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SEMI-IMPLICIT CENTRED OCTAGON

by

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SEMI-IMPLICIT CENTRED OCTAGON

1. INTRODUCTION

A semi-implicit formulation of the ten level model octagon has been written using a centred leap-frog time step, and this note discusses aspects of the computational procedure and stability of the scheme.

The model is based on the Bushy-Timpson model (see Benwell et al (1971)) as reformulated by Burridge (1975) with the two step Lax-Wendroff scheme replaced by the single step leap-frog scheme. Unlike Burridge's scheme the present model is not split, the operators for the non-linear advection step and the linear gravity wave step being additive rather than multiplicative.

Only the first two gravity waves are dealt with implicitly, the others being treated by the centred leap-frog scheme. (In Burridge's split model a forward time step is used for all but the first two gravity waves). As with the split scheme, in the h equation only the ICAO value of the $\beta\omega$ term is dealt with implicitly, the departure from ICAO being done explicitly. In the current scheme the coriolis terms are dealt with explicitly.

Diffusion is applied using a forward double time step.

The grid is staggered in space, but not in time.

2. BASIC EQUATIONS

The equations are as described in Burridge (1975).

$$\frac{\partial u}{\partial t} + \mu\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \omega\frac{\partial u}{\partial p} + \frac{1}{2}\left(u^2+v^2\right)\frac{\partial \mu}{\partial x} + g\frac{\partial h}{\partial x} - fv = 0 \quad (1)$$

$$\frac{\partial v}{\partial t} + \mu\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \omega\frac{\partial v}{\partial p} + \frac{1}{2}\left(u^2+v^2\right)\frac{\partial \mu}{\partial y} + g\frac{\partial h}{\partial y} + fu = 0 \quad (2)$$

$$\frac{\partial h'}{\partial t} + \mu\left(u\frac{\partial h'}{\partial x} + v\frac{\partial h'}{\partial y}\right) + \beta\omega = 0 \quad (3a)$$

$$\beta = p^{K-1}\frac{\partial}{\partial p}\left(p^{1-K}h'\right) = \frac{\partial h'}{\partial p} + \frac{(1-K)}{p}h'$$

$$\frac{\partial h_{10}}{\partial t} + \mu\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(h_{10} - H\right) + \beta_{10}\omega_{10} = 0 \quad (3b)$$

$$\beta_{10} = \frac{\partial h_{10}}{\partial p}$$

$$\frac{\partial r}{\partial t} + \mu\left(u\frac{\partial r}{\partial x} + v\frac{\partial r}{\partial y}\right) + \omega\frac{\partial r}{\partial p} = 0 \quad (4)$$

(u, v) are the velocities $(u^*, v^*) = \frac{1}{m}(u, v)$ of Burridge, where m is the map factor, and $\mu = m^2$. ($m = \frac{2}{1+\sin\phi} = \sec^2(\frac{\pi}{4} - \frac{\phi}{2})$ where ϕ is the latitude). h', h_{10}, r and ω are the thickness, 1000 mb height, humidity and vertical velocity respectively. Pressure p is used as the vertical co-ordinate.

The right hand sides of all the equations (the "physics"), though here set to zero, are not neglected but are treated exactly as in the split scheme. (See Burridge and Gadd (1975)).

The continuity equation is

$$\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial \omega}{\partial p} = 0 \qquad (5)$$

3. GRID AND TIME STEPPING PROCEDURE

The grid shown in fig (1) is used at each time level n.

The time step (single time step) Δt is the time advanced during one leap-frog step of the forecast (ie from n to n+1). The double time step is $2\Delta t$.

It is necessary to know the fields at time levels n-1 and n to compute those at n+1. To start a forecast, a forward time step from $t = 0$ to $t = \Delta t$ is applied to all fields.

For the simple equation $\frac{\partial \alpha}{\partial t}(x, t) = f(x, t)$

the finite difference scheme takes the form

$$\frac{1}{2\Delta t}\left(\alpha^{n+1} - \alpha^{n-1}\right) = f^n$$

except for the first time step where

$$\frac{1}{\Delta t}\left(\alpha^1 - \alpha^0\right) = f^0$$

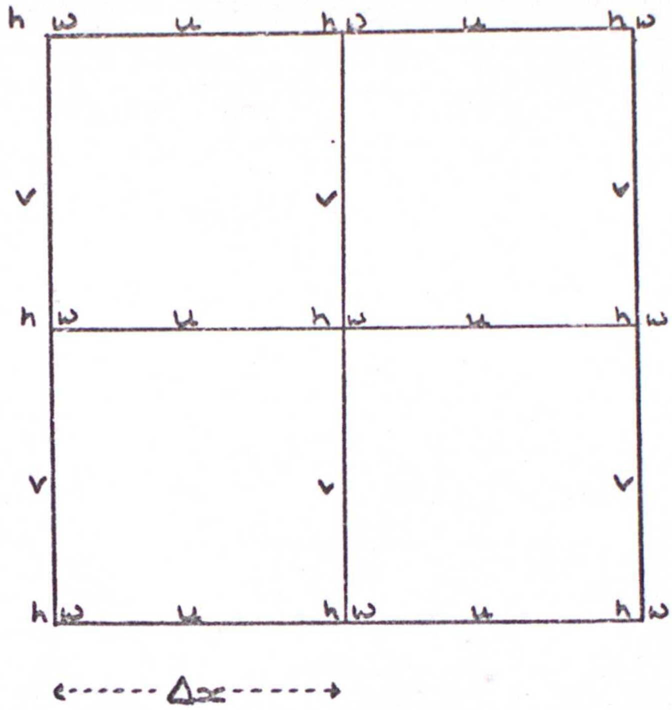


fig (1)

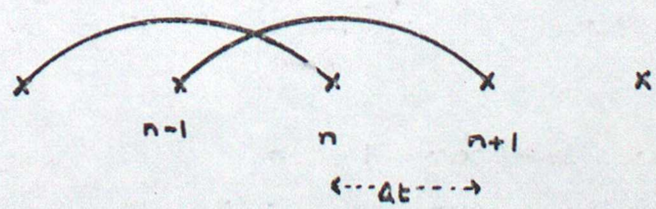


fig (2)

4. INTEGRATION PROCEDURE

The integration is carried out in two steps

(i) Advection step

To the fields at time level n-1 add an advective term computed at time level n to obtain preliminary fields at time level n+1.

Denoting these preliminary fields by $\hat{u}, \hat{v}, \hat{h}', \hat{h}_{10}, \hat{r}$ one has

$$-\frac{1}{2\Delta t}\left[\hat{u} - u^{n-1}\right] = \left[\mu \underline{v}.\nabla u + \omega \frac{\partial u}{\partial p} + \frac{1}{2}(u^2+v^2)\frac{\partial \mu}{\partial x} - fv\right]^n \tag{6}$$

$$-\frac{1}{2\Delta t}\left[\hat{v} - v^{n-1}\right] = \left[\mu \underline{v}.\nabla v + \omega \frac{\partial v}{\partial p} + \frac{1}{2}(u^2+v^2)\frac{\partial \mu}{\partial y} + fu\right]^n \tag{7}$$

$$-\frac{1}{2\Delta t}\left[\hat{h}' - h'^{\,n-1}\right] = \left[\mu \underline{v}.\nabla h' + (\beta - \beta_I)\omega\right]^n \tag{8a}$$

$$-\frac{1}{2\Delta t}\left[\hat{h}_{10} - h_{10}^{n-1}\right] = \left[\mu \underline{v}.\nabla (h_{10} - H) + (\beta - \beta_I)_{10}\,\omega_{10}\right]^n \tag{8b}$$

$$-\frac{1}{2\Delta t}\left[\hat{r} - r^{n-1}\right] = \left[\mu \underline{v}.\nabla r + \omega \frac{\partial r}{\partial p}\right]^n \tag{9}$$

where $\beta_I = \frac{(1-K)}{p} h'_{ICAO}$ and $\beta_{I,10}$

are the ICAO values of β and β_{10}

(ii) Semi-Implicit Step

There remain the gravity wave terms to be taken into account.

Equations (1) to (4) can now be written

$$u^{n+1} - \hat{u} + 2\Delta t\, g \frac{\partial h}{\partial x} = 0 \qquad (10)$$

$$v^{n+1} - \hat{v} + 2\Delta t\, g \frac{\partial h}{\partial y} = 0 \qquad (11)$$

$$h'^{\,n+1} - \hat{h}' + 2\Delta t\, \beta_I \omega = 0 \qquad (12a)$$

$$h_{10}^{n+1} - \hat{h}_{10} + 2\Delta t\, \beta_{I,10}\, \omega_{10} = 0 \qquad (12b)$$

$$r^{n+1} - \hat{r} = 0 \qquad (13)$$

It is convenient to introduce vectors $\underline{u}$, $\underline{v}$, $\underline{\omega}$, $\underline{h}$ whose elements are the 10 values of u, v, ω, h at each of the levels.

The finite difference form of Eq (5) is then

$$\frac{1}{\Delta p} \underline{\underline{A}}^T \underline{\omega} = -\mu\, \underline{\underline{D}} \left(\frac{\partial \underline{u}}{\partial x} + \frac{\partial \underline{v}}{\partial y} \right) \qquad (14)$$

and Equations (12a) and (12b) can be combined and written as a vector equation

$$\underline{h}^{n+1} - \hat{\underline{h}} + 2\Delta t \frac{\mu}{g} \underline{\underline{G}}_I \left(\frac{\partial \underline{u}}{\partial x} + \frac{\partial \underline{v}}{\partial y} \right) = 0 \qquad (15)$$

where

$$\underline{\underline{G}}_I = g \Delta p \, \underline{\underline{A}}^{-1} \underline{\underline{\Gamma}}_I \left(\underline{\underline{A}}^T \right)^{-1} \underline{\underline{D}}$$

Equations (10) and (11) can similarly be written in vector form

$$\underline{u}^{n+1} - \hat{\underline{u}} + 2\Delta t \, g \frac{\partial \underline{h}}{\partial x} = 0 \qquad (16)$$

$$\underline{v}^{n+1} - \hat{\underline{v}} + 2\Delta t \, g \frac{\partial \underline{h}}{\partial y} = 0 \qquad (17)$$

The matrices $\underline{\underline{A}}$, $\underline{\underline{\Gamma}}_I$, $\underline{\underline{D}}$ and $\underline{\underline{G}}_I$ are introduced in Burridge and are given in the Appendix. (p.14)

To compute the adjustment necessary for the first two gravity waves, decompose $\underline{u}$, $\underline{v}$ and $\underline{h}$ into eigenmodes by introducing

$$\underline{\alpha} = \underline{\underline{E}}^{-1} \underline{u}$$

$$\underline{\beta} = \underline{\underline{E}}^{-1} \underline{v}$$

$$\underline{\gamma} = \underline{\underline{E}}^{-1} \underline{h}$$

where $\underline{\underline{E}}^{-1} \underline{\underline{G}}_I \underline{\underline{E}} = \underline{\underline{C}}_I^2$ a diagonal matrix, whose diagonal elements $c_{I,i}^2$ (all real and positive definite) are the squares of the gravity wave speeds (arranged in descending order) of the ICAO atmosphere. (The $c_{I,i}^2$ are the eigenvalues of the matrix $\underline{\underline{G}}_I$, and the columns of $\underline{\underline{E}}$ are the right hand eigen functions of $\underline{\underline{G}}_I$).

Equations (16), (17) and (15) then become

$$\underline{\alpha}^{n+1} - \hat{\underline{\alpha}} + 2\Delta t g \frac{\partial \underline{\gamma}}{\partial x} = 0 \tag{18}$$

$$\underline{\beta}^{n+1} - \hat{\underline{\beta}} + 2\Delta t g \frac{\partial \underline{\gamma}}{\partial y} = 0 \tag{19}$$

$$\underline{\gamma}^{n+1} - \hat{\underline{\gamma}} + 2\Delta t \frac{\mu}{g} \underline{\underline{C}}_I^2 \left(\frac{\partial \underline{\alpha}}{\partial x} + \frac{\partial \underline{\beta}}{\partial y} \right) = 0 \tag{20}$$

The first two modes (ie the first two components of the vectors $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$) are treated implicitly ie derivative $\frac{\partial}{\partial x}$ is approximated by $\frac{1}{2} \frac{\partial}{\partial x}\Big|_{n+1} + \frac{1}{2} \frac{\partial}{\partial x}\Big|_n$ and similarly $\frac{\partial}{\partial y}$. The other eight modes are treated explicitly ie $\frac{\partial}{\partial x}$ is approximated by $\frac{\partial}{\partial x}\Big|_n$ and similarly $\frac{\partial}{\partial y}$.

By introducing new fields

$$\hat{\underline{\alpha}}^* = \hat{\underline{\alpha}} - 2\Delta t g \frac{\partial \underline{\gamma}^n}{\partial x} \tag{21}$$

$$\hat{\underline{\beta}}^* = \hat{\underline{\beta}} - 2\Delta t g \frac{\partial \underline{\gamma}^n}{\partial y} \tag{22}$$

$$\hat{\underline{\gamma}}^* = \hat{\underline{\gamma}} - 2\Delta t \frac{\mu}{g} \underline{\underline{C}}_I^2 \left(\frac{\partial \underline{\alpha}^n}{\partial x} + \frac{\partial \underline{\beta}^n}{\partial y} \right) \tag{23}$$

the explicit contributions to the gravity wave terms are now taken into account. (Computationally, these terms can be evaluated at the advection stage and $\hat{u}^*$ etc obtained directly from u^{n-1} etc without the need to compute $\hat{u}$ etc).

One finds for the first two modes $(i = 1, 2)$

$$\alpha_i^{n+1} - \hat{\alpha}_i^* + 2\Delta t . \tfrac{1}{2} g \frac{\partial}{\partial x}\left(\gamma_i^{n+1} - 2\gamma_i^n + \gamma_i^{n-1}\right) = 0 \qquad (24)$$

$$\beta_i^{n+1} - \hat{\beta}_i^* + 2\Delta t . \tfrac{1}{2} g \frac{\partial}{\partial y}\left(\gamma_i^{n+1} - 2\gamma_i^n + \gamma_i^{n-1}\right) = 0 \qquad (25)$$

$$\gamma_i^{n+1} - \hat{\gamma}_i^* + 2\Delta t . \frac{1}{2}\frac{\mu}{g} c_{I,i}^2 \left[\frac{\partial}{\partial x}\left(\alpha_i^{n+1} - 2\alpha_i^n + \alpha_i^{n-1}\right) + \frac{\partial}{\partial y}\left(\beta_i^{n+1} - 2\beta_i^n + \beta_i^{n-1}\right)\right] = 0 \qquad (26)$$

and for the other eight modes $(i = 3, 4, \cdots\cdots, 10)$

$$\alpha_i^{n+1} - \hat{\alpha}_i^* = 0 \qquad (27)$$

$$\beta_i^{n+1} - \hat{\beta}_i^* = 0 \qquad (28)$$

$$\gamma_i^{n+1} - \hat{\gamma}_i^* = 0 \qquad (29)$$

Introducing the matrix $\underline{\underline{H}}$ = diag (1,1,0,0,0,0,0,0,0,0,)

the above two sets of equations may be combined back into vector form

$$\underline{\alpha}^{n+1} - \hat{\underline{\alpha}}^{*} + 2\Delta t.\tfrac{1}{2} g \,\underline{\underline{H}} \frac{\partial}{\partial x}\left(\underline{\gamma}^{n+1} - 2\underline{\gamma}^{n} + \underline{\gamma}^{n-1}\right) = 0 \tag{30}$$

$$\underline{\beta}^{n+1} - \hat{\underline{\beta}}^{*} + 2\Delta t.\tfrac{1}{2} g \,\underline{\underline{H}} \frac{\partial}{\partial y}\left(\underline{\gamma}^{n+1} - 2\underline{\gamma}^{n} + \underline{\gamma}^{n-1}\right) = 0 \tag{31}$$

$$\underline{\gamma}^{n+1} - \hat{\underline{\gamma}}^{*} + 2\Delta t.\tfrac{1}{2} \frac{\mu}{g} \underline{\underline{H}}\, \underline{\underline{C}}_{I}^{2} \left\{ \frac{\partial}{\partial x}\left(\underline{\alpha}^{n+1} - 2\underline{\alpha}^{n} + \underline{\alpha}^{n-1}\right) + \frac{\partial}{\partial y}\left(\underline{\beta}^{n+1} - 2\underline{\beta}^{n} + \underline{\beta}^{n-1}\right)\right\} = 0 \tag{32}$$

which on transforming back to $\underline{u}$, $\underline{v}$, $\underline{h}$ become

$$\underline{u}^{n+1} - \hat{\underline{u}}^{*} + g\Delta t \,\underline{\underline{E}}\,\underline{\underline{H}}\,\underline{\underline{E}}^{-1} \frac{\partial}{\partial x}\left(\underline{h}^{n+1} - 2\underline{h}^{n} + \underline{h}^{n-1}\right) = 0 \tag{33}$$

$$\underline{v}^{n+1} - \hat{\underline{v}}^{*} + g\Delta t \,\underline{\underline{E}}\,\underline{\underline{H}}\,\underline{\underline{E}}^{-1} \frac{\partial}{\partial y}\left(\underline{h}^{n+1} - 2\underline{h}^{n} + \underline{h}^{n-1}\right) = 0 \tag{34}$$

$$\underline{h}^{n+1} - \hat{\underline{h}}^{*} + \frac{\mu}{g}\Delta t \,\underline{\underline{E}}\,\underline{\underline{H}}\,\underline{\underline{C}}_{I}^{2}\,\underline{\underline{E}}^{-1} \left\{ \frac{\partial}{\partial x}\left(\underline{u}^{n+1} - 2\underline{u}^{n} + \underline{u}^{n-1}\right) + \frac{\partial}{\partial y}\left(\underline{v}^{n+1} - 2\underline{v}^{n} + \underline{v}^{n-1}\right)\right\} = 0 \tag{35a}$$

i.e. $$\underline{h}^{n+1} - \hat{\underline{h}}^{*} - \Delta t \,\underline{\underline{E}}\,\underline{\underline{H}}\,\underline{\underline{E}}^{-1}\underline{\underline{A}}^{-1}\underline{\underline{\Gamma}}_{I}\left(\underline{\omega}^{n+1} - 2\underline{\omega}^{n} + \underline{\omega}^{n-1}\right) = 0 \tag{35b}$$

Now eliminate α_i^{n+1} and β_i^{n+1} by substituting (24) and (25) into (26) to get a second order differential equation for the quantity

$$\psi_i^{n+1} = \gamma_i^{n+1} - 2\gamma_i^{n} + \gamma_i^{n-1}$$

$$-\mu(\Delta t)^2 c_i^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi_i^{n+1} + \psi_i^{n+1} = R_i \qquad (36)$$

where

$$R_i = \left(\hat{\gamma}_i^* - 2\gamma_i^n + \gamma_i^{n-1}\right) - \frac{\mu}{g}\Delta t\, c_i^2 \left\{ \frac{\partial}{\partial x}\left(\hat{\alpha}_i^* - 2\alpha_i^n + \alpha_i^{n-1}\right) + \frac{\partial}{\partial y}\left(\hat{\beta}_i^* - 2\beta_i^n + \beta_i^{n-1}\right) \right\}$$

This is a set of uncoupled Helmholtz equations for the ψ_i^{n+1} and the right hand sides R_i are given by the expression

$$R_i = \underline{e}_i^T \underline{\underline{E}}^{-1}\left[\left(\hat{\underline{h}}^* - 2\underline{h}^n + \underline{h}^{n-1}\right) + \Delta t\, \underline{\underline{A}}^{-1} \underline{\underline{\Gamma}}_I \left(\hat{\underline{\omega}}^* - 2\underline{\omega}^n + \underline{\omega}^{n-1}\right)\right] \qquad (36a)$$

where

$\underline{e}_1^T = (1,0,0,0,0,0,0,0,0,0)$

$\underline{e}_2^T = (0,1,0,0,0,0,0,0,0,0)$ etc

so $\underline{e}_i^T \underline{\underline{E}}^{-1}$ is the ith row of $\underline{\underline{E}}^{-1}$

and $\gamma_i = \underline{e}_i^T \underline{\gamma}$

Note that the left hand sides of equation (36) are the same as in Burridge's split scheme, except that $\frac{1}{2}\Delta t$ is replaced by Δt.
One can therefore use the same ADI parameters, for example, for a 15 minute timestep in the centred model as for a 30 minute time step in the split model.

After obtaining ψ_i^{n+1} and hence γ_i^{n+1} the complete solution for $\underline{u}^{n+1}, \underline{v}^{n+1}, \underline{h}^{n+1}$ is obtained from the following relations.

$$\underline{h}^{n+1} - \hat{\underline{h}}^{*} = \underline{\underline{E}}\,\underline{\underline{H}}(\underline{\gamma}^{n+1} - \hat{\underline{\gamma}}^{*}) = \underline{\underline{E}}\,\underline{\underline{H}}(\underline{\psi}^{n+1} - \hat{\underline{\psi}}^{*}) \tag{37}$$

where $\hat{\underline{\psi}}^{*} = \hat{\underline{\gamma}}^{*} - 2\underline{\gamma}^{n} + \underline{\gamma}^{n-1}$

$$\underline{u}^{n+1} - \hat{\underline{u}}^{*} = -g\Delta t\,\underline{\underline{E}}\,\underline{\underline{H}}\,\frac{\partial}{\partial x}\,\underline{\psi}^{n+1} \tag{38}$$

$$\underline{v}^{n+1} - \hat{\underline{v}}^{*} = -g\Delta t\,\underline{\underline{E}}\,\underline{\underline{H}}\,\frac{\partial}{\partial y}\,\underline{\psi}^{n+1} \tag{39}$$

Writing

$$\underline{\underline{E}} = \begin{pmatrix} E1(1) & E2(1) & E3(1) & \cdots \\ E1(2) & E2(2) & E3(2) & \cdots \\ E1(3) & E2(3) & E3(3) & \cdots \\ \vdots & \vdots & \vdots & \\ E1(10) & E2(10) & E3(10) & \cdots \end{pmatrix}$$

then

$$\underline{\underline{E}}\,\underline{\underline{H}} = \begin{pmatrix} E1(1) & E2(1) & 0 & 0 & \cdots \\ E1(2) & E2(2) & 0 & 0 & \cdots \\ E1(3) & E2(3) & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ E1(10) & E2(10) & 0 & 0 & \cdots \end{pmatrix}$$

To summarise, the order of computation to solve equations (1) to (4) is

(a) evaluate the right hand sides of equations (6) to (9) at time level n, and obtain $\hat{u}$ etc.

(b) add on the explicit contribution to the gravity wave terms (see equations (10) to (13)) to obtain $\hat{u}^*$ etc.

(c) evaluate the right hand sides of the Helmholtz equations (36a) and the quantities $\hat{\psi}_i^*$ ($i=1,2$)

(d) solve the Helmholtz equations to obtain ψ_i^{n+1} ($i=1,2$)

(e) Add on the implicit contributions to the gravity wave terms (see equations (37) to (39)) to obtain u^{n+1} etc.

APPENDIX

The matrices $\underline{\underline{A}}$, $\underline{\underline{\Gamma}}$ and $\underline{\underline{D}}$ are as defined by Burridge.

$$\underline{\underline{A}} = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \quad \text{so} \quad \underline{\underline{A}}^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$\underline{\underline{\Gamma}} = \operatorname{diag}(-\beta_i) \qquad (i = 1, \cdots, 10)$$

$$\underline{\underline{D}} = \operatorname{diag}(1,1,1,1,1,1,1,1,1,\tfrac{1}{2})$$

and so

$$-\underline{\underline{A}}^{-1}\,\underline{\underline{\Gamma}}\,(\underline{\underline{A}}^{T})^{-1}\,\underline{\underline{D}}$$

$$= \begin{pmatrix} S_1 & S_2 & S_3 & \cdots & S_9 & \tfrac{1}{2}S_{10} \\ S_2 & S_2 & S_3 & \cdots & S_9 & \tfrac{1}{2}S_{10} \\ S_3 & S_3 & S_3 & \cdots & S_9 & \tfrac{1}{2}S_{10} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ S_9 & S_9 & S_9 & \cdots & S_9 & \tfrac{1}{2}S_{10} \\ S_{10} & S_{10} & S_{10} & \cdots & S_{10} & \tfrac{1}{2}S_{10} \end{pmatrix}$$

where $S_i = \sum_{j=i}^{10} \beta_j$

5. STABILITY ANALYSIS

A stability analysis of the linearised two dimensional shallow water wave equations

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} + g\frac{\partial \phi}{\partial x} = 0 \qquad (40)$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + g\frac{\partial \phi}{\partial y} = 0 \qquad (41)$$

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + V\frac{\partial \phi}{\partial y} + \frac{c^2}{g}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (42)$$

has been done for both the explicit and the implicit cases on the space staggered, time unstaggered grid of fig (3).

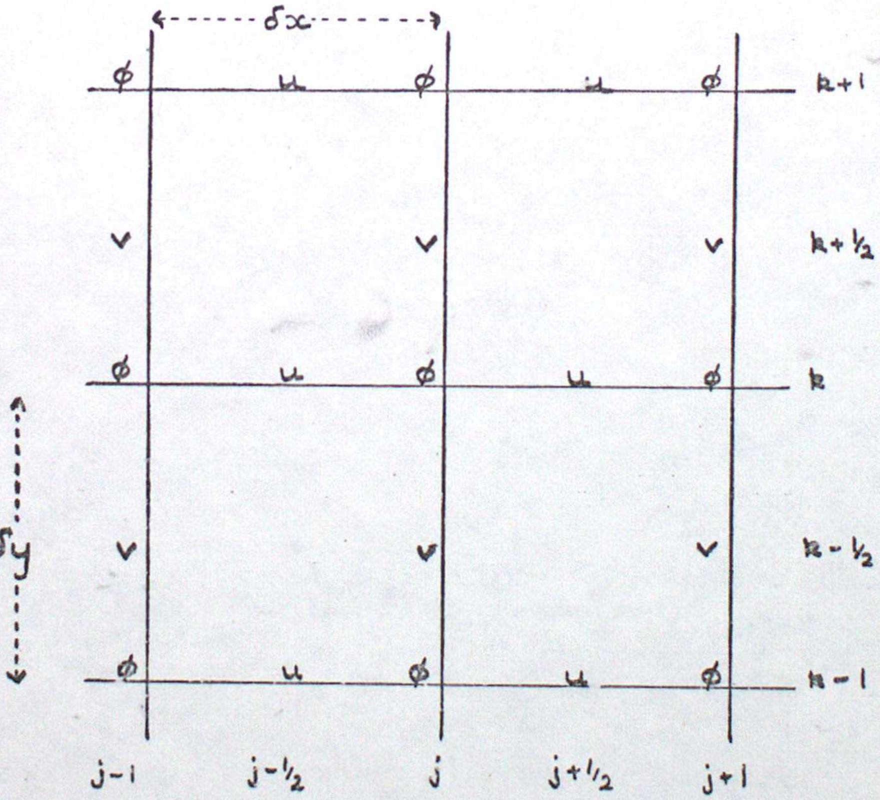


Fig. 3.

Using the finite fourier expansions

$$u(x, y, t) = u_r(t)\, e^{i(nj\delta x + mk\delta y)} \qquad \text{etc}$$

with $t = r\delta t,\ x = j\delta x,\ y = k\delta y$ n and m being the wave numbers in the x and y directions respectively and further assuming $u_r(t) = \omega^r u_o(t)$ where ω is the (complex) amplification factor, one has the following expressions for the derivatives.

$$\left(\frac{\partial u}{\partial t}\right)_{r,j+1/2,k} = \frac{1}{2\delta t}\left(u_{r+1} - u_{r-1}\right)_{j+1/2,k} = \frac{1}{2\delta t}\left(\omega - \omega^{-1}\right) u_r(t)\, e^{in(j+1/2)\delta x}\, e^{imk\delta y}$$

$$\left(\frac{\partial u}{\partial x}\right)_{r,j+1/2,k} = \frac{1}{2\delta x}\left(u_{j+3/2,k} - u_{j-1/2,k}\right)_r = \frac{1}{2\delta x}\, 2i\sin(n\delta x)\, u_r(t)\, e^{in(j+1/2)\delta x}\, e^{imk\delta y}$$

$$\left(\frac{\partial \phi}{\partial x}\right)_{r,j+1/2,k} = \frac{1}{\delta x}\left(\phi_{j+1,k} - \phi_{j,k}\right)_r = \frac{1}{\delta x}\, 2i\sin(\tfrac{1}{2}n\delta x)\, \phi_r(t)\, e^{in(j+1/2)\delta x}\, e^{imk\delta y}$$

etc.

The two cases under consideration are

(i) Explicit case where all space derivatives in equations (40), (41) and (42) are taken at time level r.

(ii) Implicit case where the gravity wave terms $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}$ and the ICAO contribution to $c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ are treated implicitly while the advection terms and the departure from ICAO are treated explicitly.

The equations for case (i) are

$$\frac{\partial u}{\partial t} + U\left(\frac{\partial u}{\partial x}\right)_r + V\left(\frac{\partial u}{\partial y}\right)_r + g\left(\frac{\partial \phi}{\partial x}\right)_r = 0$$

$$\frac{\partial v}{\partial t} + U\left(\frac{\partial v}{\partial x}\right)_r + V\left(\frac{\partial v}{\partial y}\right)_r + g\left(\frac{\partial \phi}{\partial y}\right)_r = 0$$

$$\frac{\partial \phi}{\partial t} + U\left(\frac{\partial \phi}{\partial x}\right)_r + V\left(\frac{\partial \phi}{\partial y}\right)_r + \frac{c^2}{g}\left[\left(\frac{\partial u}{\partial x}\right)_r + \left(\frac{\partial v}{\partial y}\right)_r\right] = 0$$

and for case (ii)

$$\frac{\partial u}{\partial t} + U\left(\frac{\partial u}{\partial x}\right)_r + V\left(\frac{\partial u}{\partial y}\right)_r + \frac{1}{2}g\left[\left(\frac{\partial \phi}{\partial x}\right)_{r+1} + \left(\frac{\partial \phi}{\partial x}\right)_{r-1}\right] = 0$$

$$\frac{\partial v}{\partial t} + U\left(\frac{\partial v}{\partial x}\right)_r + V\left(\frac{\partial u}{\partial y}\right)_r + \frac{1}{2}g\left[\left(\frac{\partial \phi}{\partial y}\right)_{r+1} + \left(\frac{\partial \phi}{\partial y}\right)_{r-1}\right] = 0$$

$$\frac{\partial \phi}{\partial t} + U\left(\frac{\partial \phi}{\partial x}\right)_r + V\left(\frac{\partial \phi}{\partial y}\right)_r + \frac{1}{2}\frac{c_o^2}{g}\left[\left(\frac{\partial u}{\partial x}\right)_{r+1} + \left(\frac{\partial v}{\partial y}\right)_{r+1} + \left(\frac{\partial u}{\partial x}\right)_{r-1} + \left(\frac{\partial v}{\partial y}\right)_{r-1}\right]$$

$$+ \varepsilon\frac{c_o^2}{g}\left[\left(\frac{\partial u}{\partial x}\right)_r + \left(\frac{\partial v}{\partial y}\right)_r\right] = 0$$

where $c^2 = c_o^2(1+\varepsilon)$

On substituting the finite difference approximation for a single fourier mode and eliminating u_0, v_0, and ϕ_0 one obtains in both cases a quartic for ω

Case (i) (explicit case)

$$\left\{ \omega^2 - 1 + 2i\delta t\omega \left[\frac{U}{\delta x} \sin(n\delta x) + \frac{V}{\delta y} \sin(m\delta y) \right] \right\}^2$$

$$= (4ic\delta t\omega)^2 \left[\frac{1}{\delta x^2} \sin^2(\tfrac{1}{2}n\delta x) + \frac{1}{\delta y^2} \sin^2(\tfrac{1}{2}m\delta y) \right]$$

This immediately factorises to give

$$\omega^2 + 2i\lambda\omega - 1 = 0 \qquad (43)$$

where $\lambda = \delta t \left[\frac{U}{\delta x} \sin(n\delta x) + \frac{V}{\delta y} \sin(m\delta y) \right.$

$$\left. \pm 2c \sqrt{\frac{1}{\delta x^2} \sin^2(\tfrac{1}{2}n\delta x) + \frac{1}{\delta y^2} \sin^2(\tfrac{1}{2}m\delta y)} \right]$$

Now the roots of eq (43) are $\omega = -i\lambda \pm \sqrt{1-\lambda^2}$ which satisfy the condition $|\omega| \leq 1$ if and only if $\lambda^2 \leq 1$ (in fact $|\omega| = 1$ if this condition holds)

The strongest inequality is obtained by taking the postive sign if U and V are both positive, and the negative sign if U and V are both negative.

The stability criterion is thus $|\lambda| \leq 1$

Taking $U = V$ and $\delta y = \delta x$ gives

$$\lambda = \frac{\delta t}{\delta x}\left\{U\left[\sin(n\delta x) + \sin(m\delta y)\right] + 2c\sqrt{\sin^2(\tfrac{1}{2}n\delta x) + \sin^2(\tfrac{1}{2}m\delta y)}\right\}$$

It is easily seen that a sufficient condition for $|\lambda| \leq 1$ is that

$$\frac{\delta t}{\delta x} \leq \frac{1}{2U + 2\sqrt{2}\,c}$$

and detailed computation shows that this can be relaxed by about 10%. With U = 100 m sec^{-1}, and δx = 300 km the maximum time step possible is about 5 minutes if c = 300 m sec^{-1} or about 16 mins if c = 50 m sec.$^{-1}$

Inclusion of the map factor results in a lower limit for δt. (Replace δt by $\mu^{1/2}\delta t$ throughout the above analysis.)

Case (ii) (implicit case)

$$\left\{\omega^2 - 1 + 2i\delta t\omega\left[\frac{U}{\delta x}\sin(n\delta x) + \frac{V}{\delta y}\sin(m\delta y)\right]\right\}^2$$

$$= (2ic_0\delta t)^2(\omega^2+1)(\omega^2+2\varepsilon\omega+1)\left[\frac{1}{\delta x^2}\sin^2(\tfrac{1}{2}n\delta x) + \frac{1}{\delta y^2}\sin^2(\tfrac{1}{2}m\delta y)\right]$$

Computation shows that provided $-1 \leqslant \varepsilon \leqslant +1$ the roots of this equation satisfy $|\omega| = 1$ ie the scheme is unconditionally stable, but for $|\varepsilon| > 1$ one root has $|\omega| > 1$ so the scheme is unconditionally unstable.

In practice $|(c^2 - c_0^2)/c_0^2| = |\varepsilon|$ is small for the high speed gravity modes, but can become greater than unity $(c^2 > 2c_0^2)$ for the slow moving modes. Thus one should <u>not</u> consider all the modes implicitly in the centred leap-frog scheme.

$\varepsilon < -1$ (ie. $c^2 < 0$) corresponds to static instability.

6. PROGRAMMING CONSIDERATIONS

Buffer Arrangement

The buffers are arranged as in fig (4). Three lines of the octagon are required for the calculation of the advection step, and one line for the calculation of the implicit adjustment.

Some preliminary remarks are necessary :

(1) Register 6 contains the address of a list of constants which can be referred to either by their individual labels eg MID888 or as a displacement on from register 6 eg 888(6). The contents of register 6 is never changed (except in OCTHH where the MID list is not referred to)

(2) 888(6) contains the line number of the octagon row currently being read into the buffer. On entry to OLDMAIN, register 10 contains this value.

(3) 656(6) contains the number of time steps completed. On entry to OLDMAIN, register 11 contains this value.

(4) The line buffers in fig (4) are referenced by their addresses, which are stored in the MID list. The addresses are cycled when a new line of the octagon is read in. The displacements of the three working lines (TOP,MID,BOT) are also stored in the MID list.

With reference to fig (4), for example, 44(6) contains the address of the buffer currently containing the TOP line of n-1 time step. 892(6), 896(6) and 900(6) contain the displacements in bytes from the beginning of the buffer of the (first point -1), (last point -1) and (last point -2) of the TOP line of n-1, n or n+1 time step.

There follows a brief description of each CSECT

OMAIN is the controlling program for the forecast (mainly OMLOOP)

and also controls all the I/O (using RVDATA,RHDATA,OCTFW,CFWRITE)

Briefly as follows :

(1) Reads start data set (this includes the time step DT)

(2) Sets up the MID constants eg line lengths, line displacements, initial addresses of line buffers

(3) Reads initial data set line by line into store and transfers to fixed head disk (FHD). For a start (as opposed to a restart) each line is written twice. For a restart, two initial data sets are required (at time levels n-1 and n) and both are transferred together as line pairs onto fixed head disk.

All subsequent reads and writes from and to FHD transfer a line pair of data.

(4) Reads line 1 of octagon from FHD into store

(5) Reads line 2

(6) Reads line 3, sets 888(6) = 3

(7) Calls OLDMAIN which cycles buffer addresses - Now line 1 is in BOT.

(8) Reads line 4, sets 888(6) = 4

(9) Calls OLDMAIN which cycles buffer addresses - Now lines 1, 2 are in MID,BOT respectively.

(10) Reads line 5, sets 888(6) = 5

(11) Calls OLDMAIN which cycles buffer addresses. Now lines 1,2,3 are in TOP,MID,BOT

(12) Reads line 6 sets 888(6) = 6

(13) Calls OLDMAIN which cycles buffer addresses - Now lines 2,3,4 are in TOP,MID,BOT

(14) Enters OMLOOP, writes line 1, reads line 7, sets 888(6) = 7, calls OLDMAIN which cycles buffer addresses - Now lines 3,4,5, are in TOP,MID,BOT

(15) Continues to cycle OMLOOP until end of forecast.

Write ups are done line by line from OFWRITE to two data sets VDATAMnn (time level n) and VDATAMXi (time level n-1)

nn is a data set reference number calculated from NUMDD given in the start data set.

i is alternatively 0 or 1 at succesive write up times.

The buffers which are written up are those containing BOT and OFWRITE is called after all the computation has been completed for that line.

The start data set is updated in OFWRITE after the last line of the octagon has been written.

OLDMAIN	cycles buffer addresses, sets up displacments in locations 892(6) to 924(6) and controls calls to all subroutines including OFWRITE.
OCTEM	is the controlling subroutine for computation of the advection step. Uses TOP,MID,BOT rows of n time step and puts result in MID row n+1 time step. This routine is not called when MID contains line 1 or line 61 of octagon. OCTEM calls ADDTHU and ADDTHV once for each point of the row, and calls EVENU, EVENV, EVENTH, EVENR and EVNH1000 separately for each level at each point.
ADDTHU	computes heights from thickness for MID(n) and stores in sigma work space.
ADDTHV	computes heights from thickness for BOT(n) and stores in sigma work space.
EVENU	computes advection increment in u for all terms in eq(1) including $g\frac{\partial h}{\partial x}$
EVENV	" " " " v " " " " eq(2) " $g\frac{\partial h}{\partial y}$
EVENR	" " " " r " " " " eq(4)
EVENTH	" " " " h' " " " " eq(8a)
EVNH1000	" " " " h_{10} " " " " eq(8b)
OCTEW	computes ω from eq (14). It is called twice, firstly using MID,BOT(n) to compute ω^n for BOT, and secondly using TOP,MID(n+1) to compute $\hat{\omega}$ for MID. ω cannot be computed for line 1 or line 61 of octagon.
OCTREP	moves the complete 80 word column from MID(n-1) to MID(n+1). It also 9 point smoothes the surface exchange increments in the n-1 time step and puts the result in the n+1 time step columns.
EXPADJ	computes the right hand sides R_i $(i=1,2)$ of the Helmholtz equations and stores the results in EXPBUFF. It also computes $\hat{\psi}_i^*$ $(i=1,2)$ (needed in IMPHTS) and stores in words 73, 74 of the column. It uses MID (n-1, n, ^) and is called for all lines of the octagon. It replaces

thickness in the column by heights and adds on the explicit part of the term $\beta_I \omega^n$ (see equations (3), (8) and (23)) and also the surface exchange thickness increment.

IMPMAST is the controlling routine for the computation of the implicit terms. It uses BOT(n), and is called for all lines of the octagon.

IMPMAST calls IMPU, IMPV, IMPHTS, IMPTH.

IMPU adds to $\hat{u}^*$ the adjustment term (see eq (38)) to obtain u^{n+1}

IMPV " " $\hat{v}^*$ " " " (see eq (39)) " " v^{n+1}

IMPHTS " " $\hat{h}^*$ " " " (see eq (37)) " " h^{n+1}

IMPTH replaces heights in column by thickness

TMEAN1, THEAN2 are time smoothing routines. u, v, h', h_0, ω, r are all smoothed like

$$\tilde{\phi}^n = \phi^n + \alpha(\tilde{\phi}^{n-1} - 2\phi^n + \phi^{n+1})$$

in two stages

TMEAN2 does $\phi'^n = \phi^n + \alpha(\tilde{\phi}^{n-1} - 2\phi^n)$

TMEAN1 does $\tilde{\phi}^{n-1} = \phi'^{n-1} + \alpha\phi^n$

$\alpha = 0.005$ is used.

NSHT, EWHT, MOVEH are boundary smoothing routines.

OCTPH stores the thickness (after adding an ICAO values) in sigma work space, calls CONDEVAP and CONVECT, and restores the new thickness (less ICAO values) back in column. It uses BOT(n).

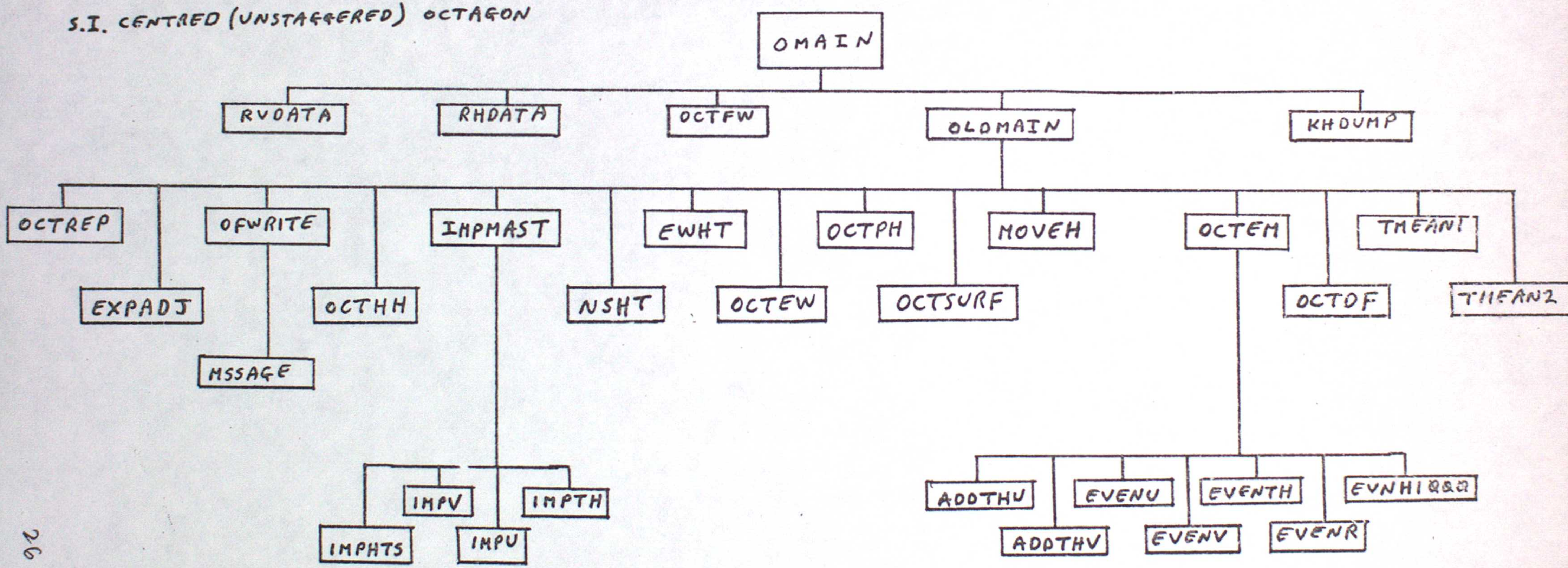
OCTSURF computes the surface exchange thickness and humidty increments and stores them in the 80 word column. It uses BOT(n-1)

OCTDF computes the diffusion (for u, v, h, r) and surface friction (for u_{1000}, v_{1000}) It uses TOP,MID,BOT(n-1) to compute increments which are added to MID(^), and is not called for the boundary points.

		n-1	n	$\widehat{n+1}$	
WRITE					
TOP		44(b)	20(b)	56(b)	892/896/900
MID		48(b)	24(b)	60(b)	904/908/912
BOT		52(b)	28(b)		916/920/924
READ					
READ	888(b)				

fig (4)

S.I. CENTRED (UNSTAGGERED) OCTAGON



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