

A Matrix Rank-finding Technique

R. DIXON

1. Introduction

Problems and procedures in numerical Meteorology can often be put into matrix-vector form. Once cast in this style the problem can be treated by the powerful techniques of matrix formalism and may often be penetrated to an extent which is difficult by other means. In many cases the validity or effectiveness of a computational sequence depends on the rank of a matrix. This note comments on the question of rank as it relates to computation and also presents a rank-finding algorithm by N.N. Gupta which deserves to be better known.

Met.O.11 (Forecasting Research)
Meteorological Office,
London Road
Bracknell, Berks. U.K.

N.B. This paper has not been published. Permission to quote from it must be obtained from the Assistant Director of the above Meteorological Office Branch.

2. The Problem

A matrix A, rectangular or square, is of rank r if all minors of order r + 1 are zero while at least one minor of order r is not zero. A minor of order r is obtained from the parent matrix A by striking out quantum sufficit rows and columns to leave an r-square matrix. The determinant of this r-square matrix is the minor.

This means that the rank of a matrix depends directly upon the values of the determinants of the various sub-matrices which can be formed by the deletion of rows and columns. But the value of a determinant is notoriously liable to be very sensitive to the actual values of the component elements of the determinant. For example

$$\begin{vmatrix} -73 & 78 & 24 \\ 92 & 66 & 25 \\ -80 & 37 & 10 \end{vmatrix} = +1 \quad (1)$$

but

$$\begin{vmatrix} -73 & 78 & 24 \\ 92 & 66 & 25 \\ -80 & 37 & 10.01 \end{vmatrix} = -118.94 \quad (2)$$

two orders of magnitude larger and of opposite sign. Furthermore, inspection of (1) and (2) suggests that somewhere between 10 and 10.01 there is a number which would make this determinant zero. There is, but the best I can do with an HP 25 is to say that it lies somewhere between 10.00008337 and 10.00008338.

Another illustration is afforded by considering say the determinant

$$\begin{vmatrix} -73.1234 & -73.1243 & -24.0000 \\ 92.1234 & 92.1243 & 25.0000 \\ -80.1234 & -80.1243 & 100.0000 \end{vmatrix} = 2.1267 \quad (3)$$

As it stands any matrix having the matrix of (3) as a sub-matrix would be of at least rank 3. However, it is seen that if the numbers in (3) could be entered only to two decimal places the determinant would be zero, since the first two columns would be identical. Thus insofar as it depended upon this particular sub-matrix the parent matrix would not be of rank 3.

So, if a matrix is being discussed mathematically the statement that it is of rank r is precise, but if it is to be used in a computer the statement should rather be that it is of rank r with wordlength m.

3. The Rank-finding Method

For any square matrix A the similarity transformation

$$A = R \cdot B \cdot R^{-1} \quad (4)$$

may exist. If A is symmetric it does exist, and furthermore it is possible to find an orthogonal matrix T, remembering that in this case $T^{-1} = \tilde{T}$, such that

$$A = T \cdot D \cdot \tilde{T} \quad (5)$$

where D is the diagonal matrix of eigenvalues of A. From (5), because

$$\tilde{T} \cdot T = I \quad (6)$$

it follows that

$$\begin{aligned} A^2 &= T \cdot D^2 \cdot \tilde{T} \\ A^3 &= T \cdot D^3 \cdot \tilde{T} \end{aligned} \quad (7)$$

$$\begin{aligned} &----- \\ &----- \end{aligned}$$

etc.

Also needed is the fact that

$$\text{Tr} A = \text{Tr} (T \cdot D \cdot \tilde{T}) = \text{Tr} D \quad (8)$$

where $\text{Tr} A$ means the trace of A, the sum of the diagonal elements of A.

Even if A is not square and symmetric, $\tilde{A} \cdot A$ or $A \cdot \tilde{A}$ will be. So consider the case where A is a $p \times n$ matrix, ($p > n$) of rank r. The maximum rank which a rectangular matrix like this can have is $r = n$, the dimension of the shortest side, in which case it is said to be of full rank, but for the purpose of this discussion we will assume that A is not of full rank so that $r < n$. We now have

$$\begin{matrix} \tilde{A} & \cdot & A & = & T & \cdot & D & \cdot & \tilde{T} \\ np & & pn & & nn & & nn & & nn \end{matrix} \quad (9)$$

where

$$D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r, \dots, \lambda_n) \quad (10)$$

the λ_i being the eigenvalues of $\tilde{A} \cdot A$. But, of course

$$\lambda_{r+1} = \lambda_{r+2} = \dots = \lambda_n = 0 \quad (11)$$

Consider now the trace of the matrix expression $(I - \alpha \tilde{A} \cdot A)^2$. We have

$$\text{Tr} (I - \alpha \tilde{A} \cdot A)^2 = \text{Tr} [(I - \alpha \tilde{A} \cdot A) \cdot (I - \alpha \tilde{A} \cdot A)] \quad (12)$$

which by (9) is

$$\text{Tr} (I - \alpha \tilde{A} \cdot A)^2 = \text{Tr} [(I - \alpha T \cdot D \cdot \tilde{T}) \cdot (I - \alpha T \cdot D \cdot \tilde{T})] \quad (13)$$

i.e.

$$T_r(I - \alpha \tilde{A} \cdot A)^2 = T_r[I - 2\alpha T \cdot D \cdot \tilde{T} + \alpha^2 T \cdot D^2 \cdot \tilde{T}] \quad (14)$$

and, using (8), this is

$$T_r(I - \alpha \tilde{A} \cdot A)^2 = T_r I - 2\alpha T_r D + \alpha^2 T_r D^2 \quad (15)$$

Evaluating the traces on the RHS, taking note of (10) and (11), we have

$$T_r(I - \alpha \tilde{A} \cdot A)^2 = n - 2\alpha(\lambda_1 + \lambda_2 + \dots + \lambda_r) + \alpha^2(\lambda_1^2 + \lambda_2^2 + \dots + \lambda_r^2) \quad (16)$$

But appropriately adding and subtracting r on the RHS this comes to

$$T_r(I - \alpha \tilde{A} \cdot A)^2 = n - r + (1 - \alpha\lambda_1)^2 + (1 - \alpha\lambda_2)^2 + \dots + (1 - \alpha\lambda_r)^2 \quad (17)$$

Inspection of the algebra from (12) to (17) reveals that evaluating $T_r(I - \alpha \tilde{A} \cdot A)^4$ would lead to

$$T_r(I - \alpha \tilde{A} \cdot A)^4 = n - r + (1 - \alpha\lambda_1)^4 + (1 - \alpha\lambda_2)^4 + \dots + (1 - \alpha\lambda_r)^4 \quad (18)$$

and so on. The general result is that

$$T_r(I - \alpha \tilde{A} \cdot A)^{2k} = n - r + \sum_{i=1}^{i=r} (1 - \alpha\lambda_i)^{2k} \quad (19)$$

This means that if α is chosen to be $0 < \alpha < 2/\lambda_1$, the sequence (19) for $k = 1, 2, 3, \dots$, converges to the nullity $(n-r)$ of A .

It may seem that to use this result it is first necessary to find the largest eigenvalue λ_1 , but this is avoided by using Gerschgorin's theorem which states that λ_1 is less than or equal to the maximum row sum of magnitudes of the elements of $\tilde{A} \cdot A$. This enables a sufficiently good choice for α to be made.

3. Practical Examples

The algebra of Section 2 was translated into a Fortran algorithm "Subroutine Ranker" by Mrs Anne Jackson. The listing is given in Appendix 1. The Subroutine was applied to two matrices. Matrix 1, given in Appendix 2, was an 11 x 50 weighting matrix used by Met.O.19 in connection with the inversion of satellite radiance data. Matrix 2, given in Appendix 3, was a 7 x 7 matrix taken from a book by Cherkasova. The results are shown in Tables 1 and 2.

Table 1 - Matrix 1

Iteration k	Trace
1	9.76
2	9.19
3	8.57
4	7.88
5	7.10
6	6.25
7	5.40
8	4.55
9	3.75
10	3.02
11	2.30
12	1.64
13	1.13
14	0.82
15	0.62
16	0.36
17	0.13
18	0.02
19	0.00
20	0.00
21	0.00
22	0.00
23	0.00

Table 2 - Matrix 2

Iteration k	Trace
1	5.16
2	4.41
3	3.56
4	2.71
5	2.05
6	1.66
7	1.40
8	1.16
9	1.02
10	1.00
11	1.00
12	1.00
13	0.99
14	0.99
15	0.97
16	0.95
17	0.90
18	0.81
19	0.65
20	0.43
21	0.18
22	0.03
23	0.00

As Matrix 1 is a matrix which has been used successfully, in the purely computational sense, in the inversion of satellite data it was a foregone conclusion that this matrix would be found to be of full rank, in this case rank 11. Table 1 and Figure 1 confirm this, showing the successive traces descending steadily and unequivocally to zero, indicating that Matrix 1 is of nullity zero.

With Matrix 2 the picture is somewhat different. The traces converge eventually to zero but the process clearly has difficulty in getting past the value 1. The possibility exists that the renewed convergence after iteration 12 may be a consequence of round-off effects in the algorithm itself. Even if this possibility can be ruled out the figures in Table 2 and the Figure 2 graph give a strong indication that, with single-length working in the 360/195, Matrix 2 is nearly of nullity 1, i.e. it is nearly of rank 6.

R DIXON
Met.O.11
September 1977

R Dixon

Reference

N.N. GUPTA:- "An optimum Iterative Method for the Computation of Matrix Rank"
IEEE Transactions on Systems, Man, and Cybernetics, July 1972.

APPENDIX 1

FORTRAN IV G LEVEL 21

RANKER

```

0001      SUBROUTINE RANKER(A,M,N)
C      THIS ROUTINE FINDS THE RANK OF THE M*N MATRIX A
0002      DIMENSION A(M,N),B(15,15),C(15,15)
0003      MAXMN=MAXO(M,N)
0004      MINMN=MING(M,N)
0005      2  FORMAT('0'/'0')
0006      3  FORMAT('0',8F12.3)
0007      WRITE(6,2)
0008      DO 130 I=1,M
0009      WRITE(6,3)(A(I,J),J=1,N)
0010      130 CONTINUE
0011      DO 10 I=1,MINMN
0012      DO 10 J=1,MINMN
0013      D=0
0014      IF(MAXMN.EQ.M)GOTO 110
0015      DO 20 K=1,MAXMN
0016      D=D+A(I,K)*A(J,K)
0017      20 CONTINUE
0018      GOTO 120
0019      110 DO 25 K=1,MAXMN
0020      D=D+A(K,I)*A(K,J)
0021      25 CONTINUE
0022      120 B(I,J)=D
0023      10 CONTINUE
0024      WRITE(6,2)
0025      DO 140 I=1,MINMN
0026      140 WRITE(6,3)(B(I,J),J=1,MINMN)
0027      E=0.0
0028      DO 30 I=1,MINMN
0029      D=0.0
0030      DO 35 J=1,MINMN
0031      D=D+ABS(B(I,J))
0032      35 CONTINUE
0033      IF(D.GT.E)E=D
0034      30 CONTINUE
0035      WRITE(6,2)
0036      WRITE(6,3)E
0037      S=1.0/E
0038      WRITE(6,2)
0039      WRITE(6,3)S
0040      DO 40 I=1,MINMN
0041      DO 40 J=1,MINMN
0042      B(I,J)=-S*B(I,J)
0043      IF(I.EQ.J)B(I,J)=1.0+B(I,J)
0044      40 CONTINUE
0045      WRITE(6,2)
0046      DO 150 I=1,MINMN
0047      150 WRITE(6,3)(B(I,J),J=1,MINMN)
0048      D=0.0
0049      DO 50 I=1,MINMN
0050      D=D+B(I,I)
0051      50 CONTINUE
0052      WRITE(6,1)D
0053      1  FORMAT(' TRACE=',F6.2)
0054      DO 100 IJ=1,100
0055      DO 60 I=1,MINMN

```



```

0056      DO 60 J=1,MINMN
0057      D=0.0
0058      DO 70 K=1,MINMN
0059      D=D+B(I,K)*B(K,J)
0060      70 CONTINUE
0061      C(I,J)=D
0062      60 CONTINUE
0063      WRITE(6,2)
0064      DO 160 I=1,MINMN
0065      160 WRITE(6,3)(C(I,J),J=1,MINMN)
0066      D=0.0
0067      DO 80 I=1,MINMN
0068      D=D+C(I,I)
0069      80 CONTINUE
0070      WRITE(6,1)D
0071      DO 90 I=1,MINMN
0072      DO 90 J=1,MINMN
0073      B(I,J)=C(I,J)
0074      90 CONTINUE
0075      100 CONTINUE
0076      RETURN
0077      END

```


APPENDIX 2 - MATRIX 1

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
217.000	296.000	405.000	632.000	860.000	1107.000	119.000	119.000	168.000	1531.000
1838.000	2124.000	2391.000	2697.000	2974.000	3290.000	3547.000	3547.000	3853.000	4910.000
4159.000	4446.000	4604.000	4742.000	4851.000	4940.000	4959.000	4959.000	2144.000	2144.000
4673.000	4406.000	4149.000	3823.000	3458.000	3053.000	2579.000	2579.000		
1867.000	2816.000								

1st row

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
533.000	710.000	888.000	1199.000	15090.000	18650.000	23530.000	23530.000	28860.000	400.000
34180.000	39960.000	46610.000	51940.000	55940.000	59930.000	6038.000	6038.000	5949.000	2575.000
5461.000	4883.000	4351.000	3907.000	3507.000	3152.000	2841.000	2841.000	1110.000	1110.000
2353.000	2131.000	1909.000	1731.000	1554.000	1421.000	1243.000	1243.000		
977.000	844.000								

2nd row

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	90.000	288.000	587.000	826.000	1094.000	1323.000	1323.000	1602.000	5372.000
1989.000	2328.000	2765.000	3193.000	3780.000	4367.000	4675.000	4675.000	4118.000	1313.000
5750.000	6038.000	5939.000	5759.000	5491.000	5133.000	1562.000	1562.000	60.000	60.000
3690.000	3183.000	2716.000	2298.000	1999.000	1781.000	239.000	239.000		
1104.000	925.000	686.000	507.000	398.000					
20.000	0.0								

3rd row

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
220.000	650.000	1200.000	2180.000	3380.000	4360.000	5010.000	5560.000		
5880.000	6100.000	5990.000	5770.000	5450.000	5120.000	4790.000	4360.000		
3920.000	3600.000	3380.000	3050.000	2830.000	2610.000	2400.000	2180.000		
1960.000	1740.000	1420.000	1200.000	980.000	760.000	550.000	440.000		
330.000	250.000	190.000	150.000	70.000	0.0	0.0	0.0		

4th row

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
650.000	1510.000	2910.000	4420.000	5610.000	6690.000	7340.000	7770.000		
7980.000	7660.000	6900.000	5830.000	5070.000	4420.000	3780.000	3240.000		
2700.000	2370.000	2160.000	1830.000	1620.000	1400.000	1190.000	970.000		
860.000	760.000	650.000	540.000	430.000	320.000	220.000	110.000		
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		
0.0	0.0								

5th row

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5460.000	7070.000	8240.000	8780.000	8890.000	8240.000	7390.000	6420.000		
5570.000	4820.000	4180.000	3530.000	3000.000	2460.000	2030.000	1710.000		
1390.000	1180.000	960.000	750.000	640.000	430.000	320.000	210.000		
140.000	80.000	0.0	0.0	0.0	0.0	0.0	0.0		
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		
0.0	0.0								

6th row

0.0	0.0	0.0	0.0	710.000	2750.000	5910.000	7950.000	9070.000
9380.000	9070.000	8260.000	7030.000	6010.000	5200.000	4490.000	3870.000	
3360.000	2750.000	2350.000	1940.000	1630.000	1430.000	1220.000	1020.000	7 th row
920.000	820.000	710.000	610.000	510.000	410.000	310.000	200.000	
100.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
0.0	0.0							

2860.000	7300.000	10370.000	12590.000	12800.000	11110.000	8360.000	6350.000	8 th row
5080.000	4230.000	3490.000	2750.000	2120.000	1800.000	1480.000	1270.000	
1060.000	850.000	740.000	640.000	560.000	500.000	420.000	350.000	
290.000	240.000	180.000	130.000	90.000	0.0	0.0	0.0	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
0.0	0.0							

12990.000	17380.000	17670.000	12610.000	8790.000	6460.000	5060.000	3920.000	9 th row
2870.000	2200.000	1820.000	1430.000	1150.000	960.000	760.000	670.000	
570.000	500.000	460.000	400.000	360.000	290.000	210.000	170.000	
140.000	110.000	50.000	0.0	0.0	0.0	0.0	0.0	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
0.0	0.0							

APPENDIX 3 - MATRIX 2

22.1	- 24.4	- 11.3	18.4	15.3	20.1	25.2
- 24.4	16.4	16.2	14.3	10.1	- 19.5	18.8
- 11.3	16.2	26.1	- 29.3	27.4	16.3	24.4
18.4	14.3	- 29.3	19.2	29.1	31.5	11.8
15.3	10.1	27.4	29.1	14.9	14.9	11.3
20.1	- 19.5	16.3	31.5	14.9	16.8	20.8
25.2	18.8	24.4	11.8	11.3	20.8	18.8