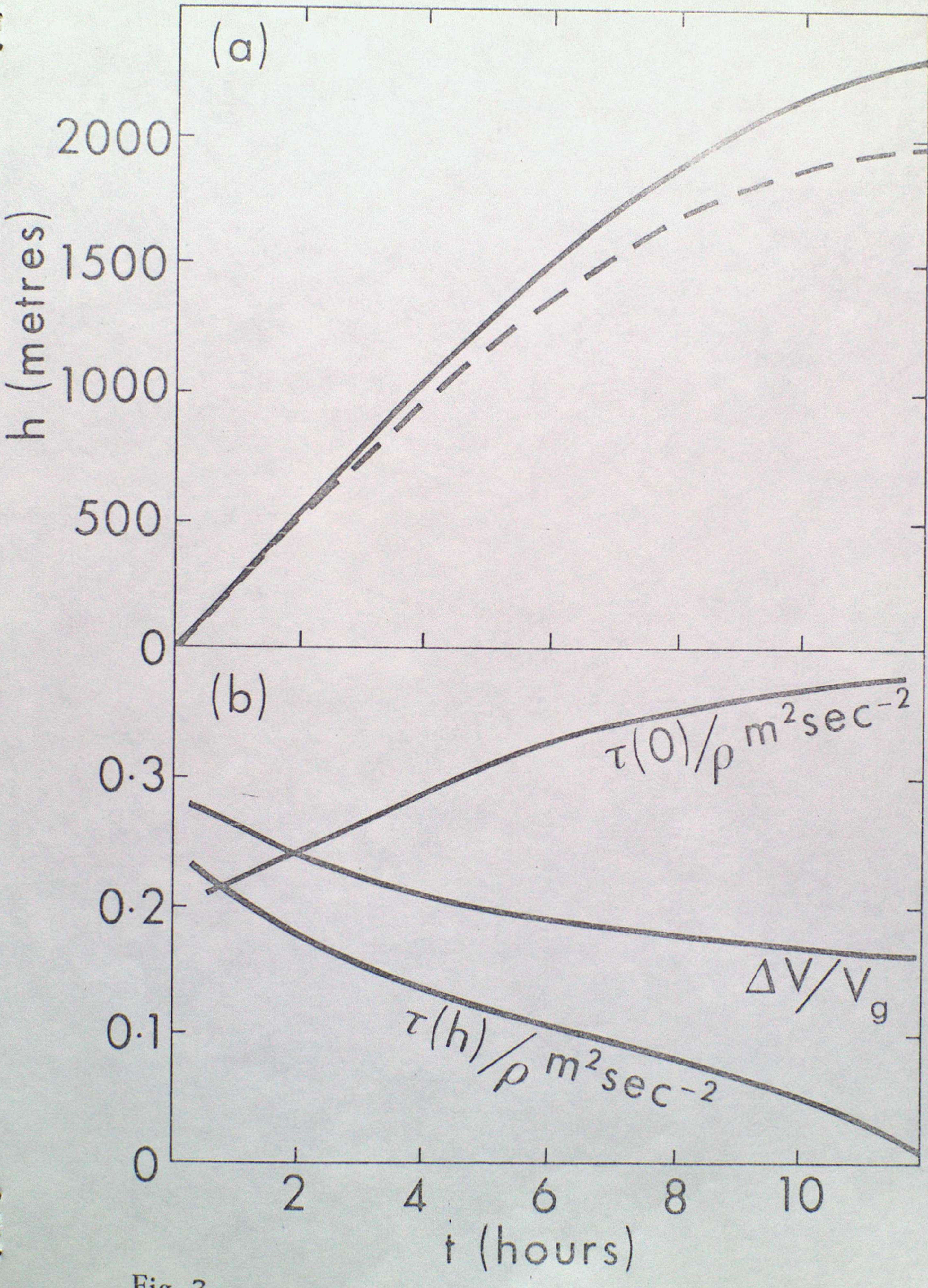

z
h
$\Delta\theta$
θ_s
θ_c
V_g
Δu
u
Δv
v
0
0
y
x
$\underset{\sim}{V}$
$\Delta\underset{\sim}{V}$
$\underset{\sim}{V}_g$
α
β

Fig. 2



(a)
h (metres)
2000
1500
1000
500
0
(b)
0·3
0·2
0·1
0
$\tau(0)/\rho$ m²sec⁻²
$\Delta V/V_g$
$\tau(h)/\rho$ m²sec⁻²
2
4
6
8
10
t (hours)

Fig. 3

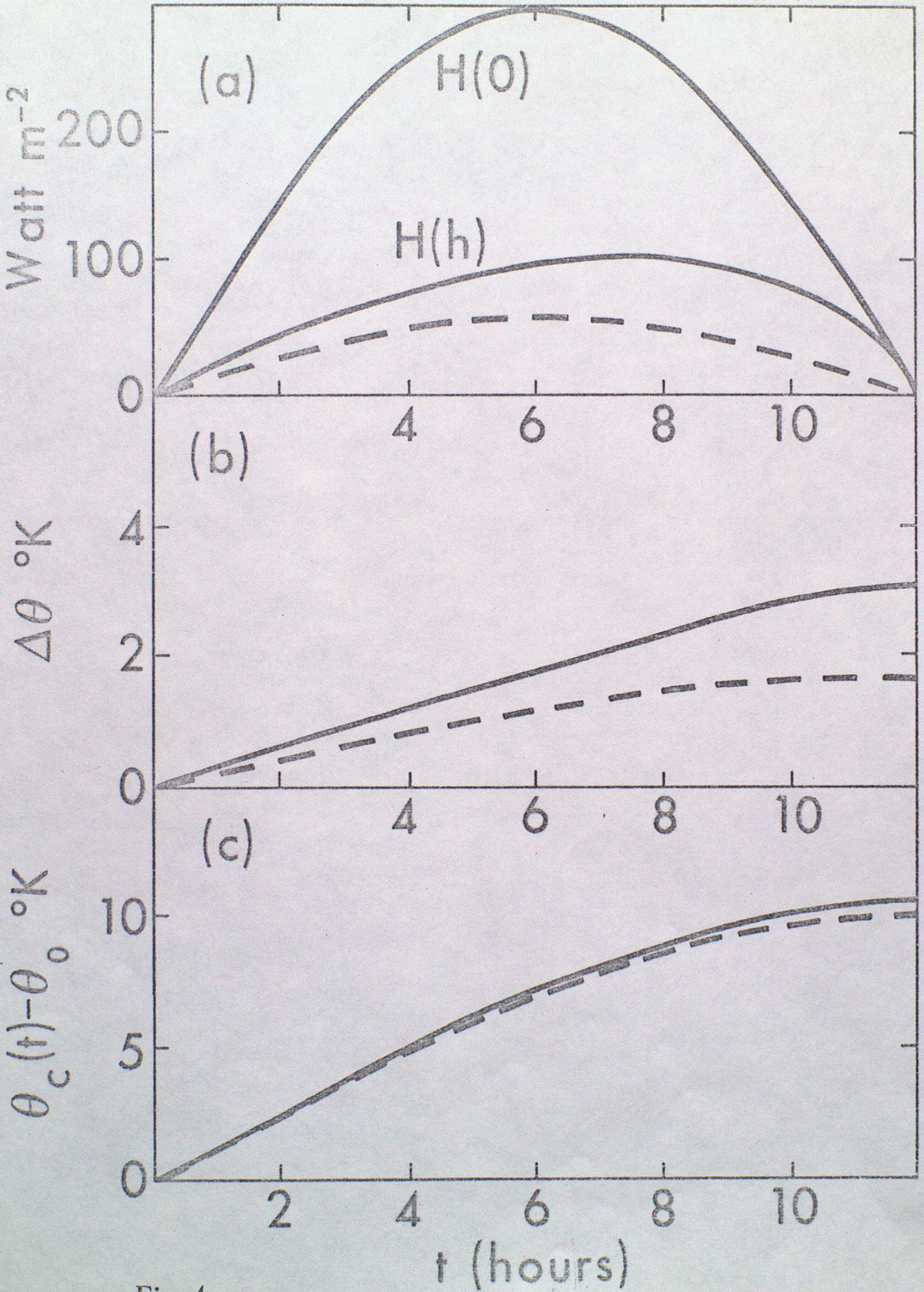
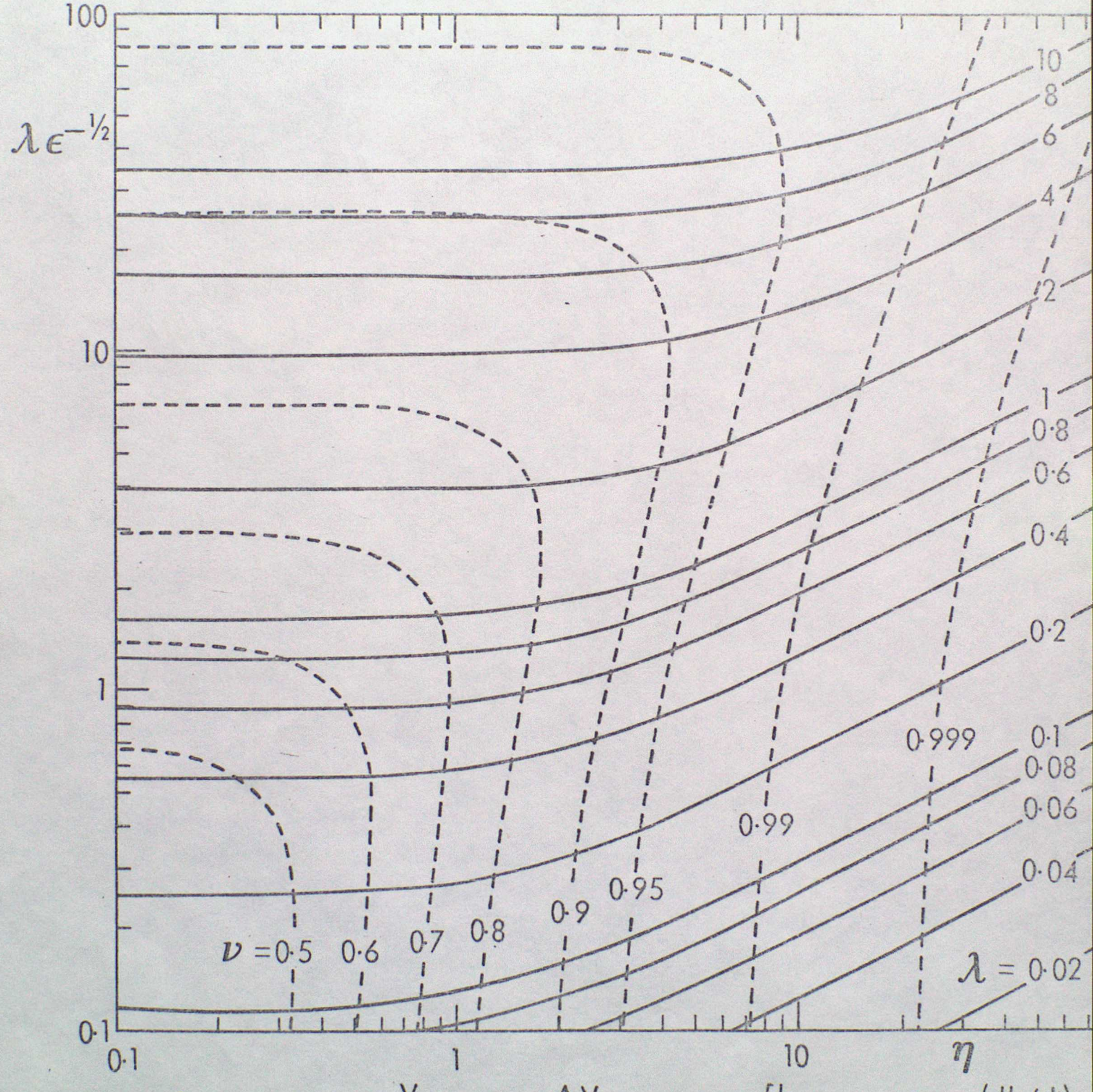

(a)
H(0)
H(h)
Watt m^{-2}
200
100
0
4
6
8
10
(b)
$\Delta\theta$ °K
4
2
0
4
6
8
10
(c)
$\theta_c(t)-\theta_0$ °K
10
5
0
2
4
6
8
10
t (hours)

Fig. 4



where $\nu = \dfrac{V}{V_g}$ $\epsilon = \dfrac{\Delta V}{V_g}$ $\eta = \dfrac{fh}{C_D V_g}$ $\lambda = \dfrac{(dh/dt)}{C_D V_g}$

Fig. 5