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Numerical Formulation of the Navier Stokes Model with
Terrain Following Coordinate System

by

C.A.Nash

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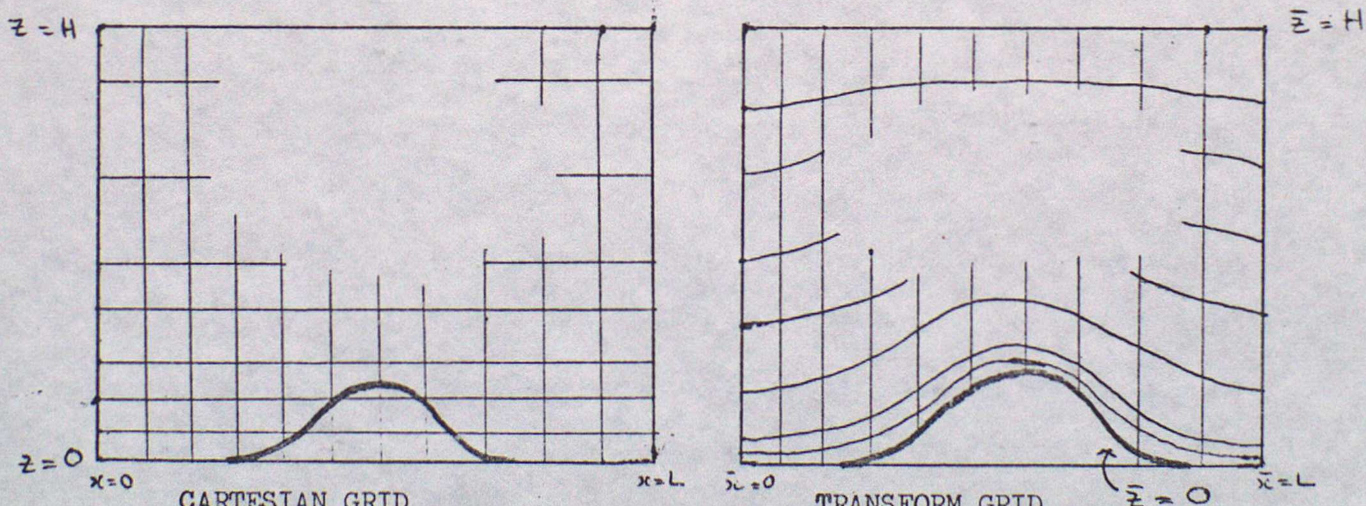
Section 1. The Model

The model described below is that of a two-dimensional laminar incompressible viscous Boussinesq fluid with an irregular lower boundary. Such a model will be used to investigate flows over simple topography and can be generalised to three dimensions and to varying levels of turbulence parameterisation. The essential difference between this model and the other programs available in the branch is the grid on which the integration is performed.

A co-ordinate transformation is used to map the domain with bottom topography onto a rectangular region in the transformed space, where a rectangular grid is imposed for the numerical formulation. In this way the physical boundary becomes a co-ordinate surface and the resolution in the grid can follow the boundary, (fig 1). This should make specification of the turbulence easier and more exact near the ground when the model is generalised.

That is the main reason for undertaking this work. However the model will be very useful in studying laminar flows too and the comparison with the ordinary Cartesian system should help indicate when the improved boundary treatment is most important.



CARTESIAN GRID
(x,z) SPACE

TRANSFORM GRID
SHOWN IN (x,z) SPACE

The model is based on that developed by Clark (1977).

The model equations in Cartesian co-ordinates (x, z) are

$$\frac{du}{dt} - fv = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x_i}\tau_{1i} - \frac{u'}{\tau_R} \qquad i = 1, 3$$

$$\frac{dv}{dt} + fu = -\frac{\partial p_0}{\partial y} + \frac{\partial}{\partial x_i}\tau_{2i} - \frac{v}{\tau_R}$$

$$\frac{dw}{dt} = -\frac{\partial p}{\partial z} + \alpha g T + \frac{\partial}{\partial x_i}\tau_{3i} - \frac{w}{\tau_R}$$

where p and τ_{ij} have been normalised by a reference density ρ_0 and a hydrostatic pressure is incorporated into p.

We have the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

and the thermodynamic equation

$$\frac{dT}{dt} = \nabla . (k \nabla T) - \frac{T'}{\tau_R}$$

where k is a diffusivity, and we write $k\,\partial T/\partial x_i = H_i$.

In using T in the equations we have defined

$$\rho = \rho_0 (1 - \alpha T)$$

and thus $T = 0$ is the reference temperature in the basic state.

The term $\partial p_0/\partial y$ is the background pressure gradient driving the geostrophic wind in the x- direction and is not inconsistent with the two-dimensionality of the flow. The terms in τ_R are Rayleigh damping terms used to suppress waves reaching the upper boundary and thereby prevent spurious reflections; this is a simple approximation to a radiation condition. τ_R is a function of height and is practically infinite below the uppermost levels in the model. u' and T' are deviations of u and T from some background profiles to be defined below.

The viscous stresses are defined conventionally as

$$\tau_{ij} = \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Now consider a rectangular domain of height H with a ground surface at $z = z_s$, as suggested in fig 1.

Now transform the co-ordinates $(x,z) \to (\bar{x},\bar{z})$ such that

$$\bar{x} = x$$

$$\bar{z} = H \cdot \frac{z - z_s}{H - z_s}$$

which clearly transforms the region containing fluid onto a rectangle in $(\bar{x},\bar{z})$ space.

If $d\vartheta$ is a volume element in (x,z) space which maps to $d\bar{\vartheta}$ the areas are related through the Jacobian

$$d\vartheta = \frac{\partial(x,z)}{\partial(\bar{x},\bar{z})} d\bar{\vartheta} = \left(1 - \frac{z_s}{H}\right) d\bar{\vartheta}.$$

Define

$$G^{1/2} = 1 - z_s/H$$

$$G^{13} = \left(\frac{\bar{z}}{H} - 1\right) \partial z_s/\partial x$$

from which we can write

$$G^{1/2}\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial \bar{x}}\left(G^{1/2}\phi\right) + \frac{\partial}{\partial \bar{z}}\left(G^{13}\phi\right)$$

$$G^{1/2}\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial \bar{z}}.$$

for any function ϕ.

An important relationship is deduced

$$\omega = \frac{d\bar{z}}{dt} = \frac{1}{G^{1/2}} \cdot \left(w + G^{13}u\right) \text{ the } \omega \text{ equation.}$$

The equations may now be rewritten in $(\bar{x},\bar{z})$ space and are reproduced in the Appendix.

Boundary conditions are provided by a no-slip bottom boundary and no-stress upper boundary. Lateral boundaries can be inflow/outflow but the initial form of the model has periodic conditions.

The numerical scheme will be described in the next section.

Returning to the Rayleigh friction terms, u' and T' are defined

$$u' = u - u_0$$
$$T' = T - T_0$$

where u_0, T_0 are the reference profiles. T_0 is largely arbitrary but

$$G^{1/2} u_0 = \left(G^{1/2} u\right)_{x=0}$$

and thus $u_0 = u_0(x, z)$. This defines a reference velocity which is consistent with mass conservation if one assumes the streamlines are approximately co-ordinate lives in $(\bar{x}, \bar{z})$ space.

Section 2. The Numerical Formulation

The domain is represented in $(\bar{x}, \bar{z})$ by an irregularly spaced rectangular grid, defined by the intersection of lines

$$\bar{x} = f_x(k) = \bar{x}_k \qquad k = -\tfrac{1}{2}, 0, \ldots, K + \tfrac{1}{2}$$

$$\bar{z} = f_z(m) = \bar{z}_m \qquad m = -\tfrac{1}{2}, 0, \ldots, M + \tfrac{1}{2}$$

where f_x and f_z are arbitrary strictly increasing functions.

The variables are held at the staggered locations of Harlow and Welch (1965) shown in fig 2.

$$\begin{array}{ccccc}
 & \tau_{13} & \begin{matrix}\tau_{23}^{i\,k+\frac{1}{2}} \\ W, \omega\end{matrix} & \tau_{13} & \\
\phi & \begin{matrix}H_1 \\ u, \tau_{12}^{i-\frac{1}{2}\,k}\end{matrix} & \phi_{ik} & u, \tau_{12}^{i+\frac{1}{2}\,k} & \phi \\
 & \tau_{13} & \begin{matrix}H_3^{i\,k-\frac{1}{2}} \\ W, \omega \\ \tau_{23}\end{matrix} & \tau_{13} &
\end{array}$$

fig 2

where ϕ is a general scalar, i.e. $T, p, \tau_{11}, \tau_{22}, \tau_{33}$

The remaining variables, H_1 is held with τ_{12} and H_3 is held with τ_{23}. $G^{1/2}$ and G^{13} are calculated everywhere they are needed but are essentially functions of single variables only $\left(G^{13} = f(\bar{x})\, g(\bar{z})\right)$.

The lateral physical boundaries run through u points and the horizontal ones through W points. It is necessary to define an extra two rows outside the horizontal boundaries and two columns beyond the lateral boundaries.

Those variables needed <u>outside</u> the domain are shown in fig 3.

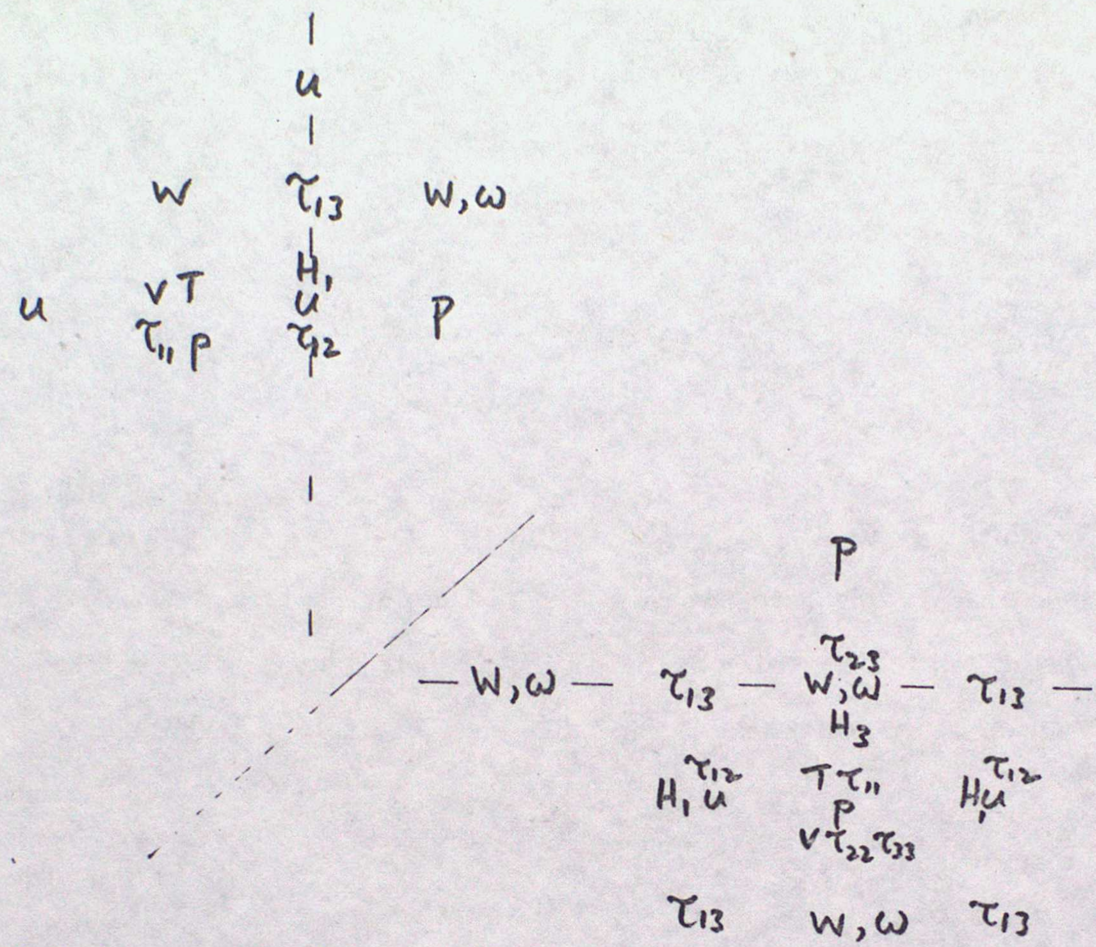


FIG. 3

which shows a general horizontal or vertical boundary.

Define the operators (Shuman 1960)

$$\overline{\phi}^{\eta_k} = \frac{\phi(\eta_{k+\frac{1}{2}}) + \phi(\eta_{k-\frac{1}{2}})}{2}$$

k a half integral index as above

$$\delta_{\eta_k}\phi = [\phi(\eta_{k+\frac{1}{2}}) - \phi(\eta_{k-\frac{1}{2}})] / [\eta_{k+\frac{1}{2}} - \eta_{k-\frac{1}{2}}]$$

where η can be any of $\bar{x}$, $\bar{z}$ or t $(t_\tau = \tau\Delta t)$

Notice

$$\delta_{\eta_k}\eta = \eta_{k+\frac{1}{2}} - \eta_{k-\frac{1}{2}} = (\Delta\eta)_k$$

the grid interval width straddling η_k.

It is a property of the differencing that it is second order accurate despite the non-uniform grid provided the grid stretching is not too severe (Kalnay de Rivas 1972).

Now the bars on co-ordinate symbols will be dropped.

The difference form of the momentum equations is

$$\delta_t \overline{(G^{1/2}u)}^t + ADVX^{\tau} - (\overline{G^{1/2}v}^x . f)^{\tau} = PFX^{\tau} + KFX^{\tau-1} + RAYX^{\tau-1}$$

$$\delta_t \overline{(G^{1/2}v)}^t + ADVY^{\tau} + (\overline{G^{1/2}u}^x . f)^{\tau} = KFY^{\tau-1} + RAYY^{\tau-1}$$

$$\delta_t \overline{(G^{1/2}w)}^t + ADVZ^{\tau} = PFZ^{\tau} + BY^{\tau} + KFZ^{\tau-1} + RAYZ^{\tau-1}$$

where the superscript denotes the time level of a source term. Notice that Clark (1977) takes his Rayleigh friction (RAYX etc.) terms at the centred time step which is unconditionally unstable when the term is examined in isolation.

The thermodynamic equation is

$$\delta_t \overline{(G^{1/2}T)}^t + ADVT^{\tau} = KFT^{\tau-1} + RAYT^{\tau-1}$$

(Clark did not use a RAYT term).

The equation of mass continuity is

$$\delta_x (G^{1/2}u) + \delta_z (G^{1/2}\omega) = 0$$

The numerical ω equation is

$$[G^{1/2}]^2 \omega = G^{1/2}w + G^{13} . \overline{G^{1/2}u}^{zx}$$

This is not the same operator as Clark and is still being investigated. Clark suggests that the form is quite crucial for good momentum and energy budgets but does not expand on this. Consistency between the ω equation and the Poisson equation for pressure is clearly essential and we have ensured this with our formulation.

The advective terms are of the Piacsek and Williams (1970) absolutely conserving form, i.e. conservative of $u^2+v^2+w^2$ and T^2 to within round off whatever the divergence of the velocity field but only conservative of linear quantities to within an error of the order of that divergence.

Define the difference operator

$$\delta^+_{\eta_k}(\phi.\psi) = \frac{\phi(\eta_{k+\frac{1}{2}})\psi(\eta_{k+1}) - \phi(\eta_{k-\frac{1}{2}})\psi(\eta_{k-1})}{2(\eta_{k+\frac{1}{2}} - \eta_{k-\frac{1}{2}})}$$

Then the advective terms are given by

$$ADVX = \delta^+_x(\overline{G^{1/2}u}^x . u) + \delta^+_z(\overline{G^{1/2}\omega}^x . u)$$

$$ADVY = \delta^+_x(G^{1/2}u . v) + \delta^+_z(G^{1/2}\omega . v)$$

$$ADVZ = \delta^+_x(\overline{G^{1/2}u}^z . w) + \delta^+_z(\overline{G^{1/2}\omega}^z . w)$$

$$ADVT = \delta^+_x(G^{1/2}u . T) + \delta^+_z(G^{1/2}\omega . T)$$

The pressure gradient terms are

$$PFX = -\delta_x(G^{1/2}p) - \delta_z(G^{13}\overline{p}^{xz})$$

$$PFZ = -\delta_z p.$$

with buoyancy $BY = \alpha g \overline{T}^z.$

The Rayleigh friction terms are given as

$$RAYX = -G^{1/2}(u-u_0)/\tau_R$$

$$RAYY = -G^{1/2}v/\tau_R$$

$$RAYZ = -G^{1/2}w/\tau_R$$

$$RAYT = -G^{1/2}(T-T_0)/\tau_R$$

The diffusion terms are given by

$$KFX = \delta_x(G^{1/2}\tau_{11}) + \delta_z(\tau_{13} + G^{13}\overline{\tau_{11}}^{xz})$$

$$KFY = \delta_x(G^{1/2}\tau_{12}) + \delta_z(\tau_{23} + G^{13}\overline{\tau_{12}}^{xz})$$

$$KFZ = \delta_x(G^{1/2}\tau_{13}) + \delta_z(\tau_{33} + G^{13}\overline{\tau_{13}}^{xz})$$

$$KFT = \delta_x(G^{1/2}H_1) + \delta_z(H_3 + G^{13}\overline{H_1}^{zx})$$

where τ_{ij} and H_i are calculated straightforwardly by the natural centred-difference formulae following

$$G^{1/2}\tau_{11} = 2\nu(\delta_x(G^{1/2}u) + \delta_z(\overline{G^{13}u}^{xz}))$$

$$G^{1/2}\tau_{12} = \nu(\delta_x(G^{1/2}v) + \delta_z(\overline{G^{13}v}^{xz}))$$

$$G^{1/2}\tau_{13} = \nu(\delta_x(G^{1/2}w) + \delta_z(u + \overline{G^{13}w}^{xz}))$$

$$G^{1/2}\tau_{23} = \nu\,\delta_z(v)$$

$$G^{1/2}\tau_{33} = 2\nu\,\delta_z(w)$$

$$G^{1/2}H_1 = K(\delta_x(G^{1/2}T) + \delta_z(\overline{G^{13}T}^{xz}))$$

$$G^{1/2}H_3 = K\,\delta_z(T)$$

In summary we have presented equations for $\delta_t u, v, w, T$ and implicitly ω. In order that these tendencies are well-defined, an elliptic equation is solved for ρ given u, v, w.

To find this equation consider $\delta_t \overline{D}^t$ where

$$D = \delta_x(G^{1/2}u) + \delta_z(G^{1/2}\omega)$$

to give

$$-\frac{D^{\tau-1}}{2\Delta t} = \delta_x\delta_t(\overline{G^{1/2}u}^t) + \delta_z\left[\frac{\delta_t(\overline{G^{1/2}w}^t) + G^{13}\overline{\delta_t(\overline{G^{1/2}u}^t)}^{xz}}{G^{1/2}}\right]$$

where $D^{\tau+1}$ has been used. The term $D^{\tau-1}$ is retained to limit growth of D during the integration (Harlow and Welch 1965).

The resulting equation may be written

$$G.\,\delta_{xx}(G^{1/2}p) + \delta_{zz}(G^{1/2}p) = \frac{G}{2\Delta t}.\,D^{\tau-1} + \,..$$

$$..+\left\{G.\,\delta_x S_u + G^{1/2}.\,\delta_z\left(S_w + G^{13}.\,\overline{S_u}^{zx}\right)\right\} + \,..$$

$$..+\left\{G.\,\delta_x p_u + G^{1/2}.\,\delta_z\left[G^{13}\overline{\left(\delta_x(G^{1/2}p) + p_u\right)}^{zx}\right]\right\}.$$

where $p_u = \delta_z(G^{13}\,\overline{p}^{xz})$

and

$$\delta_t\,\overline{(G^{1/2}u)}^{t} = S_u - \delta_x(G^{1/2}p) - p_u\,.$$

$$\delta_t\,\overline{(G^{1/2}w)}^{t} = S_w - \delta_z p\,.$$

The method of solution will be described in a later section.

3. <u>Boundary Conditions</u>

Boundary conditions are needed to close the equations. The model must calculate $\delta_t \underline{u}$ $\delta_t T$ on interior to step these fields there, $\delta_t \underline{u}$ is needed out to the first row/column outside the domain in order than the elliptic pressure equation may be calculated.

We impose a no-slip boundary condition on $\bar{z} = 0$

$u = v = w = 0 \quad \Rightarrow \quad \omega = 0$, the kinematic condition and stress-free at $\bar{z} = H$

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = W = 0 \Rightarrow \omega = 0, \text{ the kinematic condition}$$

On the lateral boundaries we assume periodicity. However inflow/outflow conditions can be applied with the same sort of considerations as on the horizontal boundaries. The choice of extrapolation scheme is the non-trivial part of the problem cf. Clark (1977).

Consider the boundary $\bar{z} = 0$ for example. We define $u_0 = -u_1$, $v_0 = -v_1$, $W_{\frac{1}{2}} = 0$ $(\Rightarrow \omega_{\frac{1}{2}} = 0)$ to give first order boundary conditions based on the specified values at $\bar{z} = 0$. Given p and necessary τ_{ij} this enables $\delta_t \overline{\underline{u}}^t$ to be calculated on all the interior points.

Now to calculate the elliptic pressure equation S_W is needed on $\bar{z} = 0$ and $\delta_t \overline{G^{1/2}u}^t$ on $\bar{z}_0$ The second follows directly from

$$\delta_t \left(\overline{G^{1/2}u}^t\right)\Big|_{\bar{z}=\bar{z}_0} = -\delta_t \left(\overline{G^{1/2}u}\right)^t\Big|_{\bar{z}=\bar{z}_1}$$

The calculation of $S_W(\bar{z}=0)$ requires $\tau_{13}(\bar{z}_{-1/2})$ and $\overline{G^{1/2}\omega}^z\Big|_{\bar{z}_0} \cdot W_{-1/2}$.

To ensure momentum conservation we define

$$\overline{G^{1/2}\omega}^z\Big|_{\bar{z}_0} \cdot W_{-1/2} = -\overline{G^{1/2}\omega}^z\Big|_{\bar{z}_1} \cdot W_{\frac{3}{2}}$$

which is equivalent to assuming zero flux through the boundary and taking first order differences.

$(\tau_{13})_{-1/2}$ and all the other stresses required outside the domain (fig 3) are calculated by extrapolation across the boundary, using their calculated values on the boundary. These values are found directly from the definitions applied naturally at $\bar{z} = 0$ to first order.

The boundary condition on the pressure equation is that on the top and bottom of the domain

$$\delta_t \overline{(G^{1/2}\omega)}^t = 0 = S_\omega - \rho\omega - \frac{\delta_z p}{G^{1/2}}.$$

with periodicity in the horizontal.

In general this is a tri-diagonal equation for p on the boundary in terms of interior values. However in this case the boundary conditions show

$$\delta_t(\overline{G^{1/2}w}^t) = 0 = S_w - \delta_z p.$$

which is an explicit relation for p on $\bar{z}_0$ and is calculable from above.

Section 4. Solution of the Pressure Equation

Write the pressure equation as

$$G\,\delta_{xx}\varphi + \delta_{zz}\varphi = G(\nabla\cdot[\underline{S}-\underline{L}]) = G\,\nabla\cdot\underline{\Phi}_s$$

where

$$\varphi = G^{1/2}p \quad , \quad \nabla\cdot\underline{S} = \frac{\partial S_{\bar{x}}}{\partial\bar{x}} + \frac{\partial S_{\bar{z}}}{\partial\bar{z}}$$

$$\underline{S} = (S_{\bar{x}}, S_{\bar{z}})$$

$$S_{\bar{x}} = \frac{G^{1/2}u^{\tau-1}}{2\Delta t} + S_u \quad . \quad S_{\bar{z}} = \frac{G^{1/2}\omega^{\tau-1}}{2\Delta t} + G^{-1/2}[S_w + G^{13}\cdot\overline{S_u}^{2x}$$

$$L_x = \rho_u \qquad . \quad L_{\bar{z}} = G^{-1/2}\cdot G^{13}\overline{(\delta_x(G^{1/2}p) + \rho_u)}^{2x}$$

The pressure equation is solved iteratively following Clark (1977) subject to periodic lateral conditions and

$$G^{-1}\cdot\delta_z\varphi = (\underline{\Phi}_s)_{\bar{z}} \qquad \text{on } \bar{z} = 0, H.$$

The solution φ is easily found by a direct method given $\underline{\Phi}_s$ but since $\underline{\Phi}_s$ depends on φ, we must proceed with iteration. Given a guess for the φ^i field, the elliptic problem is solved for a new φ^{i+1}. Then $\underline{\Phi}_s^{i+1}$ is recalculated and the process continues until convergence is achieved.

The efficient solution of each

$$G\,\delta_{xx}\,\varphi^{i+1} + \delta_{zz}\,\rho^{i+1} = G\,\nabla.\,\underline{\Phi}_s^{\,i}$$

is made possible by using the boundary condition

$$G^{-1}.\,\delta_z\,\varphi^{i+1} = 0$$

whose solution is the same as that above except on the boundary. The application of

$$\delta_z\,\rho^{i+1} = S_W$$

there completes the solution. (This property of the numerical solution is discussed further in an Appendix.)

The method of solution is that of Farnell (1980).

Initial experience with this algorithm suggests that convergence is very rapid for hill slopes less than 30° (with a 40x64 grid). Two or three iterations is sufficient to achieve a relative error of 10^{-5} provided the flow is not rapidly changing. However the correcting source term $G\,\nabla.\,\underline{\Phi}_s^{\,i}$ increases with aspect ratio in such a way that the scheme will not converge for slopes of 45°. Convergence can be achieved in this case by underrelaxation. However about 45° this artifice will probably not produce convergence. This restriction should not be significant for the work envisaged with the model.

Appendix   Numerical Poisson equations - a result

Consider the numerical solution of

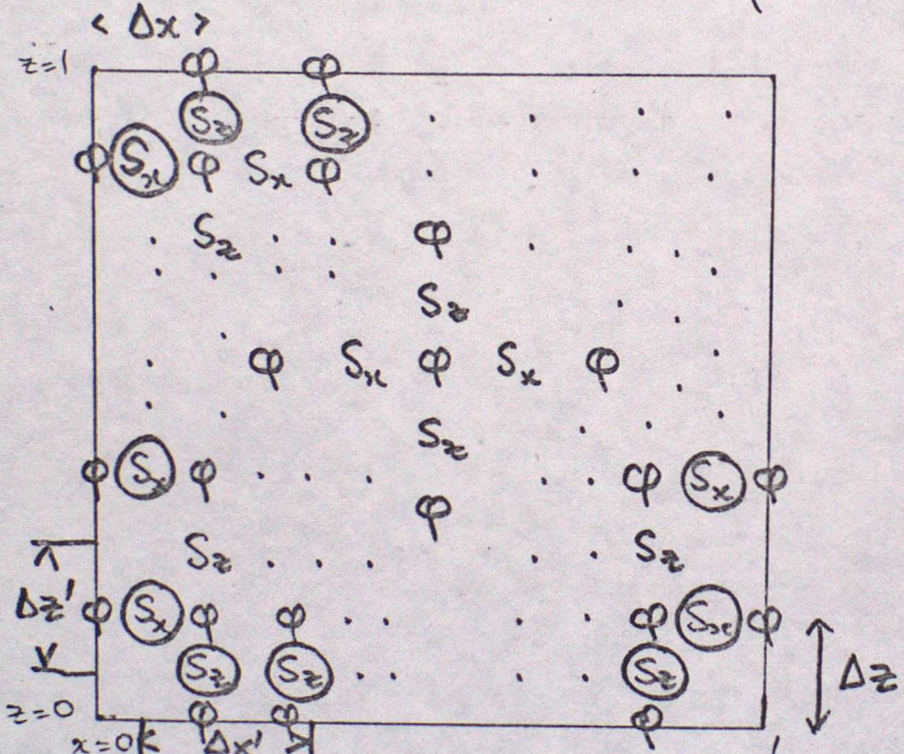
$$\delta_{xx}\varphi + \delta_{zz}\varphi = \delta_x S_x + \delta_z S_z \quad \text{on } [0,1]^2$$

$$\delta_x \varphi = S_x + B \quad \text{on } x = 0,1$$

$$\delta_z \varphi = S_z + B \quad \text{on } z = 0,1$$

where B is a general function defined on the boundary, and the grid arrangement is sketched below.

It is easily seen that the values of φ not on the boundaries are independent of the bounding S values (circled). Only the boundary values depend on these S. This convenient property of the numerics means that by setting $S_x = -B$, $S_z = -B$ on the appropriate boundary, conditions $\delta_x\varphi = 0$ and $\delta_z\varphi = 0$ can be used, and then efficient accurate direct methods may be used.



This result can be generalised so that the problem

$$\alpha(z)\,\delta_{xx}\varphi + \beta(x)\,\delta_{zz}\varphi = \delta_x S_x + \delta_z S_z$$

$$\alpha(z)\,\delta_x \varphi = S_x + B$$

$$\beta(x)\,\delta_z \varphi = S_z + B$$

has the same property.

Appendix Two-dimensional analytic equations in the transform co-ordinate system

Momentum equation

$$\frac{\partial}{\partial t}\left(G^{1/2}u\right) + \frac{\partial}{\partial x}\left(G^{1/2}u^2\right) + \frac{\partial}{\partial z}\left(G^{1/2}u\omega\right) - G^{1/2}vf$$

$$= -\frac{\partial}{\partial x}\left(G^{1/2}p\right) - \frac{\partial}{\partial z}\left(G^{13}p\right)$$

$$+ \frac{\partial}{\partial x}\left(G^{1/2}\tau_{11}\right) + \frac{\partial}{\partial z}\left(G^{13}\tau_{11} + \tau_{13}\right) - G^{1/2}u'/\tau_R$$

$$\frac{\partial}{\partial t}\left(G^{1/2}v\right) + \frac{\partial}{\partial x}\left(G^{1/2}uv\right) + \frac{\partial}{\partial z}\left(G^{1/2}v\omega\right) + G^{1/2}uf$$

$$= -\frac{\partial p_0}{\partial y} + \frac{\partial}{\partial x}\left(G^{1/2}\tau_{21}\right) + \frac{\partial}{\partial z}\left(G^{13}\tau_{21} + \tau_{23}\right) - \frac{G^{1/2}v}{\tau_R}$$

$$\frac{\partial}{\partial t}\left(G^{1/2}w\right) + \frac{\partial}{\partial x}\left(G^{1/2}uw\right) + \frac{\partial}{\partial z}\left(G^{1/2}w\omega\right) = -\frac{\partial p}{\partial z} + \alpha G^{1/2}g.T$$

$$+ \frac{\partial}{\partial x}\left(G^{1/2}\tau_{31}\right) + \frac{\partial}{\partial z}\left(G^{13}\tau_{31} + \tau_{33}\right) - \frac{G^{1/2}w}{\tau_R}$$

Continuity

$$\frac{\partial}{\partial x}\left(G^{1/2}u\right) + \frac{\partial}{\partial z}\left(G^{1/2}\omega\right) = 0$$

Thermodynamics

$$\frac{\partial}{\partial t}\left(G^{1/2}T\right) + \frac{\partial}{\partial x}\left(G^{1/2}uT\right) + \frac{\partial}{\partial z}\left(G^{1/2}\omega T\right) =$$

$$\frac{\partial}{\partial x}\left(G^{1/2}H_1\right) + \frac{\partial}{\partial z}\left(H_3 + G^{13}H_1\right) - \frac{T'}{\tau_R}$$

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