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A VIEW OF THE EQUATIONS OF METEOROLOGICAL DYNAMICS AND VARIOUS APPROXIMATIONS

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September 2000

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Summary

This article gives an account of many of the sets of equations used by theorists and numerical modellers working in meteorological dynamics. It has been written for mathematicians and physicists who want a compact introduction to the subject rather than the more extensive treatments to be found in good contemporary textbooks on meteorology. Attention is also paid to various recent developments that have received little exposure outside the research literature yet. The main areas covered are: fluid kinematics, the relevant equations of fluid dynamics and thermodynamics, and the approximate versions that are the basis of many weather forecasting and climate simulation models and of a wide range of associated theoretical studies. The approximate models include the hydrostatic primitive equations, the shallow water equations, the barotropic vorticity equation, several approximately-geostrophic models and some acoustically-filtered models that permit buoyancy modes. Conservation properties and frame invariance are given emphasis, but Hamiltonian methods of derivation are noted only in verbal summary. Elementary discussions of the hydrostatic and geostrophic approximations and their repercussions are included, as well as brief accounts of various vertical coordinate systems and local approximations to the quasi-spherical form of the Earth. A straightforward problem of small-amplitude wave motion in a rotating, stratified, compressible atmosphere is addressed in detail, with particular attention to the occurrence or non-occurrence of acoustic, buoyancy and planetary modes in the approximate models. The concluding section contains a short discussion of basic issues in numerical model construction.

1. Introduction

One of the attractions of meteorology is its many-faceted character. It invites study by mathematicians and statisticians as well as by physicists of either practical or theoretical disposition. Amongst other fields, its concerns border or overlap those of oceanography, geophysics, environmental science, biological science, agriculture and human physiology, and impinge on those of economics, politics and psychology. (Climatology, for present purposes, is counted as part of meteorology.) Its breadth can lead to a perception that meteorology is a 'soft' science. This article focuses on part of the subject's 'hard' core: the equations governing atmospheric flow, and the approximate forms used by many numerical modellers and theorists.

A discussion (in section 3) of the basic equations of meteorological dynamics is preceded by a glance at a pre-Newtonian but fundamental subject: fluid kinematics (section 2). Some of the conservation laws which the basic equations express or imply are examined in section 4. Subsequent sections deal with approximate versions of the basic equations. Consistent approximation is one of the mathematical challenges of meteorology, and the sheer range of possible (and permissible?) approximations can be a bewildering feature. The hydrostatic approximation, the hydrostatic primitive equations (HPEs) and the shallow water equations (SWEs) are considered in section 5. The HPEs are the basis of many of the numerical models used worldwide in weather forecasting and for climate simulation, and the SWEs are widely studied as a testbed for further approximations and for numerical schemes.

We pause in section 6 to discuss various vertical coordinate systems, and various approximations of Coriolis effects and the Earth's sphericity beyond those associated with the HPEs. The geostrophic approximation is considered in a diagnostic (non-evolutionary) sense in section 7. Atmospheric wave motion is discussed in linear analytical terms in section 8 – we identify acoustic, gravity (buoyancy) and Rossby (planetary) waves and note the existence of special tropical modes.

Approximations of the HPEs which result in the removal of gravity waves as well as acoustic waves are considered in section 9; the shallow water equations are a convenient vehicle for most of this discussion. The quasi-geostrophic model, QG1, is singled out for particular attention in section 10. QG1 is one of the coarsest of those models that allow time-evolution of synoptic-scale weather systems (the "Lows" and "Highs" of the weather forecaster's chart), but it succeeds in representing most of the physical content of more quantitatively accurate models. Its importance in the conceptual development of meteorological dynamics can hardly be over-stated.

In section 11 are discussed various models (other than the HPEs) which allow gravity waves but not acoustic waves. Section 12 gives a brief survey of issues in numerical modelling for weather forecasting and climate simulation, and offers some concluding remarks.

The article is based on three lectures given during various phases of the Isaac Newton Institute programme on "Mathematics of Atmosphere and Ocean Dynamics" (December 1994, July 1996, December 1997). Its approach is elementary in so far as Hamiltonian methods are noted only in brief verbal summary; they are treated at proper length elsewhere in this volume. Much of the material is mainstream, and is covered in greater depth in the texts by Lorenz (1967), Phillips (1973), Haltiner and Williams (1981), Gill (1982), Pedlosky (1987), Lindzen (1990), Carlson (1991), Daley (1991), Holton (1992), Bluestein (1992), James (1994), Dutton (1995) and Green (1999), amongst others. Some new interpretations are presented, however, and later sections deal increasingly with developments which have not yet reverberated outside the research literature. Results that are thought to be new include: a bisection theorem relating the principal directions of curvature of the height field and the dilatation axis in geostrophic flow; a geometric solution of an acoustic/gravity wave dispersion relation; and a fresh perspective on the aptly-named "omega equation" of QG1. Sections 5.5 and 8.2 contain material covered in unpublished course notes by R W Riddaway and J S A Green – notes to which I have been fortunate to have had access both as student and lecturer.

In mathematical respects, meteorological and oceanographic dynamics have much in common, and the atmosphere and oceans are closely-interacting systems, especially on climatological time-scales, but – in the interests of brevity – this article will refer only incidentally to oceanography and the oceans.

2. Fluid kinematics

Deformability is a key feature of a fluid: except in certain very simple flows, particles do not retain the fixed relative spatial relationships that are characteristic of a rigid body in motion. Our discussion in this section draws on the treatments given by Batchelor (1967), Wiin-Nielsen (1973), Ottino (1990) and Bluestein (1992).

Consider the motion of a fluid in two spatial dimensions relative to Cartesian axes Oxy; see Fig 1(a). Suppose that the velocity field $\mathbf{v} = \mathbf{v}(x,y,t) = (u(x,y,t), v(x,y,t))$ varies smoothly in space and time, so that the derivatives u_x, u_y, v_x, v_y are well defined, at least in the neighbourhood of a chosen point $P = (x_0, y_0)$ and time t_0. If a particle which is at point $Q = (x_0 + \delta x, y_0 + \delta y)$ at t_0 is at $(x_0 + \Delta x, y_0 + \Delta y)$ a short time Δt later, then it follows (from the definition of velocity as rate of change of position) that:

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = \begin{pmatrix}\delta x\\ \delta y\end{pmatrix} + \begin{pmatrix}u\\ v\end{pmatrix}\Delta t + \begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}\Delta t \,. \tag{2.1}$$

Here u, v and their first derivatives are evaluated at (x_0, y_0, t_0), and higher order terms in the Taylor expansion of $\mathbf{v}$ about (x_0, y_0, t_0), have been neglected. The second term on the right side of (2.1) represents translation with the flow at point $P = (x_0, y_0)$. Measuring position $(\delta x', \delta y')$ in a Cartesian system $O'x'y'$ (Fig 1(a)) moving with this translation velocity (i.e. $\delta x' = \Delta x - u\Delta t$, $\delta y' = \Delta y - v\Delta t$) gives

$$\delta\mathbf{x}' = \mathbf{A}\,\delta\mathbf{x}$$

where

$$\mathbf{A} = \begin{pmatrix}1+u_x\Delta t & u_y\Delta t\\ v_x\Delta t & 1+v_y\Delta t\end{pmatrix} \tag{2.2}$$

and

$$\delta\mathbf{x} = (\delta x, \delta y)\;; \qquad \delta\mathbf{x}' = (\delta x', \delta y').$$

Define divergence δ, vorticity ζ and deformation components D_1 , D_2 as

$$\begin{aligned} \delta &= u_x + v_y\;; \quad \zeta = v_x - u_y\;;\\ D_1 &= u_x - v_y\;; \quad D_2 = v_x + u_y\,. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3):

$$\mathbf{A} = \mathbf{I} + (\mathbf{R} + \mathbf{S} + \mathbf{D})\Delta t \tag{2.4}$$

where **I** is the unit diagonal matrix, and

$$2\mathbf{R} = \begin{pmatrix} 0 & -\zeta \\ \zeta & 0 \end{pmatrix}; \qquad 2\mathbf{S} = \mathbf{I}\delta \;; \qquad 2\mathbf{D} = \begin{pmatrix} D_1 & D_2 \\ D_2 & -D_1 \end{pmatrix} \tag{2.5}$$

Also, to first order in Δt, **A** can be expressed as the product of three matrices:

$$\mathbf{A} = (\mathbf{I} + \mathbf{R}\Delta t)(\mathbf{I} + \mathbf{S}\Delta t)(\mathbf{I} + \mathbf{D}\Delta t) + O(\Delta t^2) = \hat{\mathbf{R}}\hat{\mathbf{S}}\hat{\mathbf{D}} + O(\Delta t^2)$$

with

$$\hat{\mathbf{R}} = \begin{pmatrix} 1 & -\frac{1}{2}\zeta\Delta t \\ \frac{1}{2}\zeta\Delta t & 1 \end{pmatrix}; \quad \hat{\mathbf{S}} = \left(1 + \frac{1}{2}\delta\Delta t\right)\mathbf{I} \;; \quad \hat{\mathbf{D}} = \begin{pmatrix} 1 + \frac{1}{2}D_1\Delta t & \frac{1}{2}D_2\Delta t \\ \frac{1}{2}D_2\Delta t & 1 - \frac{1}{2}D_1\Delta t \end{pmatrix} \tag{2.6}$$

Consider particles which formed a *circle* centred on (x_0, y_0) at time t_0; see Fig 1(b). It is readily shown that the matrices $\hat{\mathbf{R}}$, $\hat{\mathbf{S}}$ and $\hat{\mathbf{D}}$ correspond respectively to (infinitesimal) *rotation*, *scaling* and *deformation* of the circle of particles over the time interval $[t_0, t_0 + \Delta t]$.

The rotation ($\hat{\mathbf{R}}$) is associated with vorticity (ζ), and corresponds to a turning of the initial circle through an angle $\frac{1}{2}\zeta\Delta t$ counterclockwise. The scaling ($\hat{\mathbf{S}}$) is associated with divergence (δ); it represents an isotropic change of size (a uniform magnification or minification) in which the radius of the circle changes by a factor $(1 + \frac{1}{2}\delta\Delta t)$.

The deformation ($\hat{\mathbf{D}}$) corresponds to a change of shape: the initial circle becomes an ellipse. The major axis of the ellipse (the stretching or dilatation axis) is inclined to the x axis at an angle $\frac{1}{2}\tan^{-1}(D_2/D_1)$. If the initial radius of the circle is chosen as the unit of distance, the semi-major axis of the ellipse is $1 + \frac{1}{2}D\Delta t$, where $D^2 = D_1^2 + D_2^2$ is the square of the total deformation; and the semi-minor axis of the ellipse, the contraction axis of the initial circle, is $(1 - \frac{1}{2}D\Delta t)$. The magnitudes and directions of the major and minor axes are given by the eigenvalues and eigenvectors of $\hat{\mathbf{D}}$. (The eigenvalues of **D** are $\pm\frac{1}{2}D\Delta t$. Its eigenvectors too are parallel to the axes of stretching and contraction.) Area is preserved, to order Δt, during the deformation.

As illustrated in Fig 1(b), the evolution of the initial circle of particles is (for small Δt) a combination of (i) translation, (ii) rotation, (iii) scaling, and (iv) deformation (to an ellipse).

In analytical terms, particle locations in the neighbourhood of (x_0, y_0) are transformed into locations in the neighbourhood of $(x_0 + u\Delta t, y_0 + v\Delta t)$ according to an infinitesimal general (non-conformal, non-isometric) mapping; see Klein (1938), p 105. The details of the mapping are determined by the first derivatives of u and v in the neighbourhood of (x_0, y_0).

The matrices **R**, **S** and **D** defined by (2.3) and (2.5) together constitute a decomposition of the 2D velocity gradient tensor **T**:

$$\mathbf{T} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \quad (= \text{grad}_2 \mathbf{v}) \tag{2.7}$$

R (sometimes called *the body spin matrix*) is the skew-symmetric part of **T**; **S** + **D** is the symmetric part of **T** (*the Eulerian rate of strain matrix*). Vorticity (associated with **R**) is seen to be essentially a rigid body property. Deformation (associated with **D**) is essentially a non-rigid body property; the same statement could be made about the divergence (associated with **S**), and some authors treat divergence as a special kind of deformation.

For 3-dimensional flow $\mathbf{u} = \mathbf{u}(x,y,z,t) = (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$, the treatment may be repeated for an initial *sphere* of particles and 3-D velocity gradient tensor **T**:

$$\mathbf{T} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} \quad (= \text{grad}\, \mathbf{u}) \tag{2.8}$$

The results are similar to those of the 2-D case, though more complicated in analytical terms. The sphere undergoes a translation, a rotation, a scaling and a deformation to an ellipsoid. The rotation is through an angle $\frac{1}{2}|\mathbf{Z}|\Delta t$ about the direction of $\mathbf{Z} = \text{curl}\mathbf{u}$ (the vorticity vector), and the scaling is $1 + \frac{1}{2}\Delta t \text{div}\mathbf{u}$. The deformation is specified by the orientation and magnitude of the principal axes of the ellipsoid. In general, there is a stretching axis, a contraction axis and an intermediate axis, which may be an axis of contraction or stretching; degenerate cases may occur. The components of the deformation (not given here) determine the orientation and size of the principal axes and the extents of the stretching and contraction. The spatial relationship of the velocity and vorticity vectors to the principal axes of the deformation ellipsoid will be of general kinematic and dynamic importance.

Tensor considerations obviously enter fluid dynamics at a pre-Newtonian level. The tensorial character of flow kinematics is evident also on direct physical grounds from a consideration of the effect of a deformation on a pre-existing gradient of some conserved scalar field. Figure 1(c) (representing a 2-D case) shows that a pre-existing gradient perpendicular to the stretching axis increases as a consequence of the deformation, whereas a gradient parallel to the stretching axis decreases. A pre-existing gradient at 45° to the stretching axis remains unchanged in magnitude. These effects are important in the formation of *fronts* – regions of large horizontal gradients of temperature and other properties – in the atmosphere and oceans [see Hoskins (1982) and Hewson (1998)]. Tensor considerations also play an important role in the proper representation of viscous effects, in the analysis of interactions between eddies and mean flows, and in the parametrization of subgridscale Reynolds stresses in numerical models; see Williams (1972), Hoskins *et al.* (1983) and Adcroft and Marshall (1998). It turns out, however, that vorticity – a vector quantity – figures more prominently than deformation in the dynamics of meteorological flows. Although we shall refer again in this article to deformation, the bulk of the treatment will involve nothing more complicated than vector analysis and the manipulation of vector differential operators.

3. FLUID DYNAMICS AND THERMODYNAMICS

This section gives an elementary account of those equations of thermodynamics and fluid dynamics from which the future state of the atmosphere may be forecast, given its present state.

3.1 Local and total time derivatives; advection

Consider some meteorological field $\Im$. $\Im$ might be a scalar quantity, such as temperature, or a vector, such as the flow velocity $\mathbf{u}$. Assume that $\Im$ is a function of time t and position $\mathbf{r}$ in some chosen coordinate frame:

$$\Im = \Im(\mathbf{r}, t)$$

Assume also that $\Im(\mathbf{r}, t)$ is differentiable with respect to each argument. Then first-order Taylor expansion of $\Im$ about $\Im(\mathbf{r}, t)$ gives

$$\delta\Im \equiv \Im(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - \Im(\mathbf{r}, t) = (\delta\mathbf{r}.\mathrm{grad})\Im + (\partial\Im/\partial t)\delta t \qquad (3.1)$$

Equation (3.1) applies to any (infinitesimal) choice of $\delta\mathbf{r}$, δt. Choose $\delta\mathbf{r}$ to be the displacement in time δt corresponding to the velocity $\mathbf{u}$ of the air currently at position $\mathbf{r}$. Then $\delta\mathbf{r}/\delta t = \mathbf{u}$, and (3.1) becomes

$$\frac{D\Im}{Dt} \equiv \frac{\delta\Im}{\delta t} = (\mathbf{u}.\mathrm{grad})\Im + \frac{\partial\Im}{\partial t} \qquad (3.2)$$

$D\Im/Dt$ is the rate of change of $\Im$ following a parcel of air; it is known as the *total* (or material, or substantial, or individual, or Lagrangian) time derivative of $\Im$. $\partial\Im/\partial t$ is the *local* (or Eulerian) time derivative of $\Im$; it is the rate of change of $\Im$ at a point fixed in the chosen coordinate frame.

Some important physical laws (such as Newton's second law of motion) give information about material time derivatives. The users of weather forecasts are usually – not always – interested in the consequences of the local rate of change of $\Im$. A Grampian farmer may wish to know what the temperature of the air in the neighbourhood of the farm will be tomorrow, but is unlikely to want to know what the temperature of the air which is at the farm now will be tomorrow; that body of air may be over the North Sea by then. Hence the trivial re-expression of (3.2) as

$$\frac{\partial\Im}{\partial t} = \frac{D\Im}{Dt} - (\mathbf{u}.\mathrm{grad})\Im \qquad (3.3)$$

is of fundamental importance in meteorology. Within its generality, (3.3) expresses the key physical notion that when $\Im$ is conserved on fluid particles ($D\Im/Dt = 0$) the value of $\Im$ at a fixed point in our coordinate frame will nevertheless be changing ($\partial\Im/\partial t \neq 0$) if fluid having a different value of $\Im$ is being brought in, or *advected*, by the flow ($-(\mathbf{u}.\mathrm{grad})\Im \neq 0$). The term $-(\mathbf{u}.\mathrm{grad})\Im$ represents the (rate of) advection of $\Im$. A vexed issue of terminology will be side-stepped in this article by using the expression "advection term" to describe both $-(\mathbf{u}.\mathrm{grad})\Im$ (as in (3.3)) and $+(\mathbf{u}.\mathrm{grad})\Im$ (as in (3.2)).

We now consider how various choices of $\Im$, and the application of various physical laws, lead to expressions for the local rates of change of meteorological fields. With the needs of our Grampian farmer in mind, we begin by choosing $\Im = T =$ temperature.

3.2 First law of thermodynamics

Suppose that a parcel of air having unit mass, temperature T and (specific) volume α undergoes a change of (specific) entropy δs. According to the first law of thermodynamics, the concomitant changes δT and $\delta\alpha$ of T and α are related by

$$c_v \delta T + p\delta\alpha = T\delta s \tag{3.4}$$

Here c_v is the specific heat at constant volume and p is the pressure of the parcel of air. Since the first law of thermodynamics applies to the parcel of air as it moves, it follows from (3.4) that

$$c_v \frac{DT}{Dt} + p\frac{D\alpha}{Dt} = T\frac{Ds}{Dt} \equiv Q \tag{3.5}$$

In meteorology, $Q \equiv T\,Ds/Dt$ is usually thought of as the total heating rate per unit mass; strictly, it is the heating rate that would achieve, by reversible processes, the same rates of change of T and α as those occurring in the actual irreversible system (Lorenz 1967, p14). Equation (3.5) can be written in terms of density $\rho\,(= 1/\alpha)$ as

$$c_v \frac{DT}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = Q \tag{3.6}$$

In either form, however, the first law of thermodynamics gives only a relationship between the material derivatives of T and a density variable.

3.3 Mass continuity

Information about the material derivative of density, $D\rho/Dt$ (see (3.6)), may be obtained from mass conservation. The mass within a volume τ (fixed relative to the chosen coordinate frame) changes only to the extent that there is net inflow or outflow of mass at the boundary S of the volume. Hence

$$\frac{\partial}{\partial t}\int_\tau \rho d\tau = -\int_S \rho\mathbf{u}.\mathbf{dS} = -\int_\tau \mathrm{div}\rho\mathbf{u}d\tau \tag{3.7}$$

by the divergence theorem. Equation (3.7) applies to any volume τ, so the local equality

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\rho\mathbf{u} = 0 \tag{3.8}$$

must hold. Equation (3.8) is a form of the (mass) continuity equation. By using (3.3), we may deduce an alternative form:

$$\frac{D\rho}{Dt} + \rho\mathrm{div}\mathbf{u} = 0 \tag{3.9}$$

3.4 Perfect gas law

If taken together with (3.3) in the form

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - (\mathbf{u}.\mathrm{grad})T ,$$

(3.6) and (3.9) enable us to evaluate the local rate of change $\partial T/\partial t$ so long as we know the current values of Q, p, ρ and the flow vector $\mathbf{u}$. The current value of ρ can be found from observations of p and T by using the perfect gas law in the form

$$p = \rho R T \tag{3.10}$$

where R is the gas constant per unit mass. Eq (3.10) has no time derivatives. In meteorological parlance, it is a *diagnostic* equation; equations involving time derivatives are called *prognostic*. We have now set up the apparatus to evaluate $\partial T/\partial t$, and hence (knowing the current values of T, p and $\mathbf{u}$) to calculate T at our chosen location at a later time $t + \delta t$. If we were content to take δt = 24 hours, we could calculate an expected value of T at the chosen location tomorrow. The calculated value would probably be very inaccurate, and for this reason (and others) the calculation of a 24-hour temperature forecast proceeds in practice by performing a number of *time steps* δt which are much shorter than 24 hours. (Typically, the time steps are of the order of 10 minutes). This process requires values of Q, p, ρ and $\mathbf{u}$ at each time step. Hence we require a prognostic equation for the flow $\mathbf{u}$; in general, we cannot forecast the temperature accurately for more than (say) an hour ahead without forecasting the flow too. In any case, many users of weather forecasts – including our Scottish farmer, if there are new lambs on the hill – will want to know what tomorrow's wind speed and direction are likely to be.

3.5 Newton's second law

Newton's second law of motion relates the inertial acceleration of an element of air to the net force acting on it. Contributory forces include the pressure gradient force, gravity, and friction. If (as is usually convenient) velocities and accelerations are measured relative to the rotating frame of the solid Earth, Coriolis and centrifugal 'forces' must be introduced to allow for the transformation from inertial to accelerating (rotating) frame; see Stommel and Moore (1989) and Persson (1998) for discussion.

The Lagrangian rate of change of the velocity $\mathbf{u}$ of an element of air, relative to the rotating Earth, is then given by

$$\frac{D\mathbf{u}}{Dt} = \underbrace{-2\boldsymbol{\Omega}\times\mathbf{u}}_{\text{Coriolis}} \;-\; \underbrace{\alpha\,\mathrm{grad}\,p}_{\text{Pressure gradient}} \;-\; \underbrace{\mathrm{grad}\,\Phi}_{\text{Apparent gravity}} \;+\; \underbrace{\mathbf{F}}_{\text{Friction and all other forces}} \tag{3.11}$$

Eq (3.11) is the Navier-Stokes equation for motion and acceleration relative to the Earth, whose rotation vector is $\boldsymbol{\Omega}$. "Apparent gravity", with potential function Φ, consists of the contribution (dominant in the atmosphere) of true Newtonian gravity and the contribution of the centrifugal force $-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$; here $\mathbf{r}$ is position vector relative to a frame rotating with the Earth, and having its origin at the centre of the Earth – see Fig 2. In (3.11) all forces are expressed per unit mass of air.

Equation (3.11) may be used in conjunction with

$$\frac{\partial\mathbf{u}}{\partial t} = \frac{D\mathbf{u}}{Dt} - (\mathbf{u}.\mathrm{grad})\mathbf{u} \tag{3.12}$$

(the appropriate form of (3.3)) to give an expression for the local rate of change of $\mathbf{u}$, i.e. $\partial\mathbf{u}/\partial t$. The advection term $-(\mathbf{u}.\mathrm{grad})\mathbf{u}$ is nonlinear in $\mathbf{u}$; the pressure gradient term, $\alpha\,\mathrm{grad}\,p$, is also in a certain sense nonlinear (as are the advection terms which arise from the first law of thermodynamics and the continuity equation).

3.6 The full set of forecasting equations

An audit of (3.6), (3.9), (3.10) and (3.11), together with appropriate forms of (3.3), shows that we have six equations from which p, ρ, T and the three components of $\mathbf{u}$ may be forecast by repeated time-stepping, so long as friction $\mathbf{F}$ and heating rate Q are known. For convenience and future reference we gather together the relevant equations:

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}.\mathrm{grad})\mathbf{u} - 2\mathbf{\Omega}\times\mathbf{u} - \alpha\,\mathrm{grad}\,p - \mathrm{grad}\,\Phi + \mathbf{F} \tag{3.13}$$

$$c_v\frac{\partial T}{\partial t} = -c_v(\mathbf{u}.\mathrm{grad})T - \frac{p}{\rho}\mathrm{div}\,\mathbf{u} + Q \tag{3.14}$$

$$\frac{\partial \rho}{\partial t} = -(\mathbf{u}.\mathrm{grad})\rho - \rho\,\mathrm{div}\,\mathbf{u} \tag{3.15}$$

$$p = \rho RT \tag{3.16}$$

To obtain (3.14) we have used the continuity equation (3.15) for $D\rho/Dt$.

Numerical techniques are needed for the practical time integration of (3.13) – (3.16). Hence only finite spatial and temporal resolution is possible. This means that Q and $\mathbf{F}$ include the effects of unresolved motions as well as physical processes such as radiative flux convergence, latent heat release/uptake and friction. The difficulties thus introduced are various and profound; see section 12 and Cullen (2000) for further discussion.

In addition to (3.14) – (3.16) and approximations to the components of (3.13), climate simulation models and many weather prediction models include prognostic equations for the local concentration of water substance in some or all of its phases. Water substance is a key quantity in practice – not only because humidity, cloud and precipitation are important meteorologically and climatologically – but because its distribution has a central effect on the distribution of the heating rate Q. We shall not discuss water conservation equations further. Neither shall we treat the variations in gas constant R and principal specific heats c_p and c_v which accompany variations in the amount of water substance present; Gill (1982) gives a concise account.

The equations (3.13) - (3-16) may be written in many alternative forms by using either other equations of the set, or various thermodynamic relations. One of the most important is an alternative form of the thermodynamic equation involving the potential temperature θ defined by

$$\theta = T(p_{ref}/p)^{R/c_p} \tag{3.17}$$

Here p_{ref} is a reference pressure (conventionally 1000hPa) and c_p is the specific heat at constant pressure. θ is the temperature that an element of air would have if it were to be brought adiabatically and reversibly to pressure p_{ref}. In terms of θ, (3.14) takes the simpler form

$$\frac{D\theta}{Dt} = \left(\frac{\theta}{Tc_p}\right)Q \tag{3.18}$$

(upon use of (3.15), (3.16) and the relation $c_p - c_v = R$). From (3.18) it is clear that θ remains constant following an element of air if the motion is adiabatic ($Q = 0$). θ is related to the specific entropy s by $\ln\theta = s/c_p$.

Potential temperature, a thermodynamic quantity, is conserved in adiabatic flow. A dynamic/ thermodynamic quantity that is conserved in adiabatic, frictionless flow is potential vorticity, which is of central importance in meteorology. We discuss potential vorticity in the next section.

A useful alternative form of (3.17) arises if (3.16) is used to eliminate T:

$$\ln\theta \quad \left[= \ln T + \frac{R}{c_p}\ln\left(\frac{p_{ref}}{p}\right)\right] \quad = \quad \frac{1}{\gamma}\ln p - \ln\rho + \text{constant} \tag{3.19}$$

Here $\gamma \equiv c_p / c_v$, and the relation $c_p - c_v = R$ has again been used.

Eq (3.13) provides 3 prognostic equations (for the three components of **u**). It is usual to define the vertical direction by $\nabla\Phi$, the gradient of apparent geopotential; this direction, which is known as *apparent vertical*, is the direction indicated by a plumb line hanging at rest relative to the Earth. [Since both apparent gravity and the direction of apparent vertical depend on the rotation rate of the coordinate frame, which we have chosen to be that of the Earth, they are both frame-dependent quantities.] Also, the slightly spheroidal geopotential surfaces are customarily represented by spheres – an approximation which is amply justified by the smallness (for terrestrial parameter values) of the centrifugal contribution to apparent gravity; see Gill (1982) and White (1982). Convenient horizontal coordinates are then latitude ϕ and longitude λ; see Fig 2. Isolating the three components of (3.13) is not straightforward because the unit vectors change direction over the sphere and so metric (curvature) terms arise. The results are well-known (see Phillips 1973), but we postpone presentation of them until section 4, where conservation properties will be used to provide a rationalisation.

4. CONSERVATION PROPERTIES

Eqs (3.13), (3.14) and (3.15) express conservation of momentum, thermodynamic energy and mass. Other quantities obey other conservation laws, and all such laws appear in various forms expressing, for example, the budget of a quantity in a fixed finite or infinitesimal volume (Eulerian form) or in an identifiable mass of fluid (Lagrangian form). When approximate versions of the governing equations are being set up, the fate of the conservation properties is naturally of interest and importance.

In this section we consider mass, total energy and axial angular momentum conservation, and obtain the components of (3.13) by using conservation arguments. We then derive the material conservation law for potential vorticity – which is implied by (3.13) - (3.16) but is by no means obvious. A Hamiltonian treatment which unifies the conservation laws is noted in conclusion.

4.1 Mass conservation

Eq (3.8) is a mass conservation law of Eulerian form. Eq (3.9) is of Lagrangian form, relating the material derivative of density to the divergence of the flow **u**; it can be obtained directly by considering conservation of the mass $\rho\delta\tau$ of a parcel of air, upon noting that $D(\delta\tau)/Dt \;=\; \mathrm{div}\mathbf{u}$.

A global mass conservation law can be obtained from (3.7) by taking τ to be the entire volume of the atmosphere:

$$\frac{\partial}{\partial t}\int_{\substack{\text{whole}\\ \text{atmosphere}}}\rho d\tau \quad = \quad -\int_{\text{boundaries}}\rho\mathbf{u}.\mathbf{dS} \quad = \quad 0 \tag{4.1}$$

(The second equality assumes there is no net mass transfer into or out of the atmosphere.)

4.2 Total energy conservation

By taking the scalar product of **u** with (3.13), and by using (3.14), one readily obtains a Lagrangian conservation law for the total energy E per unit mass ($E = \tfrac{1}{2}\mathbf{u}^2 + \Phi + c_v T$ is the sum of the specific kinetic, potential and internal energy):

$$\rho \frac{DE}{Dt} = -\operatorname{div}(p\mathbf{u}) + \rho(Q + \mathbf{u}.\mathbf{F}) . \tag{4.2}$$

Hence

$$\frac{\partial}{\partial t}(\rho E) = -\operatorname{div}[(\rho E + p)\mathbf{u}] + \rho(Q + \mathbf{u}.\mathbf{F}) , \tag{4.3}$$

which is the Eulerian version of (4.2). Since it acts at right angles to **u**, the Coriolis force in (3.13) does not figure directly in the energetics. Equation (4.3) may be regarded as a statement of the conservation of energy; for the case $\mathbf{F} = 0$, Holton (1992) derives (3.6) from (4.3).

Atmospheric energetics is a large subject; White (1978a) gives an elementary account. An important issue is the extent to which potential and internal energy may be converted into flow kinetic energy ($\tfrac{1}{2}\mathbf{u}^2$ per unit mass). Availability in this sense is the subject of continuing study – see Shepherd (1993), Marquet (1993), Kucharski (1997) and references in these papers.

4.3 Axial angular momentum conservation

The components of (3.13) in the zonal, meridional and vertical directions may be derived by considering the rates of change of unit vectors over the sphere. One finds (see Phillips (1973))

$$\frac{Du}{Dt} = 2\Omega v \sin\phi - 2\Omega w \cos\phi + \frac{uv\tan\phi}{r} - \frac{uw}{r} - \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial \lambda} + F_\lambda \tag{4.4}$$

$$\frac{Dv}{Dt} = -2\Omega u \sin\phi - \frac{u^2\tan\phi}{r} - \frac{vw}{r} - \frac{1}{\rho r}\frac{\partial p}{\partial \phi} + F_\phi \tag{4.5}$$

$$\frac{Dw}{Dt} = +2\Omega u \cos\phi + \frac{(u^2 + v^2)}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} + F_r - g \tag{4.6}$$

The arrangement of the terms has a purpose, as will be seen in section 4.4. By multiplying (4.4) by $r\cos\phi$, and noting that $u = r\cos\phi\, D\lambda/Dt$, $v = r\, D\phi/Dt$ and $w = Dr/Dt$, it follows that

$$\rho\frac{D}{Dt}[(\Omega r\cos\phi + u)r\cos\phi] = -\frac{\partial p}{\partial \lambda} + \rho F_\lambda r\cos\phi \tag{4.7}$$

Eq (4.7) relates the rate of change of the axial component of absolute angular momentum (per unit mass of air) to the axial components of the torques acting (see Fig 3(a)); it is a Lagrangian conservation law for axial angular momentum. Local and global versions are readily derived. The total axial angular momentum of the atmosphere is by no means constant. Changes of day-length of milliseconds over a few days are detectable by astronomical methods and reflect exchange of axial angular momentum between atmosphere and solid Earth – see Hide *et al* (1997). Small changes of the direction of the Earth's rotation vector also occur; Barnes *et al* (1983) give an account of the vectorial angular momentum dynamics involved. A notable aspect of angular momentum conservation is that it determines the frame invariance of the energy conservation laws (White 1989a).

4.4 Spherical polar components of the equation of motion – a derivation via conservation

Perhaps the most direct way of obtaining the three spherical polar components of (3.13) reverses the above argument by using (4.7) to derive (4.4), and then notes that the Coriolis and metric terms in the components of (3.13) must disappear when a kinetic energy equation is formed (see (4.2)). We outline the reasoning. By expanding the material derivative on the left side of (4.7) and multiplying by $1/\rho r\cos\phi$, one readily obtains (4.4). Multiplication by u then gives

$$u\frac{Du}{Dt} = 2\Omega uv\sin\phi - 2\Omega uw\cos\phi + \frac{u^2 v\tan\phi}{r} - \frac{u^2 w}{r} - \frac{u}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} + uF_\lambda \tag{4.8}$$

Eq.(4.8) contains two Coriolis and two metric terms which must cancel with corresponding terms in the expressions for $v\,Dv/Dt$ and $w\,Dw/Dt$. Hence the meridional (ϕ) component of (3.13) must contain a Coriolis term $-2\Omega u\sin\phi$ and the radial (r) component a Coriolis term $+2\Omega u\cos\phi$; also, the meridional component must contain a metric term $-(u^2/r)\tan\phi$ to ensure cancellation with $+(u^2 v/r)\tan\phi$ in (4.8). The remaining metric term in (4.8), $-u^2 w/r$, must cancel with a term in the expression for $w\,Dw/Dt$, so the radial component must contain a term $+u^2/r$. To ensure isotropy with respect to horizontal flow direction, a term $+v^2/r$ must accompany $+u^2/r$ in the radial component. A term $-vw/r$ must then appear in the meridional component. This reasoning reproduces all the Coriolis and metric terms seen in (4.4) – (4.6).

4.5 Potential vorticity conservation

Eqs (4.2) and (4.7) show that neither the total energy nor the axial angular momentum is generally conserved following the flow, even if it is frictionless and adiabatic. Axial angular momentum is conserved in this sense in frictionless flow if the pressure field is independent of longitude, but such axisymmetric flow is rather special. (It can be engineered in laboratory systems – see Hide and Mason (1975).) Even for axial angular momentum, then, the Lagrangian conservation law might more accurately be called a non-conservation law.

Since (3.13) contains two gradient terms (albeit one of them multiplied by α) a reasonable strategy for deriving a Lagrangian conserved quantity is to take the curl of (3.13). By using

$$(\mathbf{u}.\mathrm{grad})\mathbf{u} = \mathrm{grad}(\mathbf{u}^2/2) - \mathbf{u}\times\mathrm{curl}\mathbf{u} \tag{4.9}$$

and various other vector differential identities, one obtains from (3.13):

$$\frac{D}{Dt}\{\mathbf{Z}+2\mathbf{\Omega}\} = -(\mathbf{Z}+2\mathbf{\Omega})\mathrm{div}\mathbf{u} + [(\mathbf{Z}+2\mathbf{\Omega}).\mathrm{grad}]\mathbf{u} + \frac{1}{\rho^2}\mathrm{grad}\rho\times\mathrm{grad}p + \mathrm{curl}\mathbf{F} \tag{4.10}$$

Here $\mathbf{Z} \equiv \mathrm{curl}\ \mathbf{u}$ is the relative vorticity, and $(\mathbf{Z}+2\mathbf{\Omega})$ is the absolute vorticity. Eq (4.10) is the vorticity equation. In spite of its complexity, it is an important equation, and we have not space to do it justice here; see Batchelor (1967) and Pedlosky (1987) for detailed treatments.

Suppose there is no motion ($\mathbf{u} = 0$ everywhere) at some instant. If curl $\mathbf{F}$ vanishes when $\mathbf{u} = 0$, which will be the case if $\mathbf{F}$ consists entirely of the contribution of (Newtonian) friction, then (4.10) shows that motion will develop ($D\mathbf{Z}/Dt \neq 0$) if the surfaces of constant density and constant pressure do not coincide. Fluids having $\rho = \rho(p)$ are called barotropic; their surfaces of constant density and constant pressure coincide. Fluids *not* having $\rho = \rho(p)$ are called baroclinic; their constant density and constant pressure surfaces intersect. We deduce Jeffreys' theorem (see Hide (1977)): motion must develop, or already be present, in a baroclinic fluid.

From our perspective of wishing to derive a Lagrangian conserved quantity, (4.10) might seem to represent several steps backwards. However, if we:

i) multiply (4.10) by $1/\rho$ and apply the continuity equation in the form (3.9);

ii) use the vector identity, valid for any vector **A** and scalar S,

$$\mathbf{A}.\frac{D}{Dt}(\mathrm{grad}S) \;=\; \mathbf{A}.\mathrm{grad}\left(\frac{DS}{Dt}\right) \;-\; \mathrm{grad}S.[\mathbf{A}.\mathrm{grad}]\mathbf{u} \;;$$

iii) note that ρ can be expressed (via (3.10) and (3.17)) as a function of p and θ;

then we find that

$$\rho\frac{D}{Dt}\left[\frac{(\mathbf{Z}+2\mathbf{\Omega}).\mathrm{grad}\theta}{\rho}\right] \;=\; \mathrm{div}\left[(\mathbf{Z}+2\Omega)\frac{D\theta}{Dt} + \theta\mathrm{curl}\mathbf{F}\right] \qquad (4.11)$$

Hence if (but not only if) the motion is frictionless ($\mathbf{F} = 0$) and adiabatic ($D\theta/Dt = 0$) then

$$DP/Dt \;=\; 0\,, \qquad \text{where} \qquad P \;\equiv\; [(\mathbf{Z}+2\mathbf{\Omega}).\mathrm{grad}\theta]/\rho \qquad (4.12)$$

The quantity P, which is called Ertel's potential vorticity (Ertel 1942) or simply EPV, *is materially conserved in frictionless, adiabatic flow*. The form of (4.11) implies that any local creation of EPV by heating and friction will tend to be balanced by destruction elsewhere. Eq.(4.11) is a central result in fluid dynamics, especially rotating fluid dynamics; see Hoskins *et al* (1985), Haynes and McIntyre (1987), Hoskins (1991), Lait (1995) and Viúdez (1999).

Result (4.12) may be obtained by applying Kelvin's circulation theorem in isentropic surfaces (surfaces of constant potential temperature, θ). Kelvin's theorem takes the form

$$\frac{D}{Dt}\oint_C[\mathbf{u}+\mathbf{\Omega}\times\mathbf{r}].\mathbf{dl} \;\left(= \frac{D}{Dt}\oint_S \mathrm{curl}[\mathbf{u}+\mathbf{\Omega}\times\mathbf{r}].\mathbf{dS} = \frac{D}{Dt}\oint_S[\mathrm{curl}\mathbf{u}+2\mathbf{\Omega}].\mathbf{dS}\right) \;=\; -\oint_C\frac{dp}{\rho} + \oint_C\mathbf{F}.\mathbf{dl} \qquad (4.13)$$

Here C is any closed loop of material particles and S is any surface bounded by C. If C lies in an isentropic surface, then ρ is a function of p on C. Hence, if the motion is frictionless and adiabatic, one can apply (4.13) to a small material area δS within an isentrope $\theta = \theta_1$ to obtain

$$\frac{D}{Dt}\{([\mathrm{curl}\mathbf{u}+2\mathbf{\Omega}].\mathrm{grad}\theta)\delta S/|\mathrm{grad}\theta|\} \;=\; 0 \qquad (4.14)$$

Also, the quantity $\delta M \equiv \rho\delta S\delta\theta/|\mathrm{grad}\theta|$ – which is the mass within a right cylinder having bases δS on isentropes θ_1 and $\theta_1 + \delta\theta$ (see Fig 3(b)) – remains constant. So $\delta M/\delta\theta$ may be taken outside the material derivative in (4.14), and (4.12) is revealed.

4.6 Lagrangian symmetries and conservation properties

In the analytical dynamics of rigid-bodies it is well known that conservation laws correspond to symmetries of the Hamiltonian functional that appears in the variational formulation (Noether's theorem). For example, conservation of energy corresponds to time-parametrization invariance. Fluid dynamics is a more complicated problem, partly because of the choice between Eulerian and Lagrangian descriptions, but the theoretical position is now understood. Potential vorticity conservation (see (4.12)) corresponds to the symmetry whereby the Hamiltonian is invariant to the coordinates used to label particles (Ripa 1981, Salmon 1982). Noether's theorem offers a systematic method for deriving consistent approximate models: one approximates the Hamiltonian (whilst preserving its symmetries) and is then assured that the implied evolution equations reproduce the various conservation laws. See Salmon (1983), (1988) and Shepherd (1990). Some applications of the method are noted in section 9.5. Mobbs (1982), Wang (1984) and Sewell (1990) discuss other key aspects of variational formulations of fluid dynamics.

5. THE HYDROSTATIC APPROXIMATION, THE HYDROSTATIC PRIMITIVE EQUATIONS AND THE SHALLOW WATER EQUATIONS

Jeffreys' theorem (see section 4.6) shows that motion must occur if the pressure and density surfaces in a fluid are not parallel, and the occurrence of motion in the atmosphere is evident even to the most casual observer. Nevertheless, on a wide range of time and space scales, the vertical component of the momentum equation is dominated by the contributions of gravity and the pressure gradient force; the atmosphere is close to *hydrostatic balance*. [The adjective *aerostatic* would seem more appropriate than *hydrostatic*, but the latter is irreversibly established in meteorological usage.] We begin this section by examining the relationships which exist between the thermodynamic fields when hydrostatic balance is precise. Having noted elementary static stability criteria, we then consider how, and under what conditions, we may construct equations describing the motion of an atmosphere that is close to hydrostatic balance. We present and discuss the hydrostatic primitive equations (HPEs), which are widely used in numerical weather prediction and climate simulation, and note the shallow water equations (SWEs), which are widely used as a testbed in both theory and numerical practice.

5.1 Hydrostatic atmospheres

In the absence of motion and of forcing, the governing equations (3.14) - (3.16) and (4.4) – (4.6) are satisfied so long as

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \tag{5.1}$$

and there are no horizontal variations of pressure. Here z is height above mean sea level; see section 5.4. Eq (5.1) is the *hydrostatic equation*. Integration with respect to height gives (since $p \to 0$ as $z \to \infty$)

$$p(z) = \int_0^\infty \rho g dz \tag{5.2}$$

The pressure at height z is equal to the "weight" of the air above unit area. By using the perfect gas law (3.16) it also follows from (5.1) that

$$p(z) = p(z_s)\exp\left[-\int_{z_s}^{z}(g/RT)dz'\right] \tag{5.3}$$

where z_s is the height of the Earth's surface above mean sea level. In a hydrostatic atmosphere, the pressure field is determined by the variation of temperature with height, and temperature must vary only with height. (Spatial variations of g are neglected in this simple treatment.)

Knowing $T(z)$, one can find $p(z)$ from (5.3), $\rho(z)$ from the perfect gas equation (3.16), and $\theta(z)$ from (3.17). For illustration and later reference we list the results obtained in the special case of an isothermal atmosphere ($T = T_0$), assuming $z_s = 0$, uniform g and $p(0) = p_{ref}$ (see (3.17)):

$$p(z) = p(0)\exp\{-z/H_0\} \quad ; \quad H_0 = RT_0/g \tag{5.4}$$

$$\rho(z) = \rho(0)\exp\{-z/H_0\} \quad ; \quad \rho(0) = p(0)/RT_0 \tag{5.5}$$

$$\theta(z) = \theta(0)\exp\{+gz/c_pT_0\} \quad ; \quad \theta(0) = T_0 \tag{5.6}$$

$$\Rightarrow \quad N^2 \; [\equiv (g/\theta)\, d\theta/dz] = g^2/c_pT_0 \tag{5.7}$$

The quantity $H_0 = RT_0/g$ is called the scale height of the isothermal atmosphere; it is the height over which the pressure and density decrease by a factor of e. If temperature varies with height - as it does, of course, in the real atmosphere (see Fig 4(a)) - then (5.4) - (5.7) are not valid. Nevertheless, substituting an appropriate mean temperature gives a useful measure of the rate of decrease of pressure and density with height: taking $T_0 = 250$K gives $H_0 \approx 7.4$km.

5.2 Static stability and the buoyancy frequency

If a parcel of air is displaced vertically (see Fig 5) it will experience a change of hydrostatic pressure, with consequent (adiabatic) changes of temperature and density. From the first law of thermodynamics in the form (3.4), the perfect gas law (3.16) and the relation $c_p - c_v = R$, it follows that the changes δT and δp in temperature and pressure of the parcel will be related by

$$c_p \delta T \; - \; \alpha \delta p \; = \; 0 \tag{5.8}$$

in adiabatic displacement. Assuming that δp is equal to the change in hydrostatic pressure of the surrounding air over the distance δz (i.e. $\alpha \delta p = -g\delta z$), (5.8) becomes

$$c_p \delta T \; + \; g\delta z \; = \; 0 \tag{5.9}$$

The rate (with respect to height) at which the temperature of a parcel of air decreases on upward displacement or increases on downward displacement is therefore g/c_p (a quantity known as the dry adiabatic lapse rate). An atmosphere at rest will be stable to vertical displacements of parcels if its temperature $T(z)$ decreases less rapidly with respect to height than g/c_p, i.e. if $dT/dz \geq -g/c_p$. (A parcel of air displaced upwards will then become cooler than, and hence more dense than, its environment; and a parcel of air displaced downwards will become warmer than, and hence less dense than, its environment.) An atmosphere at rest will be unstable to vertical displacements if its temperature decreases more rapidly than g/c_p, i.e. if $dT/dz \leq -g/c_p$.

Air saturated with water vapour suffers a decrease of temperature smaller than $g\delta z/c_p$ upon upward displacement because the inevitable cooling brings about condensation and the release of latent heat (so long as condensation nuclei are present and prevent supersaturation). We shall not discuss further this important effect, which is one of the major complications and fascinations of meteorology; see Gill (1982) and Emanuel (1994) for clear discussion.

The conditions for stability to vertical displacement of unsaturated air are most easily expressed in terms of the vertical gradient of potential temperature. Use of (3.17), (5.1) and (5.9) shows that: if $\partial\theta/\partial z > 0$, unsaturated air is stable to vertical displacement; if $\partial\theta/\partial z < 0$ it is unstable; if $\partial\theta/\partial z = 0$, it is neutrally stable. Large positive values of $\partial\theta/\partial z$ tend to inhibit vertical motion.

In the Earth's atmosphere, well away from the surface, the vertical gradient of potential temperature is generally much greater numerically than the horizontal gradient; the atmosphere is said to be *stratified*. (Horizontal gradients are not, of course, negligible; the difference of potential temperature between different horizontal locations is a key driving agency of the circulation, as Jeffreys' theorem suggests.) Well away from the Earth's surface, values of $\partial\theta/\partial z$ are typically 4×10^{-3} K m^{-1} in the troposphere, and about a factor of 4 greater in the stratosphere;

see the schematic climatological section shown in Fig 4(b). Considerable spatial and temporal variations occur, however, especially in the troposphere. The transition region between the troposphere and the stratosphere – the tropopause – across which $\partial\theta/\partial z$ and potential vorticity both change markedly (see Thuburn and Craig (2000)) tends to act as a quasi-horizontal lid to motions beneath. Locally, the tropopause exhibits major variations in height associated with the passage of weather systems (see Keyser and Shapiro (1986) and Browning and Reynolds (1994)) and a general decrease with latitude is evident in Fig 4(b), but a typical value is 10km.

The physical significance of the quantity $\partial\theta/\partial z$ is further illuminated by considering the vertical displacement of a parcel of air in dynamic terms (Fig 5). Upon neglecting the (small) metric and Coriolis terms in (4.6) (terms which vanish if the motion is purely vertical), and assuming again that the displaced parcel experiences the pressure field $\overline{p}(z)$ of its surroundings, we find

$$\frac{Dw}{Dt} + g + \frac{1}{\rho}\frac{d\overline{p}}{dz} = 0 \tag{5.10}$$

Since $d\overline{p}/dz = -\overline{\rho}g$, the second and third terms in (5.10) combine to give $(\rho-\overline{\rho})g/\rho$. Given $p = \overline{p}(z)$ and small displacements δz, we have (from (3.16) and (3.17)):

$$\left(\frac{\rho-\overline{\rho}}{\rho}\right) = \left(\frac{\overline{T}-T}{\overline{T}}\right) = \left(\frac{\overline{\theta}-\theta}{\overline{\theta}}\right) \cong \frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{dz}\delta z \tag{5.11}$$

But $w = D/Dt(\delta z)$, so (5.10) becomes

$$\frac{D^2}{Dt^2}\delta z + N^2\delta z = 0 \tag{5.12}$$

where $N^2 = \left(g/\overline{\theta}\right)d\overline{\theta}/dz$ is the *buoyancy frequency* (also known as the Brunt-Väisälla frequency). The period of small vertical oscillations in a stable atmosphere is thus $2\pi/N$. If $N^2 < 0$, small vertical displacements amplify with time as $\exp(Nt)$; this is consistent with our earlier identification of $d\overline{\theta}/dz < 0$ as the condition for instability to vertical displacements.

5.3 The hydrostatic approximation for an atmosphere in motion

Vertical accelerations in air motions are typically much less than g. Even in the most violent cumulonimbus circulations one might find ascent rates of at most 10 ms^{-1} being attained in times of order 1000s; then $Dw/Dt \sim 10^{-2}$ ms^{-2}, compared with $g \approx 10$ ms^{-2}. In a *diagnostic* sense, therefore, the hydrostatic approximation - of assuming hydrostatic balance - is very good indeed.

But this is a naïve view. Our deduced hydrostatic balance mainly reflects the contributions of pressure and density fields varying only with height, which are not associated with motion; our analysis in section 5.1 considered the behaviour of such precisely hydrostatic states. We enquire to what extent and under what conditions the hydrostatic approximation applies to the deviations of all fields from a state of precise static balance: is the hydrostatic approximation valid when we have subtracted out some background static balance? To achieve this we write

$$\begin{aligned} p &= p_0(z) + p'(\lambda,\phi,z,t) \\ \rho &= \rho_0(z) + \rho'(\lambda,\phi,z,t) \end{aligned} \quad \text{with} \quad \frac{dp_0}{dz} = -\rho_0 g. \tag{5.13}$$

(The "background" state could be defined by horizontal and temporal averages of density ρ at each height and of mean sea level pressure.) Given the decomposition (5.13), the horizontal components (4.4), (4.5) of the momentum equation change only in that p' replaces p. The vertical component (4.6) becomes

$$\frac{Dw}{Dt} - 2\Omega u\cos\phi - \left(\frac{u^2+v^2}{r}\right) + g\frac{\rho'}{\rho} + \frac{1}{\rho}\frac{\partial p'}{\partial z} = 0$$

Our criterion for the validity of the hydrostatic approximation is therefore that Dw/Dt be small compared with $g\rho'/\rho$ or $(1/\rho)\partial p'/\partial z$ (we neglect the other terms in this simple treatment; see section 11.3 for a formulation that includes them). This is a much more testing condition than we had earlier, since typically $|\rho'/\rho| << 1$. Supposing the motion to be adiabatic, the thermodynamic equation gives

$$\frac{D\theta'}{Dt} + w\frac{d\theta_0}{dz} = 0.$$

Here $\theta_0 = \theta_0(z)$ is the potential temperature variation implied by $p_0(z)$ and $\rho_0(z)$, and $\theta' = \theta - \theta_0$. Since, *to order of magnitude*, $(\theta'/\theta) \sim (\rho'/\rho)$ we obtain the criterion

$$\tau^2 >> 1/N^2 \tag{5.14}$$

if we assume $D/Dt \approx 1/\tau$, where τ is a Lagrangian time scale. Perhaps not surprisingly, our condition (5.14) is that the time-scale of the motion should be much longer than that of vertical buoyancy oscillations (which are essentially non-hydrostatic). The hydrostatic approximation is clearly not dynamically appropriate for an atmosphere that is neutrally stratified ($N=0$).

The argument leading to (5.12) is readily repeated for parcel oscillations constrained to lie in a plane making an angle α to the horizontal; the resulting frequency is $N\sin\alpha$. Small values of α are characteristic of motion having its horizontal space scale L much larger than its vertical space scale H. The often-quoted condition $H<<L$ for the applicability of the hydrostatic approximation is thus consistent with condition (5.14). However, α may be small even when L and H are comparable (see Hide and Mason 1975); (5.14) is considered to be the more fundamental condition for the applicability of the hydrostatic approximation.

5.4 The traditional approximation, the shallow atmosphere approximation and the hydrostatic primitive equations

Compared to the dimensions of the Earth (mean radius 6360km) the atmosphere is shallow: consistent with our conclusions in section 5.1, 90% of its mass lies below 17km. Shallow in this sense it is, but two caveats should be noted. First, nearly every field varies with height as well as with horizontal location. For example, winds at 10km are usually markedly different from those found near the surface, as regards both speed and direction. Second, the atmosphere is generally not shallow in relation to the Earth's topography. Mountains locally attain heights of about 8km above mean sea level, and they certainly influence the motion and behaviour of the atmosphere to an important extent, but there is little tendency for the atmosphere to be divided up into "basins" in the way that the continents divide the Earth's seas into ocean basins, or in the way that high mountains effectively divide the Martian atmosphere (see Hide 1976). [Some local phenomena, for example the East African Jet (Findlater 1969) and coastal lows in Southern Africa (Gill 1977), do depend on mountain ranges acting as lateral "walls". The effects of mountains on air flow generally depend on the stratification of the air – as measured by $\partial\theta/\partial z$ – as well as on the flow itself and the elevation of the mountains. See Baines (1995).]

With these caveats in mind, it is reasonable to seek a simplification of the equations of motion which exploits the fact that the atmosphere's depth is a small fraction of the Earth's radius – a *shallow atmosphere* approximation. We aim to replace the variable radius r by a mean value a, whilst retaining differentiations with respect to height as $\partial/\partial z$, where z is height above mean sea level. An implication of this strategy is clear if we re-consider the derivation of the components of the momentum equation given in section 4.4. The Coriolis and metric terms $-2\Omega w\cos\phi$ and $-uw/r$ in (4.4) are lost if we re-define absolute axial angular momentum per unit mass as

$$(u+\Omega a\cos\phi)a\cos\phi \tag{5.15}$$

and the material derivative as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z} \tag{5.16}$$

Pursuit of the argument given in section 4.4 shows that the terms $2\Omega u\cos\phi$ and $(u^2+v^2)/r$ in the vertical component (4.6) and $-vw/r$ in the meridional component (4.5) must then be neglected for consistent energetics. The neglect of the $\cos\phi$ Coriolis terms – known as the *traditional approximation* (Eckart 1960) – is less comfortable than neglect of the quadratic metric terms not involving $\tan\phi$, although for many purposes it turns out to be a good approximation; we shall return to this issue in section 11.3 during a discussion of acoustically-filtered global models. Accepting that the stated omissions should accompany the shallow atmosphere approximation, we obtain the *hydrostatic primitive equations* as

$$\frac{Du}{Dt} = 2\Omega v\sin\phi + \frac{uv\tan\phi}{a} - \frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda} + F_\lambda \tag{5.17}$$

$$\frac{Dv}{Dt} = -2\Omega u\sin\phi - \frac{u^2\tan\phi}{a} - \frac{1}{\rho a}\frac{\partial p}{\partial\phi} + F_\phi \tag{5.18}$$

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \tag{5.19}$$

The thermodynamic and continuity equations remain

$$\frac{D\theta}{Dt} = \left(\frac{\theta}{c_p T}\right)Q \tag{5.20}$$

and

$$\frac{D\rho}{Dt} = -\rho\nabla.\mathbf{u} \tag{5.21}$$

but, as in (5.17) - (5.19), D/Dt is defined by (5.16), and

$$\nabla.\mathbf{u} \;=\; \frac{1}{a\cos\phi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right] \;+\; \frac{\partial w}{\partial z} \tag{5.22}$$

In (5.19), g is properly considered constant. The quantity $2\Omega\sin\phi$ that appears in (5.17) and (5.18) is usually referred to as the Coriolis parameter and accredited the symbol f. The other Coriolis parameter, $2\Omega\cos\phi$, which is absent from the HPEs, has no universally accepted title.

The axial angular momentum conservation law of the HPEs, readily derived from (5.17), is

$$\rho\frac{D}{Dt}\{(u \;+\; \Omega a\cos\phi)a\cos\phi\} \;\;=\;\; \rho F_\lambda a\cos\phi \;\;-\;\; \frac{\partial p}{\partial\lambda} \tag{5.23}$$

The energy conservation law (Lagrangian form) is

$$\rho\frac{D}{Dt}\left[\frac{1}{2}\mathbf{v}^2 + gz + c_v T\right] \;=\; -\nabla.(p\mathbf{u}) \;+\; \rho(Q \;+\; \mathbf{v}.\mathbf{F}_h) \tag{5.24}$$

Here $\mathbf{v}$ is the *horizontal* flow (the "wind"). Eq (5.24) shows that the vertical motion w does not contribute to the kinetic energy in the HPEs, but it appears in the pressure-work convergence term $-\nabla.(p\mathbf{u})$ and in the definition of D/Dt (see (5.16)). We consider in 5.5 what determines w.

The HPEs' analogue of potential vorticity conservation is

$$\rho\frac{D}{Dt}\left\{\frac{\boldsymbol{\xi}.\nabla\theta}{\rho}\right\} \;=\; \boldsymbol{\xi}.\nabla\left(\frac{D\theta}{Dt}\right) \;+\; \nabla\theta.\nabla\times\mathbf{F}_h \tag{5.25}$$

where

$$\nabla\theta \;=\; \left(\frac{1}{a\cos\phi}\frac{\partial\theta}{\partial\lambda}, \frac{1}{a}\frac{\partial\theta}{\partial\phi}, \frac{\partial\theta}{\partial z}\right) \qquad \left[\equiv\left(\nabla_z\theta,\; \frac{\partial\theta}{\partial z}\right)\right] \tag{5.26}$$

and

$$\boldsymbol{\xi} \;=\; 2\Omega\mathbf{k}\sin\phi + \nabla\times\mathbf{v} \tag{5.27}$$

Here $\mathbf{k}$ = unit vector in the (upward) vertical direction, and

$$\nabla\times\mathbf{v} \;\equiv\left(-\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}, \frac{1}{a\cos\phi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\phi}(u\cos\phi)\right)\right) \tag{5.28}$$

The horizontal component equations (5.17), (5.18) of the HPEs may be written in vector form as

$$\frac{\partial\mathbf{v}}{\partial t} \;+\; \nabla_z(\mathbf{v}^2/2) \;+\; \zeta\mathbf{k}\times\mathbf{v} \;+\; w\frac{\partial\mathbf{v}}{\partial z} \;=\; -f\mathbf{k}\times\mathbf{v} \;-\; \frac{1}{\rho}\nabla_z p \;+\; \mathbf{F}_h \tag{5.29}$$

Here ∇_z is the horizontal part of ∇, as defined in (5.26), $\zeta \equiv \mathbf{k}.\nabla\times\mathbf{v}$ is the vertical component of the relative vorticity, and $\mathbf{F}_h \equiv (F_\lambda, F_\phi)$. From (5.29) may be derived prognostic equations for ζ and $\nabla_z.\mathbf{v}$, the divergence of the horizontal flow. Such *vorticity, divergence* forms are used in some HPE numerical models, particularly Eulerian spectral models (see Hoskins and Simmons (1975)). The second and third terms on the left side of (5.29) are not precisely equivalent to $(\mathbf{v}.\nabla_z)\mathbf{v}$, which contains a small vertical component when ∇_z is defined as in (5.26); see Côté (1988), Ritchie (1988) and Bates *et al.* (1990).

5.5 Richardson's Equation

A by-product of the hydrostatic approximation is that the prognostic equation for w is lost. The implication is *not* that $w = 0$, or that w does not vary with time; rather, w takes that spatial form which maintains hydrostatic equilibrium as the thermodynamic and horizontal flow fields evolve. A diagnostic equation for w may be derived in several ways. We use a route which gives physical insight and an explicit expression for $\partial w/\partial z$, and then note a second order differential equation that w obeys. Our treatment follows that of unpublished Meteorological Office College lecture notes (1981) by R W Riddaway; see also Wiin-Nielsen (1968) and Dutton (1995).

By writing the HPE continuity equation (5.21) in terms of $\partial\rho/\partial t$ and using the hydrostatic approximation (5.1) one obtains the important relation

$$\nabla_z . (\rho\mathbf{v}) + \frac{\partial}{\partial z}\left(\rho w - \frac{1}{g}\frac{\partial p}{\partial t}\right) = 0 \qquad (5.30)$$

Here, once again, the flow $\mathbf{u}$ has been separated into its horizontal part $\mathbf{v}$ and vertical part $w\mathbf{k}$; $\nabla_z .$ indicates the horizontal part of the divergence:

$$\nabla_z . (\rho\mathbf{v}) = \frac{1}{a\cos\phi}\left[\frac{\partial(\rho u)}{\partial\lambda} + \frac{\partial}{\partial\phi}(\rho v\cos\phi)\right] \qquad (5.31)$$

Integrating (5.30) over the interval $[z, \infty]$ gives

$$\frac{\partial p}{\partial t} = \rho g w - g\int_z^\infty \nabla_z . (\rho\mathbf{v})dz' \qquad (5.32)$$

Eq (5.32) states simply that the time rate of change of pressure at height z is equal to ($g\times$) the rate of convergence of mass into the column (of unit horizontal cross-section) above z.

In addition to (5.32) we have another equation for $\partial p/\partial t$. Using (5.21) and (3.10), the thermodynamic equation (3.14) can be written as

$$\frac{Dp}{Dt} = -\gamma p\nabla.\mathbf{u} + \frac{\rho R Q}{c_v} \qquad (5.33)$$

Hence (using (5.19))

$$\frac{\partial p}{\partial t} = -\mathbf{v}.\nabla_z p + \rho w g - \gamma p\nabla.\mathbf{u} + \frac{\rho R Q}{c_v} \qquad (5.34)$$

The right sides of (5.32) and (5.34) must be equal; we find that

$$\gamma p\frac{\partial w}{\partial z} = \gamma p\left[\frac{Q}{Tc_p} - \nabla_z . \mathbf{v}\right] - \mathbf{v}.\nabla_z p + g\int_z^\infty \nabla_z . (\rho\mathbf{v})dz' \qquad (5.35)$$

Eq (5.35) determines $\partial w/\partial z$ at height z in terms of p, ρ, $\mathbf{v}$ and Q at z and ρ, $\mathbf{v}$ at greater heights; $w(z)$ itself may be obtained by integrating (5.35) from $z = z_s$ upwards, assuming a reasonable lower boundary condition (such as $w = 0$ at a flat lower boundary). The explicit expression for

$w(z)$ so obtained is known as Richardson's equation from its use in the first numerical weather prediction experiment (Richardson 1922). A different treatment is necessary if an upper boundary condition is applied at a finite height (Kasahara and Washington 1967).

Differentiation of (5.35) leads to a form that does not contain a vertical integral:

$$\gamma \frac{\partial}{\partial z}\left\{ p\left[\frac{\partial w}{\partial z} + \nabla_z.\mathbf{v} - \frac{Q}{Tc_p} \right]\right\} = \frac{\partial p}{\partial z}\nabla_z.\mathbf{v} - \frac{\partial \mathbf{v}}{\partial z}.\nabla_z p \tag{5.36}$$

[Tapp and White (1976) give an equivalent form in the case $Q = 0$.] Eq (5.36), which is unchanged if any upper boundary condition is applied at a finite height, can be obtained more directly by writing (5.19) as $\rho g + \partial p/\partial z = 0$, differentiating locally with respect to t, then using (5.21) and (5.33) to substitute for $\partial\rho/\partial t$ and $\partial p/\partial t$, and finally applying (5.19) for ρ.

5.6 The shallow water equations

The HPEs describe the motion of a compressible atmosphere, and allow height variation of all fields (within the shallow atmosphere approximation, and criterion (5.14)). For both theoretical and numerical testing it is often convenient to have recourse to a set of equations which does not involve height variations or compressibility. The shallow water equations (SWEs) are such a set.

Waves on the surface of a *non*-rotating, incompressible, homogeneous liquid of mean depth d under the influence of gravity behave differently in the long and short wave limits (see, for example, Lighthill (1978)). If the wavelength λ of the surface waves obeys $\lambda \ll d$, then we are close to the deep limit: the waves are dispersive, particle paths are circles (of exponentially decreasing radius as one goes deeper into the fluid) and the motion is essentially non-hydrostatic. But if $\lambda \gg d$, then we are close to the shallow limit: the waves are non-dispersive, particle paths are horizontal, amplitude is independent of depth, and the motion is essentially hydrostatic.

With this background, consider how the HPE momentum and continuity equations may be applied to a *rotating* incompressible, homogeneous fluid of density $\hat{\rho}$ bounded by a rigid horizontal surface at $z = 0$ and having a free surface at $z = h(\lambda, \phi, t)$. In the shallow limit, the pressure is (plausibly) hydrostatic, its horizontal gradient is $\hat{\rho}g$ multiplied by the free surface gradient, and (5.17), (5.18) become

$$\frac{Du}{Dt} = 2\Omega v \sin\phi + \frac{uv\tan\phi}{a} - \frac{g}{a\cos\phi}\frac{\partial h}{\partial \lambda} + F_\lambda \tag{5.37}$$

$$\frac{Dv}{Dt} = -2\Omega u \sin\phi - \frac{u^2\tan\phi}{a} - \frac{g}{a}\frac{\partial h}{\partial \phi} + F_\phi \tag{5.38}$$

with $$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial \lambda} + \frac{v}{a}\frac{\partial}{\partial \phi} \tag{5.39}$$

If the horizontal velocity components u, v are initially independent of depth, they will remain so, since the pressure gradient is independent of depth. The time evolution of h may then be obtained by integrating the continuity equation (5.21) over the depth h and noting that $w(h) = Dh/Dt$ (and $\rho = \hat{\rho} = \text{constant}$):

$$\int_0^h \nabla_z . \mathbf{v} dz + w(h) = 0 \Rightarrow \frac{Dh}{Dt} + h\nabla_z . \mathbf{v} = 0 \tag{5.40}$$

Eqs (5.37) – (5.40) are the shallow water equations (SWEs); we shall use them in our review of approximately geostrophic models (section 9). [To avoid ambiguity, we retain the symbol ∇_z for the horizontal part of the ∇ operator – see (5.26) and (5.31) – although for the SWEs ∇ reduces to ∇_z anyway, since no height variations of u, v or h are involved.]

The SWEs are closed for u, v and h, and have the following conservation properties (see Salmon (1983) for a Hamiltonian treatment):

Axial angular momentum:
$$\frac{D}{Dt}\{(u+\Omega a \cos\phi) a \cos\phi\} = F_\lambda a \cos\phi - g\frac{\partial h}{\partial \lambda} \tag{5.41}$$

Energy:
$$h\frac{D}{Dt}\left(\frac{1}{2}\mathbf{v}^2\right) = -gh\mathbf{v}.\nabla_z h + h\mathbf{v}.\mathbf{F}_h \tag{5.42}$$

Potential vorticity:
$$h\frac{D}{Dt}\left(\frac{\zeta + 2\Omega \sin\phi}{h}\right) = \mathbf{k}.\nabla_z \times \mathbf{F}_h \tag{5.43}$$

Here $\mathbf{F}_h \equiv (F_\lambda, F_\phi)$, and ζ is the SWE relative vorticity:

$$\zeta = \frac{1}{a\cos\phi}\left(\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi}(u\cos\phi)\right) \tag{5.44}$$

An important limiting case of the SWEs occurs when variations of the depth h are negligible (we examine in section 8 the conditions under which this occurs). Then (5.40) becomes

$$\nabla_z . \mathbf{v} = 0 \tag{5.45}$$

and (5.43) reduces to

$$\frac{D}{Dt}(\zeta + 2\Omega \sin\phi) = \mathbf{k}.\nabla_z \times \mathbf{F}_h \tag{5.46}$$

which is the *barotropic vorticity equation*. The material derivative in (5.46) is given by (5.39), with $\mathbf{v} = (u, v)$ satisfying the non-divergence condition (5.45). A streamfunction ψ may be introduced for $\mathbf{v}$, whereupon (5.46) becomes a prognostic equation for $\zeta = \nabla_z^2 \psi$ in terms of the advection of the absolute vorticity $\nabla_z^2\psi + 2\Omega\sin\phi$ by the flow $\mathbf{v} = \mathbf{k} \times \nabla_z \psi$:

$$\frac{\partial}{\partial t}\left(\nabla_z^2 \psi\right) = -(\mathbf{k} \times \nabla_z \psi).\nabla_z\left(\nabla_z^2\psi + 2\Omega\sin\phi\right) + \mathbf{k}.\nabla_z \times \mathbf{F}_h \tag{5.47}$$

Eq (5.47) determines the time evolution of ψ, given appropriate boundary conditions and a specification of $\mathbf{F}_h$. Studies of (5.47) and close variants (some of them Cartesian – see section 6.3) have given insight into Rossby waves (section 8.3), steady flow structures and geostrophic turbulence in rotating fluids; see, for example, Platzman (1968), Hoskins (1973), Rhines (1975), Baines (1976), Held (1983), Shutts (1983a), McWilliams (1984), Marshall (1984), White (1990) and Verkley (1993).

6. VERTICAL COORDINATE SYSTEMS; THE "f-PLANE" AND THE "β-PLANE"

This section deals with some further aspects of formulation and approximation that are characteristic of meteorological dynamics. The use of pressure as vertical coordinate is first discussed, and other choices are noted. The use of Cartesian geometry and approximate treatments of the spatial variation of the Coriolis parameter are then briefly considered.

6.1 Use of pressure as vertical coordinate

Any quantity that bears a 1:1 relation to height z may be used as a vertical coordinate. If the hydrostatic approximation is made, then pressure is certainly such a quantity, since $\rho > 0$ ensures that $\partial p/\partial z = -\rho g$ is everywhere negative. In "pressure coordinates" the independent variables are (λ, ϕ, p, t) instead of (λ, ϕ, z, t), and z becomes a dependent variable. The material derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + \omega\frac{\partial}{\partial p} \tag{6.1}$$

Here $\omega \equiv Dp/Dt$. The horizontal and local time derivatives in (6.1) are taken at constant pressure (but with distances measured on constant height surfaces); u, v are the velocity components in constant height surfaces (*not* the components in constant pressure surfaces); and $\partial/\partial p$ is taken at constant λ, ϕ, t. Eq (6.1) may be derived either from first principles, or from (5.21) by using the following rules, valid for any well-behaved $Q = Q(\lambda, \phi, z, t)$, with $X = t$, λ, ϕ and then $Q = p$:

$$\frac{\partial Q}{\partial z} = \frac{\partial p}{\partial z}\frac{\partial Q}{\partial p}\left(= -\rho g\frac{\partial Q}{\partial p}\right) \quad ; \quad \left.\frac{\partial Q}{\partial X}\right|_z = \left.\frac{\partial Q}{\partial X}\right|_p - \frac{\partial Q}{\partial z}\left.\frac{\partial z}{\partial X}\right|_p \tag{6.2}$$

See Fig 6. The quantity $\omega \equiv Dp/Dt$ is often referred to as the pressure-coordinate "vertical velocity", although, as the material derivative of a scalar, it is frame-invariant.

The hydrostatic relation (5.19) may be written as

$$g\frac{\partial z}{\partial p} = -\frac{1}{\rho} = -\frac{RT}{p} \tag{6.3}$$

The pressure-coordinate versions of (5.17) and (5.18) are

$$\frac{Du}{Dt} = \left(2\Omega + \frac{u}{a\cos\phi}\right)v\sin\phi - \frac{g}{a\cos\phi}\frac{\partial z}{\partial\lambda} + F_\lambda \tag{6.4}$$

$$\frac{Dv}{Dt} = -\left(2\Omega + \frac{u}{a\cos\phi}\right)u\sin\phi - \frac{g}{a}\frac{\partial z}{\partial\phi} + F_\phi \tag{6.5}$$

in which the nonlinear pressure gradient terms in (5.17), (5.18) have become linear in the horizontal gradient components (on pressure surfaces) of z.

The thermodynamic equation remains as (5.20), but the material derivative is expressed as (6.1). A major simplification occurs in the continuity equation (5.21) [see Sutcliffe (1947) and Eliassen (1949)]. The form (5.29) shows that hydrostatic balance reduces the continuity equation to non-divergence even in height coordinates (accepting a suitably redefined vertical velocity). Use of

$$w \;=\; \frac{Dz}{Dt} \;=\; \frac{\partial z}{\partial t} \;+\; \frac{u}{a\cos\phi}\frac{\partial z}{\partial\lambda} \;+\; \frac{v}{a}\frac{\partial z}{\partial\phi} \;+\; \omega\frac{\partial z}{\partial p} \tag{6.6}$$

in (5.29), along with (6.2) and the hydrostatic relation, shows that (5.21) becomes simply

$$\nabla_p.\mathbf{v} \;+\; \frac{\partial\omega}{\partial p} \;=\; 0 \tag{6.7}$$

in which

$$\nabla_p.\mathbf{v} \;\equiv\; \frac{1}{a\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right) \tag{6.8}$$

Eqs (6.3) - (6.7) [with (5.20)] are exact transforms of the height coordinate HPEs. Pressure-coordinate forms of the various conservation laws are readily derived, but will not be given here.

Similar transformations of the height coordinate HPEs may be made using any suitable function of pressure as vertical coordinate. Frequent choices include p^{R/c_p} (see Hoskins and Bretherton 1972) and ln p (see Holton (1975)); these coordinates are often given the symbols z or Z, and so it is easy to lose sight of the fact that they are pressured-based coordinates.

6.2 Other choices of vertical coordinate

Given the hydrostatic approximation, pressure coordinates offer at once a simplification and a complication. From the continuity equation (6.7) one can readily derive a diagnostic equation for the "vertical velocity", ω, in the pressure system:

$$\omega(p) \;=\; -\int_0^p \nabla_p.\mathbf{v}\,dp \tag{6.9}$$

Eq (6.9) is simpler than Richardson's equation for the usual height-coordinate vertical velocity, $w = Dz/Dt$; see section 5.5. The complication is that the Earth's surface is generally not a coordinate surface in the pressure system – even in the absence of topography. The local rate of change of surface pressure $p_s = p_s(\lambda,\phi,t)$ can be calculated from (6.9) with $p = p_s$:

$$\frac{\partial p_s}{\partial t} \;=\; \frac{Dp_s}{Dt} - \mathbf{v}.\nabla p_s \;=\; -\int_0^{p_s} \nabla_p.\mathbf{v}\,dp - \mathbf{v}.\nabla p_s \tag{6.10}$$

The quantity $\partial p_s/\partial t$ is known as the surface pressure *tendency*.

The boundary condition $w = 0$ at a horizontal surface ($z = 0$, say) becomes (from (6.3) and (6.6))

$$\frac{\partial z}{\partial t} \;+\; \frac{u}{a\cos\phi}\frac{\partial z}{\partial\lambda} \;+\; \frac{v}{a}\frac{\partial z}{\partial\phi} \;-\; \frac{\omega}{\rho g} \;=\; 0 \tag{6.11}$$

on $p = p_s$. In theoretical analyses, approximations to (6.11) are often resorted to, and should be carefully justified in each case (see, for example, Haynes and Shepherd (1989)). A common procedure is to apply $\omega = 0$ on $p = p_0$, where p_0 is a horizontal average surface pressure; a more accurate approximation under certain conditions (see section 10.1) is to retain the local time derivative in (6.11) and to apply $\omega = -\rho g\partial z/\partial t$ at $p = p_0$.

In numerical weather forecasting and climate simulation models it is usual to adopt a vertical coordinate for which the Earth's surface is a coordinate surface. The prototype choice is the sigma coordinate $\sigma = p/p_s$ (Phillips 1957), for which $\sigma = 1$ at the Earth's surface whether or not topography is present. The continuity equation (6.8) becomes

$$\frac{\partial p_s}{\partial t} + \nabla_\sigma.(p_s\mathbf{v}) + \frac{\partial\dot{\sigma}}{\partial\sigma} = 0 \tag{6.12}$$

where $\dot{\sigma} \equiv D\sigma/Dt$ is the σ-coordinate "vertical velocity". Thus (since $\dot{\sigma} = 0$ at $\sigma = 0$ and $\sigma = 1$):

$$\frac{\partial p_s}{\partial t} = -\int_0^1 \nabla_\sigma.(p_s\mathbf{v})d\sigma \quad . \tag{6.13}$$

The quantity $\dot{\sigma}$ may be found by eliminating $\partial p_s/\partial t$ from (6.12), (6.13), and integrating over σ:

$$\dot{\sigma} = -\int_0^\sigma \nabla_\sigma.(p_s\mathbf{v})d\sigma' - \sigma\int_0^1 \nabla_\sigma.(p_s\mathbf{v})d\sigma' \tag{6.14}$$

Haltiner and Williams (1981) give further details.

A vertical coordinate that has particular manipulative and conceptual advantages is potential temperature, θ (Starr 1945; see also Eliassen 1987). It is a permissible choice so long as no regions of neutrality or static instability exist (i.e. so long as $\partial\theta/\partial z > 0$). The material derivative in θ-coordinates is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + \dot{\theta}\frac{\partial}{\partial\theta} \tag{6.15}$$

(with the t, λ and ϕ derivatives taken at constant θ). In the case of adiabatic motion ($\dot{\theta} = 0$) the $\partial/\partial\theta$ contribution to (6.15) vanishes and advection is purely 2-dimensional (on surfaces of constant θ). The continuity equation also takes a quasi-2-dimensional form, and (whether or not the motion is adiabatic) the pressure gradient terms in (5.18) and (5.19) become linear in the horizontal gradients of the quantity $M \equiv gz + c_pT$ (known as the Montgomery potential). A partial similarity to the shallow water equations may be noted, although the fields described by the SWEs have no vertical variation – see section 5.7. Against these (and other) considerable advantages must be weighed the disadvantage that the Earth's surface is not a constant θ surface. The difficulty is not insuperable, however; see Bleck (1984) and Hsu and Arakawa (1990).

Kasahara (1974) derived forms of the HPEs using a generalised vertical coordinate s (such that $|\partial s/\partial z| \neq 0$), and some numerical weather prediction and climate simulation models use so-called hybrid coordinates which behave like σ near the Earth's surface but like pressure at high levels (Simmons and Burridge 1981, Simmons and Strüfing 1983). Hybrid coordinates have been used that behave like σ near the Earth's surface, like θ at intermediate levels and like p at high levels (Zhu *et al.* 1992, Thuburn 1993).

As we shall see in section 11, the use of pressure and sigma coordinates is not limited to models in which the hydrostatic approximation is applied. Also, a vertical coordinate equivalent to *hydrostatic pressure* has been successfully used in fully non-hydrostatic models; see Laprise (1992), Bubnová *et al.* (1995) and Geleyn and Bubnová (1997).

6.3 Further geometric and Coriolis approximations

Geometric and Coriolis approximations beyond the shallow atmosphere and "traditional" approximations of section 5.4 are common in meteorology, especially in theoretical treatments.

For accurate modelling of the global atmosphere it is essential to represent the sphericity of the Earth and the latitude variation of the Coriolis parameter $f = 2\Omega \sin\phi$. In the study of sub-planetary scale phenomena, especially when quantitatively accurate conclusions are not required, the use of simplified geometries and coarse treatment of the Coriolis parameter are convenient and justifiable. For example, if one wishes to model the circulation of a cumulus cloud, for which time-scales are typically tens of minutes and space scales a few kilometres, the use of local Cartesian geometry and neglect of Coriolis effects are entirely reasonable simplifications.

For weather systems having a horizontal space scale of 1000 km and a time-scale of a few days – the so-called synoptic scale – the Coriolis force must be accounted for, but the latitude variation of f, and spherical geometry, may be considered unimportant. The use of Cartesian geometry with a constant Coriolis parameter is a scheme known as the "f-plane". Often, Cartesian geometry is used, but in differentiated terms the Coriolis parameter is allowed a linear variation $f = f_0 + \beta y$, where f_0 and β are constants and y is northward distance from the latitude at which $f = f_0$. This scheme is known as a "β-plane": if $f_0 \cong \pm 10^{-4}\ \mathrm{s}^{-1}$, it is a "mid-latitude β-plane"; if $f_0 = 0$, it is an "equatorial β-plane". [The latitude variation of the Coriolis parameter is itself often referred to as "the β-effect".] These approximations are often introduced in a rather *ad hoc* fashion in theoretical analyses (though with due regard to the latitudinal scale of the motion being studied); for critical discussion see Pedlosky (1987), and for a Hamiltonian approach to the issue, Ripa (1997) and Graef (1998). We shall treat a particular case in section 8.3.

For illustration and later use (see sections 7-10) we note here how the height-cooordinate HPEs (5.17) – (5.22) are modified in Cartesian "β-plane" form. The material derivative becomes

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{6.16}$$

Eqs (5.17) and (5.18) are written in vector form as

$$\frac{D\mathbf{v}}{Dt} = -f\mathbf{k}\times\mathbf{v} - \frac{1}{\rho}\nabla_z p + \mathbf{F}_h \tag{6.17}$$

with $\nabla_z \equiv (\partial/\partial x, \partial/\partial y)$, $\mathbf{F}_h \equiv (F_x, F_y)$ and implied neglect of the metric terms in (5.17), (5.18). The 3-dimensional divergence term in the continuity equation (5.21) is expressed as

$$\nabla.\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{6.18}$$

The Cartesian coordinates x and y may be regarded as $x = a\lambda\cos\phi_0$, $y = a(\phi - \phi_0)$, where ϕ_0 is the central latitude of the "β-plane"; we also have $f = f_0 + \beta y = 2\Omega\sin\phi_0 + (2\Omega/a)y\cos\phi_0$. On an "$f$-plane", $\beta = 0$ and the orientation of Oxy in the horizontal is immaterial.

7. THE GEOSTROPHIC APPROXIMATION

During our discussion of hydrostatic balance, the buoyancy frequency $N \equiv ((g/\theta)\partial\theta/\partial z)^{1/2}$ emerged in section 5.2 as a key inverse time-scale in a stratified atmosphere. Another important inverse time-scale, but one having a much more systematic spatial variation, is the inertial frequency $f = 2\Omega \sin\phi$. This is the frequency with which parcels of air may circulate in the horizontal under the action only of the horizontal component of the Coriolis force. If friction and horizontal pressure gradients are absent, and the $\tan\phi$ metric terms and the latitude variation of f are neglected, then (5.18) and (5.19) give

$$\frac{D^2u}{Dt^2} + f^2u = 0$$

The period of these inertial oscillations, $2\pi/f$ $(= \pi/\Omega\sin\phi)$, is half the local pendulum day – i.e. half the period with which a Foucault pendulum will circulate about the local vertical at latitude ϕ. See Paldor and Killworth (1988) and Stommel and Moore (1989) for detailed discussion.

Large-scale motion in the extra-tropical atmosphere, on length scales of 1000km and more and time scales of a day and more (the "synoptic scale" – as noted in section 6.3), is typified by a quite different balance: the $\sin\phi$ part of the Coriolis force is nearly balanced by the horizontal pressure gradient force. In geostrophic flow, this balance is precise (see (Fig 7(a)), and (6.17) becomes

$$-f\mathbf{k}\times\mathbf{v} - \frac{1}{\rho}\nabla_z p = 0 \qquad (7.1)$$

Consistent with (7.1), the geostrophic wind $\mathbf{v}_G$, is *defined* as

$$\mathbf{v}_G \equiv \frac{1}{\rho f}\mathbf{k}\times\nabla_z p \qquad (7.2)$$

Other definitions of geostrophic wind are sometimes useful (Blackburn 1985), and one of them will be used extensively in sections 9 and 10. A definition which combines geostrophic and hydrostatic balance, and involves the $\cos\phi$ parts of the Coriolis force as well as the $\sin\phi$ parts, has been used by Hide (1971) and others; see also Shutts (1989).

In (7.1) and (7.2) (as in section 5) $\mathbf{k}$ is unit vector in the upward vertical direction. The criterion for validity of the geostrophic *approximation*, $\mathbf{v}\approx\mathbf{v}_G$, is that the acceleration term $D\mathbf{v}/Dt$ in (6.17) should be negligible compared with the Coriolis term $-f\mathbf{k}\times\mathbf{v}$. Assuming a horizontal space scale of variation L, and a horizontal velocity scale V (i.e. the horizontal flow varies by V over horizontal distance L), then $\mathbf{v}\approx\mathbf{v}_G$ according to a simple scale analysis if

$$Ro \equiv V/fL \ll 1$$

Here Ro is a Rossby number, and it has been assumed that $D/Dt \sim V/L$. Putting $V \sim 10$ ms^{-1}, $f \sim 10^{-4}$s^{-1} and $L \sim 10^6$ m gives $Ro \sim 10^{-1}$; this is a typical value for synoptic-scale weather systems in middle and high latitudes.

We consider in this section various aspects of the geostrophic wind, and the interesting consequences of combining the geostrophic and hydrostatic approximations – which together account in a diagnostic sense for many synoptic-scale features of the extra-tropical atmosphere.

7.1 Pressure and height signatures

A third possible balance in the horizontal components of the momentum equation is between the acceleration and the pressure gradient force:

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla_z p \qquad (7.3)$$

This balance may be achieved in motion systems having a much shorter time scale than the pendulum day. By applying a scale analysis to (7.3), and assuming again that $D/Dt \sim V/L$, we find that such systems will be characterised by horizontal pressure fluctuations Δp of magnitude ρV^2 (independent of horizontal scale). On the other hand, *geostrophically balanced* flow, according to (7.2), will be characterised by pressure fluctuations Δp_G of magnitude ρfVL. Hence

$$\frac{\Delta p}{\Delta p_G} \sim \frac{V}{fL} \equiv Ro \qquad (7.4)$$

The Rossby number, Ro, therefore measures the magnitude of pressure fluctuations due to circulations characterised by (7.3) compared with pressure fluctuations due to geostrophically-balanced circulations characterised by (7.1) and similar flow speeds V. In other words, the *pressure signature* of nearly-geostrophic flows is an order of magnitude greater than that of flows (of similar strength) characterised by (7.3). To the extent that $Ro \ll 1$, a map of (say) pressure at sea-level, will be dominated by the contributions of geostrophically balanced flows. Taking $\rho \sim 1$ kg m^{-3} and $V \sim 10$ ms^{-1}, we find $\Delta p \sim 10^2$Pa = 1hPa for short time-scale circulations. Taking $L \sim 10^6$ m for synoptic-scale flow gives $\Delta p_G \sim 10^3$Pa = 10hPa. Maps of sea-level pressure are therefore expected to show fluctuations of order 10hPa about a spatial mean, and such fluctuations are indeed observed: see Fig 7(b), which shows a typical sea-level pressure map.

By use of (6.2), and assuming hydrostatic balance, the definition (7.2) of geostrophic wind can be written in terms of the gradient of the height h of pressure surfaces as

$$\mathbf{v}_G \equiv \frac{g}{f}\mathbf{k} \times \nabla_p h \qquad (7.5)$$

Height variations Δh of a pressure surface associated with geostrophic flow V_G are thus of order $fLV_G/g \sim 10^2$ m (given $g \approx 10$ ms^{-2} and other values as quoted earlier). Maps of the height of a pressure surface are widely used in meteorology. Fig 7(c) shows a typical map of the height of the 300hPa (= 300mb) surface. This surface is roughly 9 km above the Earth's surface, and Fig 7(c) shows variations of about $\pm 5 \times 10^2$ m in its local height; the flow at 300hPa attains values substantially greater than 10 ms^{-1}. Even with height fluctuations of this magnitude, the 300hPa surface is very gently sloping: $\Delta h/L < 10^{-3}$.

The geostrophic wind $\mathbf{v}_G$ is horizontally divergent on pressure surfaces only to the extent that the latitude variation of the Coriolis parameter f contributes:

$$\nabla_p . \mathbf{v}_G = -\frac{\beta v_G}{f}$$

where (as in section 6.3) $\beta = (2\Omega/a)\cos\phi$ is the rate at which f increases with distance northward.

$\nabla_p.\mathbf{v}_G$ is thus much smaller than its two constituent terms if $\beta L/f \ll 1$. In middle and high latitudes this condition reduces to $L/a \ll 1$, which is reasonably well satisfied by motion having a horizontal space scale of 10^6 m. In extra-tropical latitudes, the geostrophic wind is therefore nearly non-divergent on pressure surfaces (given $L/a \ll 1$).

7.2 The differential geometry of the height field

According to (7.5), $\mathbf{v}_G$ is directed parallel to the height contours h = constant and has magnitude $(g/f)|\nabla_p h|$. Other differential geometric properties of the height field are related to other properties of the geostrophic wind field. The vertical component of the vorticity of $\mathbf{v}_G$ is

$$\mathbf{k}.\nabla_p \times \mathbf{v}_G \quad = \quad \frac{g}{f}\nabla_p^2 h \; + \; \frac{\beta u_G}{f}$$

which is dominated by the $\nabla_p^2 h$ term so long as $L/a \ll 1$. Thus $(g/f)\nabla_p^2 h$ is a good approximation to the vertical component of the vorticity of the geostrophic wind if $L/a \ll 1$.

A less well-known property of the height field is a relationship between its principal directions of curvature and the stretching and contraction axes of the geostrophic flow. Because the height field typically has a slope much less than 10^{-3} (see section 7.1) classical expressions for the principal directions of curvature may be simplified, to a very good approximation.

Consider the height of a pressure surface as a function of horizontal Cartesian coordinates on an f-plane: $z = h(x, y)$, with $f = f_0$ = constant. Assume that h and its first and second derivatives $h_x, h_y,\ h_{xx}, h_{xy}, h_{yy}$ are continuous. The projections on the (x, y) plane of the principal directions of curvature of a surface specified in Monge form $z = h(x, y)$ are lines having dy/dx given by

$$\left(\frac{dy}{dx}\right)^2\left\{h_{xy}\left(1+h_y^2\right)-h_{yy}h_xh_y\right\}-\left(\frac{dy}{dx}\right)\left\{h_{yy}\left(1+h_x^2\right)-h_{xx}\left(1+h_y^2\right)\right\}+h_{xx}h_xh_y-h_{xy}\left(1+h_x^2\right) \;=\; 0\;; \qquad (7.6)$$

(see, for example, Bell (1912), p338). If the second derivatives h_{xx}, h_{xy}, h_{yy} are of similar order of magnitude and the slopes h_x, h_y are very small ($\ll 1$), then (7.6) reduces to

$$\left(\frac{dy}{dx}\right)^2 h_{xy} \quad - \quad \left(\frac{dy}{dx}\right)\left\{h_{yy}-h_{xx}\right\} \quad - \quad h_{xy} \quad = \quad 0 \qquad (7.7)$$

In this approximation of small slope, the horizontal projections of the principal directions are perpendicular to one another. For 2-D flow, the angle θ between the dilatation axis and the x axis is given (see section 2) by

$$\tan 2\theta \quad = \quad \frac{D_2}{D_1} \quad = \quad \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\bigg/\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) \qquad (7.8)$$

In geostrophic flow on an f-plane, $u \; = \; u_g = -(g/f_0)h_y$ and $v \; = \; v_g = (g/f_0)h_x$, and (7.8) becomes

$$\tan 2\theta \quad \equiv \quad \tan 2\theta_g \quad = \quad -\left(h_{xx}-h_{yy}\right)/2h_{xy} \qquad (7.9)$$

Choose the the axes Oxy such that x lies along the dilatation axis. In this system $\theta_g = 0$, and (from (7.9)) $h_{xx} = h_{yy}$; substitution into (7.7) now shows that $(dy/dx)^2 = 1$. Hence (given that the height field is characterised by very small slopes) *the dilatation axes and contraction axes of the geostrophic flow on an f-plane bisect the principal directions of curvature of the corresponding height field.*

7.3 The thermal wind equation

An important result follows by combining the geostrophic relation (7.5) with the hydrostatic relation (6.3):

$$-\frac{\partial \mathbf{v}_G}{\partial p} = -\frac{g}{f}\mathbf{k} \times \nabla_p\left(\frac{\partial h}{\partial p}\right) = \frac{R}{fp}\mathbf{k} \times \nabla_p T \qquad (7.10)$$

Hence

$$\frac{\partial \mathbf{v}_G}{\partial z}\left(= \frac{\partial p}{\partial z}\frac{\partial \mathbf{v}_G}{\partial p}\right) = \frac{g}{fT}\mathbf{k} \times \nabla_p T \quad \left(= \frac{g}{f\theta}\mathbf{k} \times \nabla_p \theta\right) \qquad (7.11)$$

Thus the vertical *shear* of the geostrophic wind is at right angles to the temperature gradient on pressure surfaces; see Fig 8(a). Eq.(7.10) is the differential form of the *thermal wind* equation. Hydrostatic and geostrophic balance tie the wind and thermodynamic fields together in a specific way that is one of the key features of synoptic-scale meteorology.

A useful height-integrated form of (7.10) is readily obtainable in terms of the vertical distance Δz_{12} between two pressure surfaces p_2 and p_1 ($p_2 > p_1$). From the hydrostatic approximation (5.1) and the perfect gas law (3.16):

$$\Delta z_{12} = \frac{R}{g}\int_{p_1}^{p_2} T d(\ln p). \qquad (7.12)$$

The quantity Δz_{12} is known as the "thickness" of the layer between pressures p_2 and p_1. Eq.(7.12) shows that the thickness is a measure of the mean temperature of the layer. Charts of thickness, often of the layer between 1000 and 500hPa, are part of the stock-in-trade of the synoptic meteorologist. Along with appropriate height charts, they show at a glance where warm and cold air are being advected by the geostrophic wind. From (7.10) and (7.12) we find

$$\mathbf{v}_G(p_1) - \mathbf{v}_G(p_2) = \frac{g}{f}\mathbf{k} \times \nabla(\Delta z_{12}) \qquad (7.13)$$

Thus the vector *difference* in the geostrophic flow between two pressure surfaces (at the same horizontal location) bears the same relation to the thickness contours as the geostrophic wind does to the pressure or height field. This is readily understood in physical terms; see Fig 8(b).

It is worth noting that the geostrophic wind generally changes its direction as well as its magnitude with height. From (7.5) and (7.11),

$$\mathbf{k}.\left(\mathbf{v}_G \times \frac{\partial \mathbf{v}_G}{\partial z}\right) = \frac{g^2}{f^2 T}\mathbf{k}.\left(\nabla_p h \times \nabla_p T\right) \qquad (7.14)$$

Hence the geostrophic wind shear $\partial \mathbf{v}_G / \partial z$ is parallel or anti-parallel to the geostrophic wind $\mathbf{v}_G$ only if the height gradient $\nabla_p h$ is parallel or anti-parallel to the temperature gradient $\nabla_p T$.

In this brief account we have been able to mention only a few of the diagnostic results which may be obtained by combining the hydrostatic and geostrophic relations. This is currently a rather underplayed area of meteorology, but it is well described in older textbooks (see Saucier (1955)). For forecasting or for the elucidation of forecasts generated numerically (using, say, the HPEs), a time-dependent picture is required; we consider in section 9 some models which answer this need.

7.4 Other steady, balanced flows

In strictly geostrophic flow, particle accelerations and friction are absent; Coriolis and pressure-gradient forces are in precise balance and the flow is rectilinear. Balanced *circular* motion under the influence of the Coriolis and pressure gradient forces is readily analysed, and gives what is known as *gradient flow*. The balanced circular flow around a centre of low pressure is weaker than the geostrophic flow implied by the pressure or height field (the contours of which are circular in this case): the excess of the pressure gradient force over the Coriolis force supplies the acceleration that is necessary to maintain circular motion. The balanced circular flow around a centre of high pressure is larger than the geostrophic flow implied by the pressure or height field: the excess of the Coriolis force over the pressure gradient force now supplies the acceleration necessary to maintain circular motion. A possibly less expected aspect of the problem is that the supergeostrophic flow around a centre of high pressure has an upper bound. In plausibility terms, one may argue that the required acceleration in circular motion of radius r is $|\mathbf{v}|^2/r$, but the Coriolis force varies only as $|\mathbf{v}|$; hence it is reasonable that a limit exists to the extent to which the acceleration can be supplied by the Coriolis force. See Holton (1992).

Straight flow in the presence of friction may be analysed by assuming a tractable relation between the flow and the friction. The customary example is Ekman's classical treatment of the case in which $\mathbf{F} = k\,\partial^2\mathbf{v}/\partial z^2$; this is covered in textbooks such as Holton (1992) and Gill (1982). [The assumed force balance between frictional, pressure-gradient and Coriolis forces is sometimes referred to as *geotriptic*; see Bannon (1998) and references therein.] The essential physics may be exposed by considering the case in which friction is assumed to oppose the flow according to a simple linear law (first used, according to Eliassen (1984), by Guldberg and Mohn in 1876) :

$$f\mathbf{k}\times\mathbf{v} \;-\; f\mathbf{k}\times\mathbf{v}_G \;=\; -C\mathbf{v} \qquad (7.15)$$

Hence

$$f\mathbf{k}\times\mathbf{v}_{AG} \;=\; -C\mathbf{v} \qquad (7.16)$$

where $\mathbf{v}_{AG} = \mathbf{v} - \mathbf{v}_G$ is the *ageostrophic wind*, and $\mathbf{v}_{AG}$ is perpendicular to $\mathbf{v}$. If $\mathbf{v}_G$ is plotted along the diameter of a circle, $\mathbf{v}$ (the actual horizontal wind) and $\mathbf{v}_{AG}$ will meet one another on the circle (see Fig 9); also, $|\mathbf{v}| < |\mathbf{v}_G|$ and $|\mathbf{v}_{AG}| < |\mathbf{v}_G|$ (given C > 0). A simple calculation gives

$$\mathbf{v}^2 \;=\; \frac{\mathbf{v}_G^2}{\left(1+\dfrac{C^2}{f^2}\right)} \qquad \text{and} \qquad \tan\alpha \;\equiv\; \frac{C}{f} \;, \qquad (7.17)$$

where α is the angle between $\mathbf{v}_G$ and $\mathbf{v}$. Carrying the analysis through for a quadratic friction law is straightforward. In physical terms, friction reduces the flow below the geostrophic value (which is not a general property of the Ekman solution) and directs it towards lower pressure.

8. ATMOSPHERIC WAVES

The nonlinearity of the advection terms in the equations of motion cannot safely be ignored in quantitative forecasting or simulation of the atmosphere's motion. Nevertheless, a knowledge of the small amplitude oscillations and waves that are possible in a compressible, stratified, rotating atmosphere is fundamental to an appreciation of meteorological dynamics. The nonlinear terms sometimes turn out to be less important than a crude analysis might suggest (see, for example, White (1990)) and they can in any case be regarded as a forcing agency for the linearised dynamics and thermodynamics (along with diabatic and frictional sources and sinks). The properties of the possible wave motions determine how, and how quickly, local disturbances may influence distant regions (Lighthill 1978). Their properties also affect the application of numerical schemes in weather forecasting and climate simulation models (Haltiner and Williams 1981). Perhaps most important in our context, an appreciation of the possible wave motions illuminates the various approximate formulations – which do not support all modes of oscillation.

In this section we consider small oscillations of a frictionless, adiabatic, perfect gas atmosphere about an isothermal state of rest relative to the rotating Earth. Analytical results are readily obtained by use of the *f*-plane and *β*-plane approximations (see section 6.3). Only neutral waves are found because there is no energy available apart from that initially present in the perturbations. Cases in which the initial state has available energy because of velocity or horizontal temperature gradients will not be addressed, although they have played a key role in setting up the conceptual furniture of meteorological dynamics. For discussion of relevant instability problems see Drazin and Reid (1981), Gill (1982), Held (1985), Farrell (1989) and Holton (1992), for example.

The profiles of pressure, density, potential temperature and buoyancy frequency in an isothermal atmosphere were given in section 5 (Eqs(5.4)–(5.7)). Pressure and density decrease exponentially with height, potential temperature increases exponentially with height, and the buoyancy frequency N is constant. The classical adiabatic sound speed, c_0, given by

$$c_0 = \sqrt{\gamma R T_0}\,, \tag{8.1}$$

(where $\gamma = c_p/c_v$), is also independent of height, as is the scale height H_0 $(= RT_0/g)$. From (3.19), N, H_0 and c_0 obey a relation that will be used repeatedly in this and later sections:

$$\frac{N^2 H_0}{g} + \frac{gH_0}{c_0^2} = 1 \tag{8.2}$$

8.1 Oscillations of an isothermal atmosphere: f-plane case

We begin with some comments on notation. In linearised analyses it is usual to indicate perturbations from the chosen basic state by primes: $\Im = \Im_0 + \Im'$ where $\Im$ is a generic field and $\Im_0$ its value in the basic state. The equations obtained after linearization involve only $\Im_0$ and $\Im'$, and the use of primes to indicate perturbations becomes tedious and redundant. We shall drop the primes in the linearised equations: in this section, u, v, w, p, ρ and θ are to be understood as the perturbation velocity components and thermodynamic quantities. Various combinations of the thermodynamic variables feature in the linearised equations: ρ/ρ_0, p/ρ_0 and θ/θ_0 (to exercise the notation just introduced). It is tempting to introduce new symbols for all or some of these quantities, but we shall resist the temptation: it would lead us into slightly tidier equations, but their physical content might be obscured. A final issue is the choice of symbol for the angular frequency of a wave. We shall follow common usage in mathematical physics, and denote this quantity by ω; our choice is not to be confused with the use of ω in sections 6 and 9-11 to represent Dp/Dt (which is common usage in meteorology).

Linearisation of the adiabatic, frictionless, *f*-plane equations about an isothermal rest state gives:

$$\frac{\partial \mathbf{v}}{\partial t} + f_0 \mathbf{k} \times \mathbf{v} + \nabla_z \left(\frac{p}{\rho_0} \right) = 0 \tag{8.3}$$

$$\frac{\partial w}{\partial t} - g\left(\frac{\theta}{\theta_0} \right) + \left(\frac{\partial}{\partial z} - \frac{N^2}{g} \right)\left(\frac{p}{\rho_0} \right) = 0 \tag{8.4}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0} \right) + \nabla_z . \mathbf{v} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w) = 0 \tag{8.5}$$

$$\frac{\partial}{\partial t}\left(\frac{\theta}{\theta_0} \right) + \frac{N^2}{g} w = 0 \tag{8.6}$$

$$\frac{\theta}{\theta_0} - \frac{1}{c_0^2}\left(\frac{p}{\rho_0} \right) + \frac{\rho}{\rho_0} = 0 \tag{8.7}$$

Here $\nabla_z \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$. Eq (8.3) is the linearised, frictionless form of the horizontal momentum equation (6.17). Eqs (8.5) and (8.6) are respectively the linearised continuity and (adiabatic) thermodynamic equations. Eq (8.7) may be obtained by linearising (3.19) and then using (3.10) and (8.1). Eq (8.4) is the linearised vertical component of the momentum equation, with the shallow atmosphere approximation and neglect of the Coriolis and metric terms in (5.14); (8.7) has been used to eliminate the perturbation density, and (8.2) applied.

Elimination of θ/θ_0 between (8.4) and (8.6) gives

$$\left(\frac{\partial^2}{\partial t^2} + N^2 \right) w + \left(\frac{\partial}{\partial z} - \frac{N^2}{g} \right) \frac{\partial}{\partial t}\left(\frac{p}{\rho_0} \right) = 0 \tag{8.8}$$

Another relation between w and p/ρ_0, obtainable from (8.3) and (8.5) - (8.7), is

$$c_0^2\left(\frac{\partial^2}{\partial t^2} + f_0^2 \right)\left(\frac{\partial}{\partial z} - \frac{g}{c_0^2} \right) w + \left(\frac{\partial^2}{\partial t^2} + f_0^2 - c_0^2 \nabla_z^2 \right) \frac{\partial}{\partial t}\left(\frac{p}{\rho_0} \right) = 0 \tag{8.9}$$

An important special solution of (8.8) has $w = 0$ everywhere, and hence $p/\rho_0 \propto \exp\left[N^2 z/g \right]$. If wave-like form $\exp\{i(kx + ly - \omega t)\}$ is assumed, (8.9) then requires that the angular frequency ω

should obey

$$\omega^2 \;=\; c_0^2\left(k^2 + l^2\right) + f_0^2 . \tag{8.10}$$

These horizontally-propagating waves are known as *Lamb waves* (see Lamb 1932). With them are associated fluctuations of pressure, density and horizontal velocity, but not potential temperature (or vertical velocity). They are anisotropic in character, being in hydrostatic balance in the vertical, but having the structure of classical sound waves as regards their horizontal field variations. Apart from the effect of rotation ($f_0 \neq 0$) they are non-dispersive and have the phase speed of classical sound waves.

Other modes permitted by (8.8) and (8.9) obey a partial differential equation obtained by eliminating w:

$$\left[\left(\frac{\partial^2}{\partial t^2} + N^2\right)\nabla_z^2 + \left(\frac{\partial^2}{\partial t^2} + f_0^2\right)\left(\frac{\partial^2}{\partial z^2} - \frac{1}{H_0}\frac{\partial}{\partial z} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\right]\frac{\partial}{\partial t}\left(\frac{p}{\rho_0}\right) \;=\; 0 \tag{8.11}$$

Wave-like solutions of (8.11) exist in the form

$$p/\rho_0 \propto \exp\left(z/2H_0\right)\exp\{i(kx + ly + mz - \omega t)\} \tag{8.12}$$

These solutions have angular frequencies ω that obey the dispersion equation

$$\omega\left(\omega^4 - \Omega_a^2\omega^2 + \Omega_a^2\Omega_g^2\right) \;=\; 0 \tag{8.13}$$

in which the parameters Ω_a^2 and Ω_g^2 are given by

$$\Omega_a^2 \;\equiv\; c_0^2\left(k^2 + l^2 + m^2 + \frac{1}{4H_0^2}\right) + f_0^2 \tag{8.14}$$

and

$$\Omega_a^2\Omega_g^2 \;\equiv\; N^2 c_0^2\left(k^2 + l^2\right) + f_0^2 c_0^2\left(m^2 + \frac{1}{4H_0^2}\right) \tag{8.15}$$

Eq (8.13), which is obtained by substituting (8.12) into (8.11), has five solutions. Corresponding to $\omega = 0$ is a *geostrophic mode* having $f\mathbf{v} = \mathbf{k} \times \nabla(p/\rho_0)$, with u, v, $p/\rho_0 \propto \exp\left(N^2 z/g\right)$, $\rho \propto \exp\left(-gz/c_0^2\right)$, and $w = \theta = 0$. The other four solutions consist of two pairs. One pair, having high frequencies, has

$$\omega^2 \;\approx\; \Omega_a^2 \;=\; c_0^2\left(k^2 + l^2 + m^2 + \frac{1}{4H_0^2}\right) + f_0^2 \tag{8.16}$$

These are acoustic waves modified by rotation ($f_0 \neq 0$) and static compressibility ($1/H_0 \neq 0$). Even horizontally-propagating waves ($m = 0$) of this type are distinct from the Lamb waves. They are weakly dispersive through the terms in (8.16) in f_0^2 and $1/4H_0^2$. The second pair of solutions, having lower frequencies (see Gill 1982, p174), has

$$\omega^2 \approx \Omega_g^2 = \frac{N^2\left(k^2+l^2\right)+f_0^2\left(m^2+\dfrac{1}{4H_0^2}\right)}{\dfrac{f_0^2}{c_0^2}+k^2+l^2+m^2+\dfrac{1}{4H_0^2}} \tag{8.17}$$

These are buoyancy, or *gravity* waves, modified by rotation and static compressibility; they are often called *inertio-gravity* waves. Even if $f_0 = 0$, they are dispersive.

The approximations (8.16) and (8.17) are generally very good in terrestrial parameter ranges, and are sufficiently accurate for many purposes. Exact solutions may be obtained by noting that $\Omega_a^2 > 4\Omega_g^2$ and writing

$$\frac{2\Omega_g}{\Omega_a} = \sin 2\psi \, . \tag{8.18}$$

Then
$$\omega = \omega_n = \Omega_a \sin\left(\psi + \frac{n\pi}{2}\right) \; ; \quad n = 0, 1, 2, 3 \tag{8.19}$$

and the solutions may be represented graphically, as in Fig 10. From the quartic bracket of (8.13) it follows that the exact solutions $\pm\omega_a, \pm\omega_g$ obey

$$\omega_a^2 + \omega_g^2 = \Omega_a^2 \; ; \qquad \omega_a^2\omega_g^2 = \Omega_a^2\Omega_g^2 \tag{8.20}$$

The approximation (8.16) thus overestimates the true sound wave frequency, while (8.17) underestimates the true gravity wave frequency.

Apart from the introduction of the geostrophic mode having $\omega = 0$, the presence of rotation does not lead to any new modes of motion. If $f_0 = 0$, a pair of sound waves and a pair of gravity waves are still found; rotation only serves to modify their frequencies (and generally to increase dispersion). In particular, it is noticeable that there are no modes corresponding to inertial oscillations – see section 7 – whereas the (inertio-) gravity waves can be identified with buoyancy oscillations (modified by rotation). This reflects the fact that the pressure field plays a key role in gravity waves but not in pure inertial oscillations.

8.2 Filtering approximations in the f-plane problem

The consequences of various approximations and modifications of the equations of motion may be explored by repeating the above analysis with various terms omitted. An apt way of doing this [J S A Green, unpublished lecture notes, Imperial College, 1970; G J Haltiner (1971)] is to attach multiplicative tracer parameters n_i to the target terms; then $n_i = 0$ or 1 according as the associated term is omitted or retained. Our treatment in this section closely follows Green's.

Of particular interest are the *hydrostatic approximation*, in which $\partial w/\partial t$ is omitted from (8.4), and the *anelastic approximation*, in which $\partial \rho/\partial t$ is omitted from (8.5). Our freedom to omit these terms is not complete, as an examination of the energy equation implied by (8.3) - (8.7) readily shows. Consider (8.4) and (8.5) in the forms

$$n_1 \frac{\partial w}{\partial t} - g\frac{\theta}{\theta_0} + \left(\frac{\partial}{\partial z} - n_0 \frac{N^2}{g}\right)\frac{p}{\rho_0} = 0 \qquad (8.4'')$$

$$n_2 \frac{\partial}{\partial t}\left(\frac{p}{\rho_0}\right) + \nabla_z.\mathbf{v} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w) = 0 \qquad (8.5^*)$$

A tracer n_0 for $-(N^2/g)(p/\rho_0)$ has been placed in (8.4"). The local energy conservation law is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0\left(\mathbf{v}^2 + n_1 w^2 + \frac{g^2}{N^2}\left(\frac{\theta}{\theta_0}\right)^2 + \frac{n_2}{c_0^2}\left(\frac{p}{\rho_0}\right)^2\right)\right) = -\nabla_z.(p\mathbf{v}) - \frac{\partial}{\partial z}(pw) + \frac{N^2 pw}{g}(n_0 - n_2) \qquad (8.21)$$

When $n_0 = n_1 = n_2 = 1$, this reduces to a familiar form (see Gill (1982), p170); in particular, the term in $N^2 pw$ vanishes. To ensure that we do not introduce a spurious energy source, we therefore require that n_0 take the same value as n_2. In place of (8.4") we use

$$n_1 \frac{\partial w}{\partial t} - g\frac{\theta}{\theta_0} + \left(\frac{\partial}{\partial z} - n_2 \frac{N^2}{g}\right)\frac{p}{\rho_0} = 0 \qquad (8.4^*)$$

Eqs (8.8) and (8.9) become

$$\left(n_1 \frac{\partial^2}{\partial t^2} + N^2\right)w + \left(\frac{\partial}{\partial z} - n_2\frac{N^2}{g}\right)\frac{\partial}{\partial t}\left(\frac{p}{\rho_0}\right) = 0 \qquad (8.8^*)$$

and $$c_0^2\left(\frac{\partial^2}{\partial t^2} + f_0^2\right)\left(\frac{\partial}{\partial z} + n_2\frac{N^2}{g} - \frac{1}{H_0}\right)w + \left(n_2\left(\frac{\partial^2}{\partial t^2} + f_0^2\right) - c_0^2\nabla_z^2\right)\frac{\partial}{\partial t}\left(\frac{p}{\rho_0}\right) = 0 \qquad (8.9^*)$$

The solution having $w = 0$, $p/\rho_0 \neq 0$ has vertical structure $\exp(N^2 z/g)$ if $n_2 = 1$, but [from (8.9*)] $\partial/\partial t(p/\rho_0) = 0$ if $n_2 = 0$. The Lamb wave is thus absent if the term $\partial p/\partial t$ is omitted from the continuity equation (*anelastic* approximation). The Lamb wave is still present (given $n_2 = 1$) if $n_1 = 0$ (*hydrostatic* approximation).

Elimination of w between (8.8*) and (8.9*) gives

$$\left[\left(n_1\frac{\partial^2}{\partial t^2} + N^2\right)\nabla_z^2 + \left(\frac{\partial^2}{\partial t^2} + f_0^2\right)\left(\frac{\partial^2}{\partial z^2} - \frac{1}{H_0}\frac{\partial}{\partial z} - \frac{n_1 n_2}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\right]\frac{\partial}{\partial t}\left(\frac{p}{\rho_0}\right) = 0 \qquad (8.11^*)$$

which reduces to (8.11) if $n_1 = n_2 = 1$. We examine the fate of wave-like solutions of the form (8.12) in the remaining cases.

$n_1 = 0$, $n_2 = 0$ (*Hydrostatic, anelastic*)
In this case, (8.11*) gives $\omega = 0$ (geostrophic mode) or

$$\omega^2 = \frac{N^2(k^2 + l^2) + f_0^2\left(m^2 + \frac{1}{4H_0^2}\right)}{\left(m^2 + \frac{1}{4H_0^2}\right)} \qquad (8.22)$$

There are no sound waves (or Lamb waves) in this case. Eq (8.22) represents a pair of gravity waves, whose frequencies differ even from those of the approximate solution (8.17) (section 8.1).

$n_1 = 0,\ n_2 = 1$ (*Hydrostatic, elastic*)
This gives the same results as the previous case, except that Lamb waves remain.

$n_1 = 1,\ n_2 = 0$ (*Nonhydrostatic, anelastic*)
Now (8.11*) gives $\omega = 0$ (geostrophic mode), or

$$\omega^2 = \frac{N^2\left(k^2+l^2\right)+f_0^2\left(m^2+\dfrac{1}{4H_0^2}\right)}{\left(k^2+l^2+m^2+\dfrac{1}{4H_0^2}\right)} \tag{8.23}$$

There are no sound waves (or Lamb waves). Eq (8.23) represents a pair of gravity waves; in the absence of rotation $(f_0 = 0)$, their phase speeds are as given by Ω_g (see (8.17)).

In summary, the *anelastic* approximation removes sound waves and Lamb waves and leaves the gravity wave frequencies almost intact. The *hydrostatic* approximation removes sound waves but not Lamb waves, and the frequencies of the remaining gravity waves are noticeably modified.

8.3 Hydrostatic waves in an isothermal atmosphere: mid-latitude β-plane

The treatment given in sections 8.1 and 8.2 assumed a constant value f_0 of the Coriolis parameter f (as well as Cartesian geometry). Allowing f to vary with latitude (y) opens up new possibilities and brings new problems too. These problems are typical of those that arise when one seeks to approximate the equations for motion subject to gravity on a rotating sphere. We treat the variable-f linearised case because it offers a vignette of more complicated cases as well as revealing an important new type of wave motion. The hydrostatic approximation will be applied, thus limiting attention to motion having a frequency much less than the buoyancy frequency N; wave motion having a horizontal scale large compared with its vertical scale is of this type.

We consider the linearised equations of motion with, initially, $f = f_0 + \beta y$, where f_0 and β are constants. In place of (8.3) and (8.4*), we have

$$\frac{\partial \mathbf{v}}{\partial t} + f\mathbf{k}\times\mathbf{v} + \nabla_z\left(\frac{p}{\rho_0}\right) = 0 \tag{8.24}$$

$$-g\frac{\theta}{\theta_0} + \left(\frac{\partial}{\partial z} - n_2\frac{N^2}{g}\right)\frac{p}{\rho_0} = 0 \tag{8.25}$$

The linearised continuity equation (8.5*) – with tracer parameter n_2 – remains unchanged, as does the linearised thermodynamic equation (8.6) and the linearised relation (8.7).

For the *equatorial β-plane* ($f = \beta y$), analysis of (8.5*), (8.6), (8.7), (8.24) and (8.25) can be carried through without further approximation (Gill 1982); we shall refer to this case in section 8.4. Analysis of the mid-latitude β-plane case ($f = f_0 + \beta y$; $f_0 \neq 0$) can also be pursued without further approximation, but unwieldy latitude structure functions arise. Instead of following this route, we seek to replace $f = f_0 + \beta y$ by constant values, *wherever possible in a consistent way.*

From (8.24) we form equations for the time evolution of the (vertical) relative vorticity $\zeta \equiv \partial v/\partial x - \partial u/\partial y$ and the (horizontal) divergence $\delta \equiv \partial u/\partial x + \partial v/\partial y$:

$$\frac{\partial \zeta}{\partial t} + f\delta + \beta v = 0 \tag{8.26}$$

$$\frac{\partial \delta}{\partial t} - f\zeta + \beta u + \nabla_z^2(p/\rho_0) = 0 \tag{8.27}$$

With a Helmholtz decomposition of $\mathbf{v} = (u, v)$ into rotational/non-divergent, and divergent/irrotational parts, i.e. $\mathbf{v} = \mathbf{k} \times \nabla_z \psi + \nabla_z \chi$, where ψ and χ are streamfunction and velocity potential, (8.26) and (8.27) become

$$\frac{\partial}{\partial t}\nabla_z^2\psi + f\nabla_z^2\chi + \beta\frac{\partial \psi}{\partial x} + \beta\frac{\partial \chi}{\partial y} = 0 \tag{8.28}$$

$$\frac{\partial}{\partial t}\nabla_z^2\chi - f\nabla_z^2\psi + \beta\frac{\partial \chi}{\partial x} - \beta\frac{\partial \psi}{\partial y} + \nabla_z^2(p/\rho_0) = 0 \tag{8.29}$$

A naïve application of the β-plane approximation would involve setting $f = f_0$ in (8.28) and (8.29), and then proceeding with f_0 (as well as β) constant thereafter. Grimshaw (1975) noted that the β-plane approximation, in this guise, is ill-posed because it does not commute with other operations such as differentiation with respect to latitude. In the present case we reason instead that, if we set $f = f_0$ in (8.28) and (8.29), we should also omit the terms $\beta\,\partial\chi/\partial y$ and $-\beta\partial\psi/\partial y$ to ensure that the resulting forms

$$\frac{\partial}{\partial t}\nabla_z^2\psi + f_0\nabla_z^2\chi + \beta\frac{\partial \psi}{\partial x} = 0 \tag{8.30}$$

and

$$\frac{\partial}{\partial t}\nabla_z^2\chi - f_0\nabla_z^2\psi + \beta\frac{\partial \chi}{\partial x} + \nabla_z^2\left(\frac{p}{\rho_0}\right) = 0 \tag{8.31}$$

imply an acceptable kinetic energy equation. [To obtain a kinetic energy equation, multiply (8.28) by ψ, multiply (8.29) by χ and add the results. The term $f_0\left(\psi\nabla_z^2\chi - \chi\nabla_z^2\psi\right)$ can be written in divergence form as $\nabla_z.\left(f_0\left[\psi\nabla_z\chi - \chi\nabla_z\psi\right]\right)$. The term $\beta\left(\psi\,\partial\chi/\partial y - \chi\,\partial\psi/\partial y\right)$, which arises if $\beta\,\partial\chi/\partial y$ in (8.28) and $-\beta\partial\psi/\partial y$ in (8.29) are retained, is not of the required divergence form.] The omissions can also be justified by scale analysis, as follows. We wish to represent the latitude variation of f in some WKB sense; thus the scale L_y of latitude variation of the motion must be much less than that of f - i.e. the planetary scale a = radius of the Earth. Hence (for wave-like motion which is not evanescent in the horizontal), $\beta\partial\chi/\partial y$ in (8.28) must be much less than $f\nabla_z^2\chi$ in numerical terms, since $\beta \sim f/a$. Similarly, $\beta\partial\psi/\partial y$ in (8 29) must be much less than $f\nabla_z^2\psi$. [Some published accounts achieve these omissions by assuming $\partial/\partial y = 0$, which is not the appropriate limit.]

For reasons that will soon be clear, we attach a single tracer parameter (n_3) to both the first and third terms in (8.31):

$$n_3\left(\frac{\partial}{\partial t}\nabla_z^2\chi + \beta\frac{\partial \chi}{\partial x}\right) - f_0\nabla_z^2\psi + \nabla_z^2\left(\frac{p}{\rho_0}\right) = 0 \tag{8.32}$$

From (8.5*), (8.6), (8.7) and (8.25) we obtain, after a lengthy calculation assuming that w does not vanish everywhere,

$$N^2\nabla_z^2\chi - \left[\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} - \frac{1}{H_0}\right) + n_2(1-n_2)\frac{N^4}{g^2}\right]\frac{\partial}{\partial t}\left(\frac{p}{\rho_0}\right) = 0 \tag{8.33}$$

The term in N^4/g^2 in (8.33) vanishes whether $n_2 = 0$ or 1. From (8.30), (8.32) and (8.33):

$$\left\{\left[n_3\left(\frac{\partial}{\partial t}\nabla_z^2 + \beta\frac{\partial}{\partial x}\right)^2 + f_0^2\nabla_z^4\right]\left[\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} - \frac{1}{H_0}\right)\right]\frac{\partial}{\partial t} + N^2\left(\frac{\partial}{\partial t}\nabla_z^2 + \beta\frac{\partial}{\partial x}\right)\nabla_z^4\right\}\left(\frac{p}{\rho_0}\right) = 0 \tag{8.34}$$

Solutions of (8.34) of the form (8.12) obey the dispersion equation

$$\left[f_0^2K^4 - n_3\left(\beta k + \omega K^2\right)^2\right]\omega + \left(\beta k + \omega K^2\right)N^2\left(\frac{K^4}{M^2}\right) = 0 \tag{8.35}$$

in which

$$K^2 \equiv k^2 + l^2 \tag{8.36}$$

and

$$M^2 \equiv m^2 + \frac{1}{4H_0^2}\,. \tag{8.37}$$

Eq (8.35) is a cubic in ω. We shall not give a detailed analysis of the general case ($n_3 = 1$): two of the solutions are a pair of (inertio-)gravity waves modified by the β-effect; the third solution is a lower frequency solution, a *Rossby* or *planetary wave*. When $n_3 = 0$, (8.35) becomes linear in ω ; the gravity waves disappear, but the Rossby wave remains:

$$\omega = -\frac{\beta k}{\left[K^2 + \left(f_0^2/N^2\right)M^2\right]} \tag{8.38}$$

The westward propagation of Rossby waves arises because of the latitude variation of the Coriolis parameter (the β-effect). For our present purposes, the key aspect is that gravity waves are "filtered" by omitting the term $\partial/\partial t\left(\nabla_z^2\chi\right) = \partial\delta/\partial t$ from the divergence equation (8.32) [i.e. $n_3 = 0$], but Rossby waves remain (Thompson 1956). [Putting $n_3 = 0$ in (8.32) also implies omission of $\beta\,\partial\chi/\partial x$. Separate treatment of this term unproductively complicates the analysis.]

Our derivation and discussion has assumed that $w \neq 0$. What about Lamb waves? If $w = 0$ everywhere, then (from (8.6)) $\theta = 0$ also, and use of (8.7) shows that (8.5*) becomes

$$\frac{n_2}{c_0^2}\frac{\partial}{\partial t}\left(\frac{p}{\rho_0}\right) + \delta = 0 \tag{8.39}$$

The corresponding vorticity and divergence equations are the same as before ((8.30) and (8.32)). We find, instead of (8.34),

$$\left\{\left[n_3\left(\frac{\partial}{\partial t}\nabla_z^2 + \beta\frac{\partial}{\partial x}\right)^2 + f_0^2\nabla_z^4\right]\frac{n_2}{c_0^2}\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial t}\nabla_z^2 + \beta\frac{\partial}{\partial x}\right)\nabla_z^4\right\}\left(\frac{p}{\rho_0}\right) = 0 \tag{8.40}$$

Once again, we obtain a cubic dispersion relation (if $(p/\rho_0) \propto \exp[i(kx + ly - \omega t)]$ is assumed). If $n_3 = 0$ we find

$$\omega = -\frac{\beta k}{\left[K^2 + n_2\left(f_0^2/c_0^2\right)\right]} \tag{8.41}$$

Setting $n_3 = 0$ removes the two (paired) hydrostatic Lamb waves, but leaves a Rossby mode - known as the Rossby-Lamb or external Rossby mode. This mode's frequency (see (8.41)) is then dependent on whether horizontal divergence is retained by setting $n_2 = 1$ or non-divergence is enforced by setting $n_2 = 0$; see (8.39).

8.4 Waves on shallow water: mid-latitude β-plane

The above analysis is readily repeated for the shallow water equations on a mid-latitude β-plane. Appropriate linearization of the β-plane versions of (5.36), (5.37) and (5.39) gives

$$\frac{\partial \mathbf{v}}{\partial t} + f\mathbf{k} \times \mathbf{v} + \nabla_z(gh) = 0 \tag{8.42}$$

and

$$\frac{\partial h}{\partial t} + h_0 \nabla_z . \mathbf{v} = 0 \tag{8.43}$$

Eqs (8.42) is of the same form as (8.24), with gh replacing p/ρ_0. With the same approximations and tracer scheme as before, we obtain (8.30) and (8.32), with gh replacing p/ρ_0. Eq (8.43) is much simpler than (8.33). In place of (8.34) we find

$$\left\{\left[n_3\left(\frac{\partial}{\partial t}\nabla_z^2 + \beta\frac{\partial}{\partial x}\right)^2 + f_0^2\nabla_z^4\right]\frac{\partial}{\partial t} - gh_0\left(\frac{\partial}{\partial t}\nabla_z^2 + \beta\frac{\partial}{\partial x}\right)\nabla_z^4\right\}h = 0 \tag{8.44}$$

Solutions of (8.44) of $\exp\{i(kx + ly - \omega t)\}$ form have angular frequency ω which obeys the cubic

$$\left\{f_0^2 K^4 - n_3\left(\beta k + \omega K^2\right)^2\right\}\omega + \left(\beta k + \omega K^2\right)gh_0 K^4 = 0 \tag{8.45}$$

Eq (8.45) is the same as (8.35), except that gh_0 replaces $N^2/M^2 = N^2/\left(m^2 + 1/4H_0^2\right)$. The quantity $d_E \equiv N^2/g\left(m^2 + 1/4H_0^2\right)$ is called the *equivalent depth*. Every Rossby wave or (inertio-) gravity wave in an isothermal atmosphere at rest has the same dispersion relation as a counterpart Rossby or (inertio-) gravity wave on a shallow layer of incompressible fluid having mean depth d_E. By comparison of (8.40) and (8.44), it is clear that Rossby-Lamb waves have equivalent depth $c_0^2/g = \gamma RT_0/g = \gamma H_0$.

Putting $n_3 = 0$ in (8.45) reduces it to an explicit linear expression for ω:

$$\omega = -\frac{\beta k}{\left[K^2 + \left(f_0^2/gh_0\right)\right]} \tag{8.46}$$

Gravity waves have been removed by setting $n_3 = 0$, and (8.46) gives the angular frequency of the remaining shallow water Rossby wave (which is an approximation to the corresponding root of the cubic dispersion equation (8.45) with $n_3 = 1$). If we omit the term $\partial h/\partial t$ from (8.43) then we oblige the flow to be non-divergent, and the vorticity equation derived from (8.42) is simply

$$\left(\frac{\partial}{\partial t}\nabla_z^2 + \beta\frac{\partial}{\partial x}\right)\psi = 0 \tag{8.47}$$

with, from (8.34), $\psi = gh/f_0$. Waves of $\exp\{i(kx + ly - \omega t)\}$ form have

$$\omega = -\frac{\beta k}{K^2} \tag{8.48}$$

(Comparison with (8.46) shows that the imposition of non-divergence is valid if $K^2 >> f_0^2/gh_0$). These are prototypical Rossby waves – non-divergent, barotropic waves (Rossby 1939). See Hoskins *et al.* (1985), Durran (1988) and Holton (1992) for discussion of their mechanism.

The shallow-water equations in spherical polar geometry have been the vehicle of analyses of tidal motion dating back to Laplace (see Lamb (1932) and Gill (1982)); and the linearised free-wave problem, which subsumes both equatorial (section 8.5) and mid-latitude cases, was thoroughly studied by Longuet-Higgins (1968). The mid-latitude β-plane analysis given in this section provides a straightforward illustration of the key result that omission of the term $\partial\delta/\partial t$ from the divergence equation leads to the removal, or "filtering" of gravity waves, and we have already seen (section 8.3) that the result extends to the case of an isothermal, compressible atmosphere.

8.5 Tropical modes

If $f = \beta y$ – the equatorial β-plane case – the linearised problems of section 8.3 and 8.4 can be completed without approximation; Gill (1982) gives a full account. Equatorially trapped modes, which propagate in the equatorial plane, are found. As well as gravity waves and Rossby waves, two other types occur: equatorial *Kelvin waves*. and *mixed Rossby-gravity waves*. We discuss the shallow water case (in which the waves propagate in the zonal direction). Equatorial Kelvin waves are non-dispersive, eastward propagating, and similar in many ways to the classical Kelvin waves which are permitted in middle latitudes in the presence of a vertical boundary. They are hybrid, anisotropic modes, being in geostrophic balance in the meridional direction (perpendicular to the equator), but having the character of gravity waves as regards the force balance in the zonal direction (parallel to the equator). Mixed Rossby-gravity waves behave like Rossby waves in their westward propagating branch, but like gravity waves in their eastward-propagating branch. Behaviour in the case of an isothermal, compressible atmosphere (with the hydrostatic approximation and a basic state of no motion) is similar, but with the possibility of vertical propagation.

The consequences of omitting the term $\partial\delta/\partial t$ in the divergence equation are are not obvious *a priori* because of the special character of some of the tropical modes. Results depend on which other terms are omitted from the divergence equation (Gent and McWilliams 1983). Kelvin modes are absent, but one branch of mixed Rossby-gravity waves remains, and spurious high frequency modes occur if the term $\beta\,\partial\chi/\partial x$ is retained (cf. the pairing of this term with $\partial\delta/\partial t$ by the tracer parameter n_3 in (8.32)). Such spurious modes are also found on the sphere and on a mid-latitude β-plane if $\beta\,\partial\chi/\partial x$ is retained but $\partial\delta/\partial t$ omitted (Moura 1976, Allen *et al.* 1990b).

9. APPROXIMATELY GEOSTROPHIC MODELS

There are many dynamical models that are intermediate in accuracy between the HPEs (section 5) and the diagnostic geostrophic approximation (section 7) and from which inertio-gravity waves have been filtered. Wide-ranging accounts are given by McWilliams and Gent (1980) and Allen and Newberger (1993); see also Phillips (1963) and Eliassen (1984). In this section we aim not to review, but to indicate the major types of model and the guiding principles. We use the shallow water equations (5.36) – (5.39) as a simple vehicle for discussion of each of the major types except the balance class (section 9.6), for which the HPEs in pressure coordinates are more appropriate. For simplicity we shall ignore both heating and friction.

9.1 Planetary geostrophic equations (QG2)

The planetary geostrophic equations were first discussed by Burger (1958), and are known as QG2 (following Phillips (1963)). In their shallow-water guise, they replace the horizontal momentum equations (5.37), (5.38) by the diagnostic geostrophic approximation, and retain time evolution only in the continuity equation (5.40); $\mathbf{v}$ is replaced by the geostrophic wind $\mathbf{v}_G$ in the material derivative, and spherical geometry is retained – as is the latitude variation of $f = 2\Omega \sin\phi$:

$$\mathbf{v} = \mathbf{v}_G \equiv \frac{g}{f}\mathbf{k} \times \nabla_z h \tag{9.1}$$

$$\frac{Dh}{Dt_G} + h\nabla_z . \mathbf{v}_G = 0 \tag{9.2}$$

where
$$\frac{D}{Dt_G} \equiv \frac{\partial}{\partial t} + \mathbf{v}_G . \nabla_z = \frac{\partial}{\partial t} + \frac{u_G}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v_G}{a}\frac{\partial}{\partial\phi} \tag{9.3}$$

Gravity waves are absent because the implied divergence equation lacks the term $\partial\delta/\partial t$. The vorticity equation is also necessarily diagnostic, and (in the terminology of vorticity dynamics) represents a balance between planetary vorticity advection and vortex stretching/compression:

$$\nabla_z . (f\mathbf{v}_G) = \frac{v_G}{a}\frac{df}{d\phi} + f\nabla_z . \mathbf{v}_G = 0 \tag{9.4}$$

By using the continuity equation (9.2), (9.4) can be written

$$\frac{D}{Dt_G}\left\{\frac{2\Omega a \sin\phi}{h}\right\} = 0 \tag{9.5}$$

Eq.(9.5) is a form of the potential vorticity equation in which the contribution of relative vorticity is completely neglected. This is an extreme approximation, valid to the extent that the omission in (9.4) of relative vorticity advection is justified: $V/fL << L/a$. Since $Ro \equiv V/fL << 1$ is the condition for geostrophic motion, it is required that $L \sim a$: the horizontal scale of the (nearly geostrophic) motion must be comparable with the radius of the Earth.

The energy equation of QG2 is
$$\frac{\partial}{\partial t}\left(\frac{1}{2}gh^2\right) = -\nabla_z . \left(gh^2\mathbf{v}\right) \tag{9.6}$$

Angular momentum conservation is reflected in the meridional component of (9.1) in the form

$$\frac{D}{Dt_G}\left(\Omega a^2 \cos^2 \phi\right) = -g\frac{\partial h}{\partial \lambda} \tag{9.7}$$

In the context of the shallow water equations, QG2 is of interest mainly in theoretical rather than practical terms. It is a compact model that exhibits analogues of the main conservation properties, and is in this respect a fully consistent approximation; see also section 9.5.

9.2 Quasi-geostrophic model (QG1)

QG1 originated in attempts by various meteorologists in the 1930s and 1940s to derive equations describing the time-evolution of extra-tropical weather systems having a horizontal space scale, L, of about 1000km (the "synoptic scale") and typified by horizontal flow speeds, V, of order $10\, ms^{-1}$. The term *quasi-geostrophic* was suggested by Sutcliffe (1938). For such systems the Rossby number is of order 10^{-1}, the β-plane approximation is applicable since $L << a$, and the use of Cartesian geometry is justified. A 3-D version of this important model will be discussed in section 10. Here we give an outline derivation of the shallow water version, and describe how it defines both the geostrophic and ageostrophic parts of the flow.

Suppose that the fluid exhibits variations h' about its mean depth h_0:

$$h = h(x,y,t) = h_0 + h'(x,y,t) \tag{9.8}$$

Define the geostrophic flow in terms of a *mean* Coriolis parameter, f_0, as

$$\mathbf{v}_g \equiv \frac{g}{f_0}\mathbf{k}\times\nabla_z h = \frac{g}{f_0}\mathbf{k}\times\nabla_z h' = \mathbf{k}\times\nabla_z\left(\frac{gh'}{f_0}\right) \tag{9.9}$$

[In (9.9), and throughout this section, ∇_z is the Cartesian operator $(\partial/\partial x, \partial/\partial y)$.] The use of f_0, rather than the variable f, in (9.9) is a key simplifying feature in the subsequent analysis; note that $\nabla_z.\mathbf{v}_g = 0$, so that the divergent part of the flow is contained in the ageostrophic flow $\mathbf{v}_a \equiv \mathbf{v} - \mathbf{v}_g$. (The ageostrophic flow also has a rotational part, as we shall see.) The choice (9.9) of $\mathbf{v}_g$ is a good approximation to $\mathbf{v}_G$ (see (9.1)), given $L << a$. From (9.9), the streamfunction, ψ, of the geostrophic flow is

$$\psi = \frac{gh'}{f_0} \tag{9.10}$$

The horizontal components of the momentum equation (the SWE form of (6.17)) may now be written in vector form as

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k}\times\mathbf{v}_a + \beta y\mathbf{k}\times\mathbf{v}_g = 0 \tag{9.11}$$

Since $\mathbf{v} \approx \mathbf{v}_g$ to the extent that the Rossby number is small, it is reasonable to replace $D\mathbf{v}/Dt$ in (9.9) by the geostrophically-approximated (but still nonlinear) quantity

$$\frac{D\mathbf{v}_g}{Dt_g} \equiv \left(\frac{\partial}{\partial t} + \mathbf{v}_g.\nabla_z\right)\mathbf{v}_g \tag{9.12}$$

Note that $\mathbf{v}$ has been replaced by $\mathbf{v}_g$ in both the advecting and the advected flow. The replacement of the advected flow by $\mathbf{v}_g$ (and the non-divergence of $\mathbf{v}_g$) ensures the absence of gravity waves.

Given $L << a$, the Coriolis term in (9.11) depending on the ageostrophic flow may be approximated by $f_0\mathbf{k}\times\mathbf{v}_a$, so that the latitude variation of the Coriolis parameter enters only via the term βy associated with the (much larger) geostrophic flow. Eq.(9.11) then becomes

$$\frac{D\mathbf{v}_g}{Dt_g} + f_0\mathbf{k}\times\mathbf{v}_a + \beta y\mathbf{k}\times\mathbf{v}_g = 0 \tag{9.13}$$

An equation for the time-evolution of the geostrophic vorticity $\zeta_g \equiv \left(\partial v_g/\partial x - \partial u_g/\partial y\right) = \nabla_z^2\psi$ may be formed from (9.13). Noting the non-divergence of $\mathbf{v}_g$, we find the simple result

$$\frac{D}{Dt_g}\left(\nabla_z^2\psi + \beta y\right) = -f_0\nabla_z.\mathbf{v}_a \tag{9.14}$$

This is the shallow water QG1 vorticity equation.

Consider the shallow water continuity equation (5.39) in the Cartesian form

$$\frac{Dh}{Dt} + h\nabla_z.\mathbf{v}_a = 0 \tag{9.15}$$

Eq(9.15) is replaced by

$$\frac{Dh'}{Dt_g} + h_0\nabla_z.\mathbf{v}_a = 0 \tag{9.16}$$

This step involves the same approximation of the material derivative as that made in the momentum equation to reach (9.13). Also, the term $h\nabla_z.\mathbf{v}_a$ in (9.15) has been approximated by $h_0\nabla_z.\mathbf{v}_a$, which requires that fluctuations h' about the mean depth h_0 be small, i.e. $|h'|/h_0 << 1$; see (9.18), below. Elimination of $\nabla_z.\mathbf{v}_a$ between (9.14) and (9.16), and use of (9.10), then gives

$$\frac{D}{Dt_g}\left\{\nabla_z^2\psi + \beta y - \frac{f_0^2}{gh_0}\psi\right\} = 0 \tag{9.17}$$

Since D/Dt_g (see (9.12)) involves only $\partial/\partial t$, ∇_z and $\mathbf{v}_g$, (9.17) *defines the time-evolution of the geostrophic streamfunction* ψ (given suitable initial and spatial boundary conditions).

Eq.(9.17) is the shallow-water QG1 potential vorticity equation. The advected quantity is readily seen to be an approximation to $h_0(\zeta+f)/h$, valid in the case of small Rossby number and small height deviation $|h'|/h_0 << 1$. The criterion for the latter may be deduced by simple scale analysis:

$$h' \sim \frac{f_0VL}{g} \quad\Rightarrow\quad \frac{h'}{h_0} \sim \frac{f_0VL}{gh_0}$$

Hence we require $gh_0/f_0VL >> 1$, which is equivalent to

$$\Re \equiv \frac{gh_0}{V^2} >> \frac{f_0L}{V} = Ro^{-1} \tag{9.18}$$

the applicability of which depends on the mean depth h_0 as well as quantities already discussed. Taking $V = 10\text{ms}^{-1}$ and $h_0 = 10\text{km}$ gives $\Re = 10^3$, while $h_0 = 1\text{km}$ gives $\Re = 10^2$; so for a wide range of choices of h_0 (9.18) is obeyed if $Ro = 10^{-1}$ or greater. Indeed, (9.16) shows that the dynamics reduces to that of the barotropic vorticity equation (see section 5.6) if $gh_0 / f_0^2 L^2 >> 1$ (a result also noted in section 8.4).

The derivation of Eq (9.17) depends on f_0 being a constant. If f_0 had been a function of y, a conservation law would not have resulted. If $h\nabla_z.\mathbf{v}_a$ in (9.15) had not been replaced by $h_0\nabla_z.\mathbf{v}_a$ in (9.16), a conservation law would have resulted, but not in terms of a quantity linear in ψ.

Finding the height deviation h' and geostrophic flow from the streamfunction ψ, via (9.9) and (9.10), is just a matter of multiplication and spatial differentiation. The determination of the ageostrophic flow $\mathbf{v}_a$ is more subtle. Rather than eliminating $\nabla_z.\mathbf{v}_a$ between (9.14) and (9.16) we may eliminate the local time derivatives (noting (9.10)). The result is a diagnostic partial differential equation for $\nabla_z.\mathbf{v}_a$:

$$\left(\nabla_z^2 - \frac{f_0^2}{gh_0}\right)\nabla_z.\mathbf{v}_a = \frac{f_0}{gh_0}\left\{(\mathbf{v}_g.\nabla_z)\nabla_z^2\psi + \beta v_g\right\} \qquad (9.19)$$

The r.h.s. term is known if the streamfunction is known, so (9.19) determines the irrotational part of $\mathbf{v}_a$ (given appropriate boundary conditions). Eq (9.19) may also be obtained from (9.17) by algebraic application of (9.10) and use of (9.16).

The rotational part of $\mathbf{v}_a$ may be determined from an elliptic p.d.e. obtained by taking the divergence of (9.13):

$$f_0\nabla_z^2\psi_a = \nabla_z.\left[(\mathbf{v}_g.\nabla_z)\mathbf{v}_g\right] + \beta u_g - \beta y\nabla_z^2\psi \qquad (9.20)$$

where ψ_a is the streamfunction of the ageostrophic flow.

Thus the ageostrophic flow is completely defined in QG1; it is that flow which is required to maintain geostrophic balance between the geostrophic flow and height fields as the time-evolution occurs.

An energy equation is readily derived from (9.17),

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\left[\left(\nabla_z^2\psi\right)^2 + \frac{f_0^2}{gh_0}\psi^2\right]\right\} = -\nabla_z.\left\{\frac{1}{2}\left[\left(\nabla_z^2\psi\right)^2 + \frac{f_0^2}{gh_0}\psi^2\right]\mathbf{v}_g + f_0\psi\mathbf{v}_a\right\} \qquad (9.21)$$

The axial angular momentum balance is governed by the zonal component of (9.13), which – upon restoring the terms representing geostrophic balance – may be written as

$$\frac{Du_g}{Dt_g} - f_0 v_a - \beta y v_g - f_0 v_g + g\frac{\partial h}{\partial x} = 0$$

Hence $$\frac{D}{Dt_g}\left\{u_g - \int f dy'\right\} - f_0 v_a = -g\frac{\partial h}{\partial x} \qquad (9.22)$$

This form allows for the fact that the zonal (x) average of the meridional geostrophic flow vanishes, so the contribution of the ageostrophic flow must be represented.

9.3 Models based on formal considerations of accuracy

The derivation of QG1 given in section 9.2 may be formalised by a truncated Rossby number expansion of the velocity field $\mathbf{v}$. We sketch the procedure, and note that it can be taken to higher order to generate models of higher formal accuracy than QG1. For simplicity we consider the *f*-plane case ($\beta = 0$), in which (9.11) may be written as

$$\mathbf{v} = \mathbf{v}_g + \mathbf{v}_a = \frac{g}{f_0}\mathbf{k}\times\nabla_z h' + \frac{1}{f_0}\mathbf{k}\times\frac{D\mathbf{v}}{Dt} \tag{9.23}$$

Non-dimensionalise $\mathbf{v}$, h', ∇_z and $\partial/\partial t$ by extracting factors of V, f_0VL/g, $1/L$ and V/L:

$$\mathbf{v} = V\hat{\mathbf{v}};\quad h' = \frac{f_0VL}{g}\hat{h};\quad \nabla_z = \frac{1}{L}\hat{\nabla}_z;\quad \frac{\partial}{\partial t} = \frac{V}{L}\frac{\partial}{\partial\hat{t}} \tag{9.24}$$

The non-dimensional velocity $\hat{\mathbf{v}}$, depth deviation $\hat{h}$ and operators $\hat{\nabla}_z$, $\partial/\partial\hat{t}$ are each assumed to have magnitude of order unity. Eq.(9.23) becomes

$$\hat{\mathbf{v}} = \mathbf{k}\times\hat{\nabla}_{\mathbf{z}}h' + Ro\,\mathbf{k}\times\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{v}}.\hat{\nabla}_z\right)\hat{\mathbf{v}} \tag{9.25}$$

where $Ro \equiv V/f_0L$. Eq.(9.25) formally expresses the horizontal flow as the sum of the geostrophic contribution and an ageostrophic flow that is one order of magnitude smaller in Rossby number terms. The continuity equation (9.15) becomes

$$\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{v}}.\hat{\nabla}_z\right)\hat{h} + \left(\mathrm{B}Ro^{-1} + \hat{h}\right)\hat{\nabla}_z.\hat{\mathbf{v}} = 0 \tag{9.26}$$

where

$$\mathrm{B} \equiv \frac{gh_0}{f_0^2L^2}\quad\left(= \Re Ro^2\right) \tag{9.27}$$

From (9.25), the zero order approximation to $\hat{\mathbf{v}}$ is $\mathbf{k}\times\hat{\nabla}_{\mathbf{z}}h'$, which is simply the non-dimensional geostrophic flow $\hat{\mathbf{v}}_g$. If B is of order unity, or greater, the leading order balance in (9.26) is simply $\mathrm{B}Ro^{-1}\hat{\nabla}_z.\hat{\mathbf{v}} = 0$; this is consistent with $\hat{\mathbf{v}} = \hat{\mathbf{v}}_g$, since the geostrophic flow is non-divergent.

To find the next order approximation, put

$$\hat{\mathbf{v}} = \hat{\mathbf{v}}_g + Ro\hat{\mathbf{v}}_1 \tag{9.28}$$

and isolate the coefficient of Ro in (9.25) and the coefficient of Ro^0 in (9.26) – assuming that $\mathrm{B}\left(\equiv gh_0/f_0^2L^2\right) = O(1)$:

$$\hat{\mathbf{v}}_1 = \mathbf{k}\times\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{v}}_g.\hat{\nabla}_z\right)\hat{\mathbf{v}}_g \tag{9.29}$$

$$\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{v}}_g.\hat{\nabla}_z\right)\hat{h} + \mathrm{B}\hat{\nabla}_z.\hat{\mathbf{v}}_1 = 0 \tag{9.30}$$

Eqs (9.29) and (9.30) are non-dimensional forms of the *f*-plane versions of (9.13) and (9.16); elimination of $\hat{\mathbf{v}}_1$ gives

$$\left(\frac{\partial}{\partial\hat{t}} + \hat{\mathbf{v}}_g.\hat{\nabla}_z\right)\left\{\nabla_z^2\hat{h} - \frac{1}{B}\hat{h}\right\} = 0 \tag{9.31}$$

Eq.(9.31) is a non-dimensional, f-plane form of the QG1 potential vorticity equation (9.17).

A second order approximation may be obtained by putting

$$\hat{\mathbf{v}} = \hat{\mathbf{v}}_g + Ro\hat{\mathbf{v}}_1 + Ro^2\hat{\mathbf{v}}_2 \tag{9.32}$$

and isolating the coefficient of Ro^2 in (9.25) and the coefficient of Ro in (9.26). Higher order approximations may be obtained. A broadly similar procedure has been used by Allen (1993) to obtain a hierarchy of increasingly accurate "iterated geostrophic models" of 3-D stratified flow; Allen and Newberger (1993) found that the third member of the hierarchy performed very well in numerical simulations against the (Cartesian) hydrostatic primitive equations.

Power series expansions are a useful way of systematising the derivation of approximately geostrophic models which happen to conform to a single truncation of the assumed series, and of giving a critical perspective on those that do not. Such expansion methods may be suspected of lending a cosmetic veneer to what is rather crude and restricted scale analysis. In the present case (which is typical) it has been assumed that the local time-scale is of order L/V, and that a single velocity scale (V) describes spatial and temporal variations of the flow. The method may lead to lengthy equations, especially at higher orders of accuracy; these may be amenable to numerical solution but not necessarily to analysis aimed at developing insight into the physical processes involved. The method is not guaranteed to deliver equations that reproduce conservation properties at any chosen truncation (although the order Ro truncation in the above case gives the QG1 model, which does have good conservation properties). A more subtle aspect of our chosen example is that the deviation height field h' has been given special status (Muraki *et al.* 1999); it has not been expanded in powers of Ro. Other fields may equally well be granted special status: in derivations of some of the PV-balance models noted briefly in section 9.6 the potential vorticity field is considered as central to the dynamics, and not expanded in powers of the Rossby number. Pedlosky (1987), section 3.12, expands *all* variable fields as powers of Ro; see also Pedlosky (1964).

9.4 Semi-geostrophic model: SG

In QG1 the advecting flow is replaced by the geostrophic flow $\mathbf{v}_g$ wherever it occurs. QG1 requires for its validity the replacement of f by f_0 (i.e. $L << a$) and only small deviations of height h from a mean value h_0 (as well as small Rossby number). In order to remove gravity waves, only the advected flow need be replaced by $\mathbf{v}_g$ (or some other non-divergent flow) in the horizontal momentum equation. The semi-geostrophic model (SG) takes advantage of this situation by retaining advection by the total flow throughout. The shallow water equations in SG form are

$$\frac{D\mathbf{v}_g}{Dt} + f_0\mathbf{k}\times\mathbf{v}_a = 0 \tag{9.33}$$

$$\frac{Dh}{Dt} + h\nabla_z.\mathbf{v}_a = 0 \tag{9.34}$$

with $$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{v}.\nabla_z\right) \quad \text{and} \quad \mathbf{v}_g = \frac{g}{f_0}\mathbf{k}\times\nabla_z h \tag{9.35}$$

The f-plane approximation is made, and Cartesian geometry assumed. Within this framework, the *only* approximation made in SG is the neglect of the term $D\mathbf{v}_a/Dt$ in the horizontal momentum equation. (This implies retention of some terms or order Ro^2 but neglect of others; see Fjϕrtoft

(1962), p158, and Craig (1993a).) There is no restriction on the depth h, and no need to divide it into mean and deviation parts. The definition (9.35) of geostrophic flow $\mathbf{v}_g$ is the same as in QG1.

Eq (9.33) implies the axial angular momentum principle

$$\frac{D}{Dt}\{u_g - f_0 y\} = -g\frac{\partial h}{\partial x} \tag{9.36}$$

An energy equation exists in the form

$$h\frac{D}{Dt}\{\mathbf{v}_g^2 + gh\} = -\nabla_z.(gh^2\mathbf{v}_a) \tag{9.37}$$

The prospects for the existence of a potential vorticity conservation analogue in SG do not at first look bright, since (9.33) involves $(\mathbf{v}.\nabla_z)\mathbf{v}_g$, and such mixed vector advection terms are notoriously difficult to handle by the usual differential operator methods. However, following Allen *et al* (1990a), consider the components of (9.33) as linear algebraic expressions for u and v:

$$\begin{aligned} u\left(\frac{\partial u_g}{\partial x}\right) - v\left(f_0 - \frac{\partial u_g}{\partial y}\right) &= -\frac{\partial u_g}{\partial t} - f_0 v_g \\ u\left(f_0 + \frac{\partial v_g}{\partial x}\right) + v\left(\frac{\partial v_g}{\partial y}\right) &= -\frac{\partial v_g}{\partial t} + f_0 u_g \end{aligned} \tag{9.38}$$

"Solving" (9.38) for u and v gives

$$u = \left[\left(f_0 u_g - \frac{\partial v_g}{\partial t}\right)\left(f_0 - \frac{\partial u_g}{\partial y}\right) - \left(f_0 v_g + \frac{\partial u_g}{\partial t}\right)\frac{\partial v_g}{\partial y}\right]\Big/ f_0\xi_{SG} \tag{9.39}$$

$$v = \left[\left(f_0 v_g + \frac{\partial u_g}{\partial t}\right)\left(f_0 + \frac{\partial v_g}{\partial x}\right) + \left(f_0 u_g - \frac{\partial v_g}{\partial t}\right)\frac{\partial u_g}{\partial x}\right]\Big/ f_0\xi_{SG} \tag{9.40}$$

in which

$$f_0\xi_{SG} \equiv \left(f_0 + \frac{\partial v_g}{\partial x}\right)\left(f_0 - \frac{\partial u_g}{\partial y}\right) + \frac{\partial u_g}{\partial x}\frac{\partial v_g}{\partial y} \tag{9.41}$$

Now form $\nabla_z.(\xi_{SG}\mathbf{v})$ from (9.39) - (9.41). After some easy algebra and a few exhilarating cancellations we find that

$$\frac{\partial}{\partial x}(u\xi_{SG}) + \frac{\partial}{\partial y}(v\xi_{SG}) = -\frac{\partial \xi_{SG}}{\partial t} \tag{9.42}$$

Hence

$$\frac{D}{Dt}\xi_{SG} + \xi_{SG}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{9.43}$$

From (9.34) and (9.43) the SG potential vorticity equation follows:

$$\frac{D}{Dt}\left(\frac{\xi_{SG}}{h}\right) = 0 \tag{9.44}$$

The quantity ξ_{SG} is the SG absolute vorticity. Its definition (9.41) may be re-written as

$$\xi_{SG} = f_0 + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f_0}\frac{\partial(u_g, v_g)}{\partial(x, y)} = \xi_g + \frac{1}{f_0}\frac{\partial(u_g, v_g)}{\partial(x, y)} \tag{9.45}$$

Thus ξ_{SG} is the usual geostrophic absolute vorticity, ξ_g, augmented by a Jacobian term which is small in comparison if the Rossby number is small.

From (9.33) the SG divergence equation may be derived as

$$\frac{\partial \mathbf{v}}{\partial x}.\nabla_z u_g + \frac{\partial \mathbf{v}}{\partial y}.\nabla_z v_g - f_0\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + g\nabla_z^2 h = 0$$

To a leading-order approximation $(\mathbf{v} \to \mathbf{v}_g)$ of its nonlinear terms, this implies

$$\xi \equiv f_0 + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = f_0 + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} - \frac{2}{f_0}\frac{\partial(u_g, v_g)}{\partial(x, y)} \tag{9.46}$$

From (9.45) and (9.46), we see that ξ_{SG} is a worse approximation to the absolute vorticity ξ than might have been hoped; ξ_g is in fact a better approximation to ζ.

Hoskins (1975) made notable advances in the 3-D version of SG by transforming from spatial to *geostrophic coordinates*:

$$X = x + \frac{v_g}{f_0} \quad \left(= x + \frac{g}{f_0^2}\frac{\partial h}{\partial x}\right) ; \quad Y = y - \frac{u_g}{f_0} \quad \left(= y + \frac{g}{f_0^2}\frac{\partial h}{\partial y}\right) ; \quad Z = z \tag{9.47}$$

Then:

$$\frac{DX}{Dt} = u + \frac{1}{f_0}\frac{Dv_g}{Dt} = u_g \quad \text{and} \quad \frac{DY}{Dt} = v - \frac{1}{f_0}\frac{Du_g}{Dt} = v_g$$

and it is readily shown that the Jacobian of the transformation from physical to geostrophic space is none other than the (3-D) SG absolute vorticity (divided by f_0). In our 2-D context of the shallow water equations a similar result follows for the transformation $(x, y) \to (X, Y)$:

$$\frac{\partial(X, Y)}{\partial(x, y)} = \frac{\xi_{SG}}{f_0} \tag{9.48}$$

Further, Hoskins showed that the SG potential vorticity equation can be written in terms of derivatives of an augmented potential function with respect to the geostrophic coordinates in a form nearly isomorphic to the QG1 potential vorticity equation in its usual space-coordinate form.

The SG model has given important insights into the dynamics of weather systems (in particular, the formation of fronts) and into the status of QG1. It has also excited interest in other ways, prompting various questions. We have space only to juxtapose some of the questions and some of the studies that have addressed them. What is the mathematical significance of the geostrophic coordinate transformation? [Blumen 1981, Roulstone and Sewell 1997.] Can a version of SG having a more satisfactory definition of ξ_{SG} be derived? [McIntyre and Roulstone 2000.] Can SG be extended to the case of variable Coriolis parameter? [Shutts 1980, Magnusdottir and Schubert 1991.]

9.5 Hamiltonian models

Of the nearly geostrophic models so far presented, QG2 is the only one that succeeds in retaining the conservation properties of the SWEs whilst allowing latitude variation of both the Coriolis parameter and the mean depth h_0. QG2, however, is applicable only to motion on planetary scales; it is not appropriate for motion on the synoptic scale of extra-tropical weather systems. Useful extensions of the SG model have been proposed, but neither of those cited at the end of section 9.4 represents the true latitude variation of the Coriolis parameter whilst retaining SG's accuracy.

QG1, QG2 and SG have each been proposed or derived as sets of approximate equations that represent more or less heavily approximated versions of the SWEs. Conservation properties have then been investigated as a sort of health check. A requirement of good conservation properties is useful in limiting the vast number of conceivable approximations of the SWEs (or HPEs) which present themselves, but it is unhelpful if none of the candidate models passes muster.

Salmon (1983), (1988) proposed a systematic method of deriving consistent approximate models; see also Allen and Holm (1996). As noted in section 4.6, the unapproximated equations are equivalent to a variational statement, and by Noether's theorem, the symmetries of the Hamiltonian functional in that variational statement are associated with the conservation properties of the system. Making the desired approximations in the Hamiltonian then ensures that the implied (approximate) equations have consistent conservation properties.

Salmon (1983) applied this method to the shallow water equations. The coarsest level of approximation, involving the complete neglect of the velocities *u*, *v* in the Hamiltonian, delivers the planetary geostrophic equations QG2. The next level, in which *u* and *v* are replaced by their geostrophic values, leads to forms reminiscent of the SG equations, but with further terms. For the *f*-plane case, Salmon's approximate momentum equation reduces to

$$\left(\frac{\partial}{\partial t} + \mathbf{v}.\nabla_z\right)\mathbf{v}_g + f_0\mathbf{k}\times\mathbf{v}_a \;=\; -\left(\mathbf{v}_g.\nabla_z\right)\mathbf{v}_a - \frac{g}{h}\nabla_z\left(\frac{h^2}{f_0}\zeta_a\right) \qquad (9.49)$$

(ζ_a is the relative vorticity of the ageostrophic flow.) The right-hand terms in (9.49), both of which are absent in SG, are of order *Ro* smaller than the left-hand terms. Their presence is consistent with the following potential vorticity conservation and global energy conservation laws:

$$\frac{D}{Dt}\left\{\left(f_0 + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right)\Big/ h\right\} \;=\; 0 \qquad (9.50)$$

and

$$\frac{d}{dt}\iint\left(\mathbf{v}_g^2 + gh\right)h\,dxdy \;=\; 0 \qquad (9.51)$$

Allen *et al* (1990a) give details of the derivation of (9.50) and (9.51) from (9.49).

Salmon's method can be relied upon to give consistent equations, but in the present case they are not simple or familiar ones. Salmon (1985) showed that the SG model (section 9.4) may be obtained from an augmented Hamiltonian whose extra terms are compatible with the formal accuracy of the model. This demonstration of Hamiltonian structure enabled SG *per se* to be generalised to the case of variable f; see also Purser (1993), (1999).

The variational method has been successfully applied in the derivation of a number of approximate models of rotating flows having vertical structure: see, for example, Shutts (1989), Craig (1993b), Roulstone and Brice (1995), Holm (1996) and Ripa (1997).

9.6 Balance equations

Lorenz (1960) considered how the vorticity and divergence forms of the HPEs (see section 5.4) might be approximated so as to preserve a global energy invariant. There is little point in illustrating this important technique in a shallow water model because applying it delivers only QG1 and SG (Gent and McWilliams 1982). The reason for this perhaps surprising result is that the energy integrand in the shallow water system is essentially a cubic quantity $\left(h\mathbf{v}^2/2\right)$; in the stratified flow case considered by Lorenz (1960) the integrand in pressure coordinates is quadratic $\left(\mathbf{v}^2/2\right)$. Perhaps less surprisingly, the retention of potential vorticity conservation in approximated forms of the vorticity equation is far easier to achieve in the shallow water case than in the stratified flow case (in which the potential vorticity is a scalar product of vectors rather than the absolute vorticity divided by the depth of the fluid). The SWEs thus exhibit nearly the opposite properties to the HPEs written in pressure coordinates.

Lorenz's method depends on dividing the horizontal flow into its rotational (solenoidal), non-divergent part $\mathbf{v}_\psi$ and its divergent, irrotational part $\mathbf{v}_\chi$:

$$\mathbf{v} \;=\; \mathbf{v}_\psi + \mathbf{v}_\chi \;=\; \mathbf{k}\times\nabla_p\psi \;+\; \nabla_p\chi \tag{9.52}$$

This Helmholtz decomposition differs from geostrophic/ageostrophic decompositions of $\mathbf{v}$, since the geostrophic flow has a non-zero divergence if the latitude variation of the Coriolis parameter is taken into account (see section 7.1), and the ageostrophic flow in QG1 has a rotational part (see sections 9.2 and 10.1).

Vorticity and divergence equations are obtained by taking $\mathbf{k}.\nabla_p\times$ and $\nabla_p.$ of the p-coordinate version of the HPE horizontal momentum equation (5.29):

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla_p\left(\mathbf{v}^2/2\right) + \zeta\mathbf{k}\times\mathbf{v} + \omega\frac{\partial\mathbf{v}}{\partial p} \;=\; -f\mathbf{k}\times\mathbf{v} \;-\; g\nabla_p h \tag{9.53}$$

Here ∇_p is the (spherical polar) horizontal gradient operator on pressure surfaces; $\zeta=\nabla_p^2\psi$ is the relative vorticity and $\delta=\nabla_p^2\chi$ is the divergence (both defined in p-coordinate terms). Multiplication of the two resulting equations respectively by ψ and χ, and use of the identity

$$F\frac{\partial}{\partial t}\nabla_p^2 F \;=\; \nabla_p.\left\{F\nabla_p\frac{\partial F}{\partial t}\right\} - \frac{\partial}{\partial t}\left\{(\nabla_p F)^2/2\right\}$$

(with $F=\psi$ or χ) then gives equations for the time-evolution of the rotational and divergent flow specific kinetic energies $\mathbf{v}_\psi^2/2$ and $\mathbf{v}_\chi^2/2$. The thermodynamic and continuity equations are then applied to produce a total energy equation. Associations between groups of terms in the vorticity and divergence equations which retain total energy conservation are sought. One consistent approximation of the vorticity and divergence equations which is recognised in this way is the pair

$$\frac{\partial\zeta}{\partial t} \;=\; -\mathbf{v}_\psi.\nabla_p(\zeta+f) - \nabla_p.(f\mathbf{v}_\chi) - \mathbf{v}_\chi.\nabla_p\zeta - \zeta\delta - \omega\frac{\partial\zeta}{\partial p} - \nabla_p\omega.\nabla_p\frac{\partial\psi}{\partial p} \tag{9.54}$$

$$\nabla_p.\left[\left(\mathbf{v}_\psi.\nabla_p\right)\mathbf{v}_\psi\right] - \nabla_p.\left(f\nabla_p\psi\right) + g\nabla_p{}^2 h = 0 \tag{9.55}$$

Eq.(9.54) is a nearly complete form of the vorticity equation. Eq.(9.55) is a form of the divergence equation known as the Charney balance equation. It neglects the term $\partial\delta/\partial t$ (so gravity waves are absent), several elements of $\nabla_p.\{(\mathbf{v}.\nabla_p)\mathbf{v} + \omega\,\partial\mathbf{v}/\partial p\}$, and $\nabla_p.\left(f\mathbf{k}\times\nabla_p\chi\right)$.

A further energetically-consistent pair is obtained if the last four terms on the right side of (9.54) and the first on the left side of (9.55) are omitted:

$$\frac{\partial\zeta}{\partial t} = -\mathbf{v}_\psi.\nabla_p(\zeta+f) - \nabla_p.\left(f\mathbf{v}_\chi\right) \tag{9.56}$$

$$-\nabla_p.\left(f\nabla_p\psi\right) + g\nabla_p^2 h = 0 \tag{9.57}$$

Eq. (9.57) is known as the linear balance equation. The resemblance of (9.56) and (9.57) to QG1, if the f-plane or β-plane approximations *are* applied, is noticeable. However, (9.56) and (9.57) are an energetically consistent pair (as are (9.54) and (9.55)) even when the latitude variation of f is fully represented (though potential vorticity conservation is then lost). Energy consistency requires in each case the use of the complete thermodynamic equation, and the definition of kinetic energy includes only the contribution of the rotational flow. The latter aspect shows that the filtering of gravity waves by omission of the term $\partial\delta/\partial t$ from the divergence equation is intimately related to the absence of divergent flow kinetic energy from the prognostic energy equation. The same link occurs between the kinetic energy of vertical motion and the filtering of vertically-propagating sound waves via the hydrostatic approximation (see Eq. (5.24) and section 8.2).

A variant of the vorticity/divergence equation approach that is more tractable in many respects is the use of separate momentum equations for the rotational flow and for the divergent flow. The divergent flow equation is rendered in diagnostic form in order to eliminate gravity waves. Such a momentum form of the balance equations which conserves both energy and potential vorticity has been proposed by Allen (1991). This model implies spurious high frequency modes similar to those noted in section 8.5; they may be controlled by choosing initial conditions and time integration schemes carefully.

Other workers, especially in recent years, have used what may be called PV-balance models. These use the PV equation, perhaps in complete (HPE) form, as a forecasting equation, in conjunction with the Charney balance equation (9.45), the linear balance equation (9.46) or some variant. Energy conservation is generally not reproduced, but another quadratic quantity – the potential enstrophy (PV^2) – is conserved in the global average; see Gent and McWilliams (1984). Models of this type have been constructed and used by Lynch (1989), Raymond (1992), Warn *et al* (1995) and Vallis (1996); an earlier example is that of Charney (1962). The same rationale underlies the static PV inversions (see section 10.4) carried out by Davis and Emanuel (1991), Demirtas and Thorpe (1999) and others.

10. THE 3-D QUASI-GEOSTROPHIC MODEL QG1

The shallow-water version of QG1 was discussed in section 9.2. Here we focus on a version applicable to the synoptic-scale, quasi-geostrophic evolution of a 3-dimensional, perfect gas atmosphere. We begin with an outline derivation of the model in pressure coordinates, and then note a height-coordinate version that illuminates various issues, including the status of the so-called *omega equation*. Conditions for the applicability of QG1 are then summarised. In conclusion, we note the frequent occurrence in QG1 of variants of Poisson's differential equation, and discuss the application of well-known properties of these equations, with particular regard to various forms of "PV inversion". Cartesian geometry will be assumed throughout this section, and, for simplicity, friction and diabatic forcing will be neglected.

10.1 Pressure-coordinate development of QG1

Central to the development of QG1 in pressure coordinates is a hydrostatic reference state (of no motion) in which all thermodynamic variables, and height z, are functions of pressure only. The fields themselves are expressed as deviations from the reference state values. For example:

$$T = T_s(p) + T'(x,y,p,t) \tag{10.1}$$

$$\theta = \theta_s(p) + \theta'(x,y,p,t) \tag{10.2}$$

$$z = z_s(p) + z'(x,y,p,t) \tag{10.3}$$

From (6.3), hydrostatic balance of the reference state is expressed by

$$g\frac{dz_s}{dp} + \frac{RT_s}{p} = 0 \tag{10.4}$$

From (6.3), (10.1) and (10.3), the deviations z' and T$'$ obey a similar relation:

$$g\frac{dz'}{dp} + \frac{RT'}{p} = 0 \tag{10.5}$$

The geostrophic wind, $\mathbf{v}_g$, is defined as

$$\mathbf{v}_g \equiv \frac{g}{f_0}\mathbf{k}\times\nabla_p z' = \mathbf{k}\times\nabla_p\psi\,; \qquad \psi \equiv \frac{gz'}{f_0}, \quad \nabla_p \equiv \left(\left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p\right) \tag{10.6}$$

As in the SWE case (section 9.2), $|\mathbf{v}_g| \approx |\mathbf{v}_G|$ if $L << a$ ($\mathbf{v}_G$ being defined by (7.2)). From (10.5) and (3.20), the streamfunction $\psi = \psi(x,y,p,t)$ defined in (10.6) obeys

$$-\frac{\partial\psi}{\partial p} = \frac{RT'}{f_0 p} = \left(\frac{RT_s}{f_0 p\theta_s}\right)\theta' \tag{10.7}$$

Differentiating ψ with respect to p thus gives the temperature and potential temperature deviations multiplied by functions of pressure; horizontal differentiation gives $\mathbf{v}_g$ via (10.6). In terms of the ageostrophic wind $\mathbf{v}_a \equiv \mathbf{v} - \mathbf{v}_g$, the continuity equation (6.18) becomes

$$\nabla_p.\mathbf{v}_a + \frac{\partial\omega}{\partial p} = 0 \tag{10.8}$$

Apart from the use of Cartesian geometry, no approximation of the HPE forms of section 6.1 has been made so far. Approximations *are* made in the HPE horizontal momentum and thermodynamic equations. Extraction of (10.6) from (6.17) (in which $f = f_0 + \beta y$) gives

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k}\times\mathbf{v}_a + \beta y\mathbf{k}\times\mathbf{v}_g = 0 \tag{10.9}$$

As in the shallow water case, consistent with $Ro \ll 1$ and $L \ll a$, we replace the horizontal flow $\mathbf{v}$ by the geostrophic value $\mathbf{v}_g$ in the material derivative term in (10.9), and $f\mathbf{k}\times\mathbf{v}_a$ by $f_0\mathbf{k}\times\mathbf{v}_a$. In addition, we neglect the vertical advection term $\omega\,\partial\mathbf{v}/\partial p$ by comparison with $(\mathbf{v}.\nabla)\mathbf{v}$ in $D\mathbf{v}/Dt$. This is justified if $\hat{\omega}/\hat{p} \ll V/L$ (where $\hat{\omega}$ is a typical magnitude of ω and $\hat{p}$ is a scale of pressure variation in the vertical), which requires $RiRo \gg 1$ – see section 10.3. Hence

$$\frac{D\mathbf{v}}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}.\nabla_p + \omega\frac{\partial}{\partial p}\right)\mathbf{v} \rightarrow \left(\frac{\partial}{\partial t} + \mathbf{v}_g.\nabla_p\right)\mathbf{v}_g \equiv \frac{D\mathbf{v}_g}{Dt_g} \tag{10.10}$$

and (10.9) is replaced by

$$\frac{D\mathbf{v}_g}{Dt_g} + f_0\mathbf{k}\times\mathbf{v}_a + \beta y\mathbf{k}\times\mathbf{v}_g = 0 \tag{10.11}$$

Eq.(10.11) is nonlinear through the geostrophic self-advection term $(\mathbf{v}_g.\nabla_p)\mathbf{v}_g$; see (10.10). By taking $\mathbf{k}.\nabla\times$(10.11) and using (10.6), (10.8), one obtains, without further approximation:

$$\frac{D}{Dt_g}\left(\nabla_p^2\psi + \beta y\right) = f_0\frac{\partial\omega}{\partial p} \tag{10.12}$$

which is the p-coordinate, QG1 vorticity equation.

The p-coordinate form of the thermodynamic equation (5.20) may be written (in Cartesian geometry and with $Q = 0$) as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}.\nabla_p\right)\theta' + \omega\frac{\partial}{\partial p}(\theta_s + \theta') = 0 \tag{10.13}$$

Approximations are now made in (10.13) that parallel those made in (10.9), and are justified under the same conditions: $\omega\,\partial\theta'/\partial p$ is neglected compared with $\mathbf{v}.\nabla\theta'$, and $\mathbf{v}$ is replaced by $\mathbf{v}_g$. Upon use of (10.7), the QG1 thermodynamic equation is obtained as

$$\frac{D}{Dt_g}\left\{\frac{f_0^2}{S}\frac{\partial\psi}{\partial p}\right\} + f_0\omega = 0 \tag{10.14}$$

in which
$$S = S(p) \equiv -\frac{RT_s}{p\theta_s}\frac{d\theta_s}{dp}\,. \tag{10.15}$$

Elimination of ω between (10.12) and (10.14) gives the QG1 potential vorticity equation:

$$\frac{D}{Dt_g}\{QGPV\} = 0 \quad (10.16); \qquad QGPV \equiv \nabla_p^2\psi + \beta y + f_0^2\frac{\partial}{\partial p}\left(\frac{1}{S}\frac{\partial\psi}{\partial p}\right) \quad (10.17)$$

QGPV is the (p-coordinate) quasi-geostrophic potential vorticity.

Eq.(10.16) (with (10.17)) is an approximation to the conservation of Ertel's potential vorticity (Bretherton 1966, Green 1970, Kuo 1972). The analogy is between the conservation laws and not the conserved quantities; Ertel's PV is conserved under D/Dt, but QGPV under D/Dt_g, although vertical motion is allowed for in QG1. To indicate this, QGPV is sometimes called the quasi-geostrophic *pseudo* potential vorticity (Charney 1971).

Given appropriate initial and spatial boundary conditions, Eq.(10.16) determines the time evolution of the streamfunction, ψ; it is the central prognostic equation of QG1, the "signal accomplishment" of quasi-geostrophic theory (Dutton 1974).

Elimination of the local time derivatives between (10.14) and (10.11) or (10.12), leads to a diagnostic equation for ω:

$$\frac{S}{f_0}\nabla_p^2\omega + f_0\frac{\partial^2\omega}{\partial p^2} = G \tag{10.18}$$

The source function G in (10.18), which involves ψ and its spatial derivatives, may be expressed in many different forms; see Hoskins *et al.* (1985), Sanders and Hoskins (1990), Xu (1992), Carroll (1995) and Martin (1999). One of the most useful is the Q-vector form of Hoskins *et al.*(1978):

$$G = -2\nabla.\mathbf{Q} + \nabla_p.\left(\frac{f_0}{S}\beta y\frac{\partial \mathbf{v}_g}{\partial p}\right) \quad \text{with} \quad \mathbf{Q} \equiv \frac{f_0}{S}\left(\frac{\partial \mathbf{v}_g}{\partial p}.\nabla_p\right)\nabla_p\psi \tag{10.19}$$

At any time t, ω is determined by (10.18) and appropriate boundary conditions. Knowledge of ω enables the divergent part of $\mathbf{v}_a$ to be found from (10.8) [and appropriate boundary conditions]. The rotational part of $\mathbf{v}_a$ may be found from the result of taking $\nabla_p.$(10.11):

$$f_0\nabla_p^2\psi_a = \nabla_p.\left[\left(\mathbf{v}_g.\nabla\right)\mathbf{v}_g\right] + \beta u_g - \beta y\nabla_p^2\psi \tag{10.20}$$

Here $\psi_a = \psi_a(x,y,p,t)$ is the steamfunction of the ageostrophic flow (cf. (9.20)).

A knowledge of the geostrophic streamfunction $\psi = \psi(x,y,p,t)$ at some time t thus enables all other variables of the model to be determined at that time (given appropriate boundary conditions). The relevant equations are: (10.6) [for $\mathbf{v}_g$]; (10.7) [for T' and θ']; (10.18) [for ω]; (10.8) and (10.20) [for the divergent and rotational parts of $\mathbf{v}_a$].

An energy equation may be formed by multiplying (10.14) by $\partial\psi/\partial p$, adding the result to $\nabla_p\psi.$(10.11), and using (10.8):

$$\frac{D}{Dt_g}\left\{\frac{1}{2}\left(\mathbf{v}_g^2 + \frac{f_0^2}{S}\left(\frac{\partial\psi}{\partial p}\right)^2\right)\right\} + f_0\left\{\nabla_p.(\mathbf{v}_a\psi) + \frac{\partial}{\partial p}(\omega\psi)\right\} = 0 \tag{10.21}$$

The boundary condition $\omega = 0$ on $p = p_0$ is often applied in this model (and, from (10.14), determines a boundary condition on $\partial/\partial t(\partial\psi/\partial p)$). A more accurate choice is $\omega = -(f_0p_0/RT_0)\partial\psi/\partial t$ on $p = p_0$, which introduces various interesting features, both to the time evolution and to the energetics; see White (1978b) and Rõõm (1996).

10.2 QG1 in height coordinates

It is revealing to compare the analysis and results given in section 10.1 with the development of QG1 in ordinary height coordinates. In this case, all thermodynamic variables (including pressure) are represented as deviations from a hydrostatic reference state that is a function of height z only:

$$q = q(x,y,z,t) = q_0(z) + q'(x,y,z,t) \qquad (10.22)$$

where $q = p, \rho, \theta$ or T, and

$$\frac{dp_0}{dz} = -\rho_0 g\,; \qquad p_0 = \rho_0 R T_0\,; \qquad \theta_0 = T_0\left(p_{ref}/p_0\right)^{R/c_p}. \qquad (10.23)$$

Approximations are made in the hydrostatic and continuity equations as well as in the horizontal momentum and thermodynamic equations. Pedlosky (1964) gives a power series derivation assuming $Ro << 1$, $B \equiv N^2H^2/f^2L^2 \sim 1$, $N^2H/g << 1$ where

$$N^2 \equiv \frac{g}{\theta_0}\frac{d\theta_0}{dz} \qquad (10.24)$$

and L, H are, as usual, horizontal and vertical length scales of the motion. The third of Pedlosky's conditions is readily relaxed to $N^2H/g \sim 1$ in his derivation; the result is an extended QG1 z-coordinate model which includes terms that are negligible if $N^2H/g << 1$. It has been referred to variously as the "modified", "non-Doppler" or "deep" QG1 model (Blumen 1978, White 1982, Bannon 1989) and is very similar to the formulation originally proposed by Charney (1948). The geostrophic flow $\mathbf{v}_g$ is defined as

$$\mathbf{v}_g \equiv \frac{1}{\rho_0 f_0}\mathbf{k}\times\nabla_z p' = \mathbf{k}\times\nabla_z\psi\,; \qquad \psi \equiv \frac{p'}{\rho_0 f_0}, \quad \nabla_z \equiv \left(\left.\frac{\partial}{\partial x}\right|_z, \left.\frac{\partial}{\partial y}\right|_z\right) \qquad (10.25)$$

The horizontal momentum, continuity and thermodynamic equations of the model, as obtained by White (1977), may be written in the forms

$$\frac{D\mathbf{v}_g}{Dt_g} + f_0\mathbf{k}\times\hat{\mathbf{v}}_a + \beta y\mathbf{k}\times\mathbf{v}_g = 0 \qquad (10.26)$$

$$\rho_0\nabla_z.\hat{\mathbf{v}}_a + \frac{\partial}{\partial z}(\rho_0\hat{w}) = 0 \qquad (10.27)$$

$$\frac{D}{Dt_g}\left(\frac{\partial\psi}{\partial z}\right) + \frac{N^2}{f_0}\hat{w} = 0 \qquad (10.28)$$

in which

$$\frac{D}{Dt_g} \equiv \frac{\partial}{\partial t} + \mathbf{v}_g.\nabla_z \qquad (10.29)$$

and

$$\hat{\mathbf{v}}_a \equiv \mathbf{v}_a + \frac{\rho'}{\rho_0}\mathbf{v}_g\,; \qquad \hat{w} \equiv w - \frac{f_0}{g}\frac{\partial\psi}{\partial t}. \qquad (10.30)$$

$\hat{\mathbf{v}}_a$ is an extended ageostrophic flow and $\hat{w}$ an extended vertical velocity.

Given appropriate boundary conditions, (10.26) – (10.30) imply a global energy equation having a quadratic integrand (Blumen 1978) and Hamiltonian structure may be demonstrated (Holm and Zeitlin 1998). Eqs (10.26) – (10.30) imply a prognostic equation for (height-coordinate) QGPV and a diagnostic equation for the extended vertical velocity $\hat{w}$:

$$\frac{D}{Dt_g}\{QGPV\} = 0 \quad (10.31); \qquad QGPV \equiv \nabla_z^2\psi + \beta y + \frac{f_0^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial\psi}{\partial z}\right) \qquad (10.32)$$

$$\frac{N^2}{f_0}\nabla_z^2\hat{w} + f_0\frac{\partial}{\partial z}\left\{\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\hat{w})\right\} = -2\nabla_z.\left[\left(\frac{\partial \mathbf{v}_g}{\partial z}.\nabla_z\right)\mathbf{k}\times\mathbf{v}_g\right] + \beta\frac{\partial v_g}{\partial z} \qquad (10.33)$$

The QGPV equation (10.31) [with (10.32)] is the same in the cases $N^2H/g << 1$ (Pedlosky 1964) and $N^2H/g \sim 1$ (White 1977). Equation (10.33) is also the same in both cases, but when $N^2H/g << 1$ the local time derivative term in the definition (10.30) of $\hat{w}$ becomes negligible, so that $\hat{w}$ reduces to w; $\hat{\mathbf{v}}_a$ also reduces to $\mathbf{v}_a$ when $N^2H/g << 1$.

Various aspects of this "non-Doppler" QG1 model are of interest.

(i) From (10.28) and (10.30), the condition on ψ at a rigid horizontal boundary ($w = 0$) is

$$\frac{D}{Dt_g}\left(\frac{\partial\psi}{\partial z}\right) - \frac{N^2}{g}\frac{\partial\psi}{\partial t} = 0 \qquad (10.34)$$

The term in $\partial\psi/\partial t$ (which is negligible if $N^2H/g << 1$) allows for the change of apparent vertical – see section 3.6 – that accompanies a steady zonal frame translation (Betts and McIlveen 1969, White 1982). Its presence means that the effect of adding a constant U_0 to the zonal flow is not simply to shift the evolution by U_0; hence the epithet *non-Doppler* (Lindzen 1968). The same effect is seen in the pressure-coordinate QG1 model if the boundary condition $\omega = -(f_0 p_0/RT_0)\partial\psi/\partial t$ is applied at $p = p_0$ (see section 8.1).

(ii) In terms of w and $\mathbf{v}_a$, rather than $\hat{w}$ and $\hat{\mathbf{v}}_a$, the continuity equation (10.27) becomes

$$\frac{\partial\rho'}{\partial t} + (\mathbf{v}_g.\nabla_z)\rho' + \rho_0\nabla_z.\mathbf{v}_a + \frac{\partial}{\partial z}(\rho_0 w) = 0 \qquad (10.35)$$

Eq.(10.35), which is equivalent to the continuity equation used by Charney (1948), is *not* of anelastic form (see sections 8 and 11). When $N^2H/g << 1$, the terms in ρ' are negligible, and (10.35) reduces to the anelastic form of Pedlosky's (1964) model. The two models give widely different external Rossby wave phase speeds at planetary scales (White 1978b); those predicted by the non-Doppler model are in better accord with observation.

(iii) Eq.(10.33) is diagnostic for the extended vertical velocity $\hat{w}$, and for the usual vertical velocity w only in the case $N^2H/g << 1$, when $\hat{w}$ reduces to w. One might have expected that development of QG1 in height coordinates would lead to a diagnostic equation for w that was in some way a constrained version of Richardson's equation (see section 5.5), but this is not the case. Further, from (10.23), (10.25) and (10.30) we have:

$$\hat{w} \equiv w - \frac{f_0}{g}\frac{\partial\psi}{\partial t} = w - \frac{1}{\rho_0 g}\frac{\partial p'}{\partial t} = -\frac{1}{\rho_0 g}\left(\frac{\partial p'}{\partial t} + w\frac{\partial p_0}{\partial z}\right) \approx -\frac{\omega}{\rho_0 g} \qquad (10.36)$$

(since $\mathbf{v}_g.\nabla_z p' = 0$). Hence $-\rho_0 g\hat{w}$ is an approximation to the pressure-coordinate "vertical velocity" $\omega \equiv Dp/Dt$. Clearly, (10.33) is an omega equation, although it has emerged from a height-coordinate analysis. This result suggests (as one would hope, though perhaps not expect) that the development of QG1 is essentially independent of the vertical coordinate used. [See Berrisford *et al.* (1993) for a development of QG1 in θ-coordinates.]

(iv) From (ii), (iii) and Eq (10.32), we see that compressibility may be taken into account to varying degrees in QG1 models. By using the p-coordinate form (section 10.1) we achieve the most complete treatment as regards the interior equations, but at the expense (in practice) of an approximate treatment of the boundary conditions at quasi-horizontal surfaces. Within a z-coordinate framework, formally the same interior accuracy can be achieved by using the non-Doppler model; and boundary conditions are more clearly defined. Both models represent the effect of dynamic compressibility in the continuity equation: in the p-coordinate development the full HPE form is used, whilst in the non-Doppler model the term $D\rho'/Dt$ is represented by $\partial\rho'/\partial t + (\mathbf{v}_g.\nabla_z)\rho'$ (see (10.35)). In addition to dynamic compressibility, there is also a static compressibility effect (Green 1960): the variation with height of the reference state density $\rho_0(z)$. If this is neglected in (10.32), and the buoyancy frequency N is assumed independent of height also, then the pseudo potential vorticity reduces to

$$QGPV = \nabla_z^2\psi + \beta y + \frac{f_0^2}{N^2}\frac{\partial^2\psi}{\partial z^2} \tag{10.37}$$

In this Boussinesq limit, $QGPV - \beta y$ is simply the 3-dimensional Laplacian of the stream-function, ψ, if z is scaled by N/f_0. A similar simplification occurs on the left side of (10.33).

10.3 Conditions for validity and application of QG1

A summary of the assumptions made in deriving QG1 may be timely. We consider the z-coordinate case examined in section 10.2. As before, L and H are horizontal and vertical length scales over which $|\mathbf{v}|$ and $|w|$ change by the characteristic values V and W respectively.

(a) The central condition is that the Rossby number be small: $Ro \equiv V/fL << 1$.

(b) The Lagrangian time-scale is assumed to be of order L/V, so that $|D\mathbf{v}/Dt| \sim V^2/L$. Since $|(\mathbf{v}.\nabla)\mathbf{v}| \sim V^2/L$, the *local* time scale is assumed to be of order, or greater than, L/V.

(c) $L/a \le Ro$ ensures that $\mathbf{v}_g$ is a good approximation to $\mathbf{v}_G$.

(d) The neglect of vertical advection of momentum and deviation potential temperature in comparison with the horizontal parts of the advection requires $W/H << V/L$. By noting that fractional variations of potential temperature and pressure in the horizontal are of the same order, one obtains from scale analysis of the thermodynamic equation (and previous assumptions) that $W/H \sim (V/L)(RiRo)^{-1}$, where $Ri \equiv N^2H^2/V^2$ is a Richardson number. Hence it is required that $RiRo >> 1$; $RiRo \sim Ro^{-1}$ is sufficient, i.e. the Burger number $B \equiv RiRo^2 \sim 1$. For synoptic-scale motion in mid-latitudes we have $H \sim 10^4 m$ (depth of troposphere), $L \sim 10^6 m$ (synoptic horizontal scale), $N \sim 10^{-2} s^{-1}, f \sim 10^{-4} s^{-1}$; thus $B \sim 1$.

(e) Fractional variations in pressure in the horizontal are of order $fVL/gH = (V^2/gH)Ro^{-1}$, and fractional variations of density in the horizontal will be of the same order. Hence we require that the Froude number $F \equiv V^2/gH$ should obey $F << Ro$, in order that the neglect of horizontal variations of ρ in the definition of $\mathbf{v}_g$ is to be reasonable; the values quoted earlier give $F{\sim}10^{-3}$, $FRo^{-1} \sim 10^{-2}$.

(f) Notice that $Ri \equiv N^2H^2/V^2 = (N^2H/g)F^{-1}$. The importance of the quantity N^2H/g becomes clear from a scale analysis of the continuity equation using results already obtained: $\partial w/\partial z \sim (V/L)(RiRo)^{-1}$ and $(1/\rho)D\rho'/Dt_h \sim (V/L)FRo^{-1}$. Hence dynamic compressibility is important if $N^2H/g \sim 1$. The values quoted at (d) give $N^2H/g \sim 10^{-1}$, but motion having a height scale substantially greater than the depth of the troposphere will give a substantially larger value. Also, $N^2H_0/g = 2/7$ for an isothermal, diatomic, perfect gas atmosphere.

Assumptions (a)-(d) obviously appear also in the p-coordinate case. Assumptions (e)-(f) are not required for the interior equations in the p-coordinate case, but they are required for the validity of the usual boundary conditions. See White (1977) for further discussion of (a)-(f).

Derivation of the conservation properties of QG1 depends on f_0 being a constant, and on N being a function of height only. It is tempting to apply the model in contexts for which the ranges of variation of f and N are not small – to treat f_0 and N as functions of space and time, for example within the definition of QGPV (used, perhaps, as the prognostic variable in a numerical model). Such variations, particularly of f, are sometimes allowed on the understanding that they have small fractional variations over the horizontal space scale of the motion; see, for example, Kuo (1959), Charney and Stern (1962) and Pedlosky (1987).

The conservation properties of QG1 are retained if f_0 is held constant but spherical geometry is assumed and βy is replaced (in the definition of QGPV) by the true planetary vorticity $2\Omega \sin\phi$. Such a formulation has been used by many authors: see Baer (1970), Simons (1972), Baines and Frederiksen (1978), Shutts (1983b), Wu and White (1986) and Marshall and Molteni (1993). This spherical polar version of QG1 is analytically and numerically convenient but involves gross approximation of f except in the planetary vorticity term.

If N is allowed to vary horizontally in QG1 in height coordinates (or S in a pressure coordinate version – see section 10.1), then the global potential temperature budget is disrupted (Haltiner and Williams 1981). Advection by the horizontally divergent flow should be retained in this case, with the consequence that the model ceases to be of QG1 type.

The desire to allow horizontal variations of f and N, and time variations of the latter, has been a stimulus to development of the more general nearly-geostrophic models discussed in section 9 (and their 3-dimensional relatives). Another stimulus has been a desire to remove gravity waves by less invasive surgery: to make minimal approximations in the momentum equation, and – ideally – to leave the other equations intact.

10.4 Equations of Poisson type in QG1

Although it retains the nonlinearity of advection by the geostrophic flow, QG1 yields a number of linear, elliptic partial differential equations. Two-dimensional Poisson equations arise in the determination of the ageostrophic flow; see, for example, (10.20). The omega equation ((10.18) in p-coordinates, (10.33) in z-coordinates) is a 3-dimensional elliptic p.d.e., the source function being a function of ψ and its spatial derivatives. If $QGPV - \beta y$ is regarded as known, then (in (10.17) and (10.32)) it is the source function in another 3-D Poisson-type equation, in this case for ψ. The QGPV equation itself ((10.16), (10.31)) can be written as yet another Poisson-type equation – for the streamfunction *tendency* $\partial\psi/\partial t$ (see, for example, Nielsen-Gammon and Lefevre 1996). Considering the *z*-coordinate case, we can write (10.31) as

$$\left[\nabla_z^2 + \frac{f_0^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial}{\partial z}\right)\right]\frac{\partial\psi}{\partial t} = -\mathbf{v}_g.\left[\nabla_z^2\psi + \beta y + \frac{f_0^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial\psi}{\partial z}\right)\right] \qquad (10.38)$$

for which the boundary condition at rigid horizontal surfaces is, from (10.34) and (10.29), the mixed Dirichlet-Neumann form

$$\left[\frac{\partial}{\partial z} - \frac{N^2}{g}\right]\frac{\partial\psi}{\partial t} = -\left(\mathbf{v}_g.\nabla_z\right)\frac{\partial\psi}{\partial z} \qquad (10.39)$$

From classical treatments of Newtonian gravitation, electrostatics, magnetostatics, steady-state heat conduction and elastic membranes – and, indeed, fluid dynamics – equations of Poisson type are amongst the most extensively analysed and best understood in mathematical physics. [See Eriksson *et al.* (1996), chapter 15, and Batchelor (1967), section 2.4.] All the insights gained can be used to rationalise the behaviour of the linear, elliptic QG1 problems. For example, the total solution for ω or $\partial\psi/\partial t$ can be additively *attributed* to different regions or elements of the forcing or boundary conditions. This approach has been applied by Hoskins *et al.* (1985) and Clough *et al.* (1992) to the omega equation, and to the streamfunction tendency equation by Hakim *et al.* (1996) and Nielsen-Gammon and Lefevre (1996); see also Räisänen (1997). Such methods offer a rational basis for identifying cause and effect links between fields of ω or $\partial\psi/\partial t$ (the effects) and the relevant source terms (the causes).

The problem in which (10.38) is inverted for $\partial\psi/\partial t$ is conveniently referred to as *prognostic PV inversion*, and that in which $QGPV - \beta y$ is inverted for ψ as *static PV inversion* (Hakim *et al.* 1996). Static PV inversion is of particular interest. Since QGPV is conserved in the sense that $D/Dt_g\left(QGPV\right) = 0$ [in the absence of heat sources and friction, the effects of which can be taken into account if desired] QGPV may be regarded – to use the language of gravitation or electrostatics – as a "mass-like" or "charge-like" quantity. To the extent that inverting $QGPV - \beta y$ for the streamfunction ψ may be achieved, and all other fields may be calculated from ψ, the analogy of QGPV with mass or charge becomes even closer. Generalizations of this picture to EPV (with the hydrostatic approximation) subject to a balance condition such as the Charney balance equation (see section 9.6) offer a still more compelling view. A gap in the vision is that there is no unique specification of boundary conditions that can be justified by physical arguments; the boundary conditions in static PV inversion are ultimately a matter of choice (Bishop and Thorpe 1994). [In the current QG1 case, note that (10.39) gives a well-defined boundary condition on $\partial\psi/\partial t$, but no condition on ψ, at rigid horizontal boundaries.] Hakim *et al.* (1996) have noted that some choices of boundary condition violate regularity requirements; this observation is helpful in providing a constraint on the choice of boundary conditions, but it is a non-holonomic constraint in that it does not define a particular choice. Nevertheless, given awareness of the flexibility in choice of boundary conditions on horizontal surfaces, *static* PV inversion, as well as the clearly defined *prognostic* PV inversion, is useful in the development of well-founded conceptual models of weather systems and their behaviour. For a recent application of this type, involving an approximation to EPV, see Griffiths *et al.* (2000).

11. ACOUSTICALLY-FILTERED MODELS

The HPEs (section 5.4) are not the only 3-D meteorological model that lacks vertically-propagating acoustic waves but supports gravity waves. Some other acoustically-filtered (or "soundproofed") models are briefly addressed in this section. Anelastic models are discussed in section 11.1, section 11.2 describes a model which might be seen as an anelastic variant but uses pressure as vertical coordinate, and section 11.3 discusses application of its technique to represent the non-hydrostatic effect of the vertical component of the Coriolis force rather than the relative acceleration Dw/Dt. An important nonhydrostatic model which is *not* acoustically-filtered, but retains the shallow atmosphere approximation in spherical geometry, is considered in section 11.4.

11.1 Anelastic models

The linear mode analysis presented in section 8.2 suggests that gravity waves are more accurately treated when the continuity equation is written in incompressible form than when the hydrostatic approximation is applied. Using an incompressible form of the continuity equation to remove acoustic waves is therefore an attractive proposition – the more so because Lamb waves are removed as well as vertically-propagating acoustic waves. A typical *anelastic model* uses the continuity equation in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w) = 0 \qquad (11.1)$$

in which $\rho_0 = \rho_0(z)$ is a fixed profile of mean density; see Ogura and Phillips (1962). The analysis given in section 8.2 also suggests (see (8.21)) that use of (11.1) should be accompanied by neglect of a certain term in the vertical component of the momentum equation, and this is usually done. Appropriate Boussinesq forms of the horizontal components and a form of the thermodynamic equation complete the model. Application of (11.1) to the three components of the momentum equation gives a diagnostic 3-D elliptic equation for the pressure field. The formulation is then similar in many respects to the Navier-Stokes equations for incompressible flow (Williams 1969), and indeed becomes equivalent if the height variation $\rho_0(z)$ is neglected [as is appropriate if the vertical scale of the motion is much less than the scale height $H_0 = RT_0/g$ – see, for example, Mason and Brown (1999)].

Nonlinear conservation properties are good if $\rho_0(z)$ corresponds to certain simple thermodynamic states, but more general choices require specific investigation. Bannon (1995) gives a thorough discussion of this and related issues regarding a number of models of anelastic type.

The meteorological context of the anelastic equations is commonly that of cumulonimbus-scale convection; then the Coriolis terms are usually neglected and Cartesian geometry is used. If the hydrostatic approximation is applied, and Coriolis terms are included, the anelastic model becomes in the geostrophic limit the height-coordinate QG1 model that is valid when $N^2H/g \ll 1$; see section 10.2.

11.2 Nonhydrostatic convection models using pressure coordinates

Miller (1974) and Miller and Pearce (1974) first proposed and used a pressure coordinate model to describe *non*hydrostatic motion of cumulonimbus scale. Their model incorporates a reference state in hydrostatic balance and deals with nonhydrostatic departures from this state.

The horizontal momentum equation is written

$$\frac{D\mathbf{v}}{Dt} + g\nabla_p z' = \mathbf{F}_h \tag{11.2}$$

where z' is the deviation of the height z of a pressure surface from the reference state $z_s(p)$ which is associated hydrostatically with a temperature profile $T_s(p)$:

$$z(x,y,p,t) = z_s(p) + z'(x,y,p,t) \tag{11.3}$$

$$g\frac{dz_s}{dp} + \frac{RT_s}{p} = 0 \tag{11.4}$$

The continuity and thermodynamic equations are applied in the forms

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{11.5}$$

$$\frac{DT}{Dt} - \frac{\omega RT}{pc_p} = \frac{Q}{c_p} \tag{11.6}$$

Very small terms are neglected in writing (11.2) and (11.5) according to the criterion $g >> Dw/Dt$. Nonhydrostatic effects are retained in the vertical component of the momentum equation by applying the approximation

$$w \approx -\omega/g\rho_s(p) = -\omega RT_s/gp \tag{11.7}$$

in the vertical acceleration term:

$$\frac{R}{g}\frac{D}{Dt}\left\{\frac{\omega T_s}{p}\right\} + g\frac{T'}{T_s} + \frac{g^2 p}{RT_s}\frac{\partial z'}{\partial p} = 0 \tag{11.8}$$

(See the comment after (11.12), below.) In (11.8), $T' = T - T_s(p)$. The material derivative is

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \tag{11.9}$$

with $\omega \equiv Dp/Dt$ and differentiations with respect to t, x and y taken at constant p.

Miller (1974) justified the model by considering numerical magnitudes of the (small) terms omitted, and Miller and White (1984) obtained the same equations via power series expansion. The approximation (11.7), as applied in (11.8), has the effect of eliminating vertically-propagating acoustic waves, and Lamb waves may be eliminated by applying the lower boundary condition $\omega = 0$ at $p = p_0$. Integration proceeds by time-stepping in conjunction with solution of a 3-D Poisson-like equation for the height deviation z'; it is obtained by taking $\nabla_p.$(11.2), adding $(gp/RT_s)\partial/\partial p$ (11.8), and applying (11.5). The model gives analogues of energy and potential vorticity conservation laws, and is virtually isomorphic to an anelastic model in height coordinates. It has been used in a range of numerical simulations of cumulonimbus and squall-line motion; see, for example, Miller (1978) and Brugge and Moncrieff (1985). Sigma-coordinate forms (which imply Lamb waves) have been used by Xue and Thorpe (1991) and Miranda and Valente (1997) to model flow over and around orography.

White (1989b) noted the dependence of the Miller-Pearce model on the reference state profiles $z_s(p), T_s(p)$, and pointed out that the (Cartesian) vertical component of the momentum equation can be written, without approximation, as

$$\frac{Dw}{Dt} - \frac{gp}{RT}\left(1+\frac{1}{g}\frac{Dw}{Dt}\right)\left(g\frac{\partial z'}{\partial p}+\frac{RT'}{p}\right) = 0 \qquad (11.10)$$

Use of the uncritical approximation $g >> Dw/Dt$, together with

$$w \approx -\omega/\rho g = -\omega RT/gp, \qquad (11.11)$$

then replaces (11.8) by
$$\frac{R}{g}\frac{D}{Dt}\left\{\frac{\omega T}{p}\right\} + g\frac{T'}{T} + \frac{g^2 p}{RT}\frac{\partial z'}{\partial p} = 0 \qquad (11.12)$$

which does not involve the reference state. (Setting $T = T_s(p)$ in (11.12) gives (11.8).)

Equations (11.2), (11.5) and (11.6), (11.12) constitute a modified Miller-Pearce model that retains analogues of energy and potential vorticity conservation, and implies a diagnostic, Poisson-like equation for z'; Salmon and Smith (1994) demonstrate its Hamiltonian form. Rõõm (1998) and Rõõm and Mannik (1999) describe related formulations and compare their linearised behaviour.

Economical time integration of the fully-nonhydrostatic equations, with acoustic waves present, may be achieved by using a semi-implicit scheme (Tapp and White 1976). Such a formulation was the basis of a regional, mesoscale model used operationally by the U.K. Met. Office during the 1980s. The use of semi-implicit methods requires the solution of a 3-D Helmholtz-type equation at each timestep. This illustrates a common situation: a diagnostic elliptic p.d.e. has to be solved at each timestep whether special numerical methods are used to handle high frequency modes or whether these modes are filtered by approximating the governing equations. Lie (1999) gives a survey of nonhydrostatic models in the context of mesoscale weather forecasting.

11.3 Acoustically-filtered global models having a full representation of the Coriolis force

Miller and Pearce's method can be used to represent other nonhydrostatic terms in the vertical component of the momentum balance. White and Bromley (1995) noted that the term $2\Omega w\cos\phi$ in the *zonal* component of the momentum equation (4.4) is not comfortably negligible in tropical, synoptic-scale flow systems in which diabatic heating is important. To include it requires the inclusion of other terms and effects, if conservation principles are to be respected: as discussed in section 5.4, the corresponding term in the vertical component (4.6) must be kept, the shallow-atmosphere approximation must be relaxed, and various metric terms retained. [The $2\Omega\cos\phi$ terms are negligible in *adiabatic* motion if $2\Omega << N$; see Gill (1982), p 449. This condition is obeyed given $2\Omega \sim 10^{-4}s^{-1}$ and $N \sim 10^{-2}\ s^{-1}$, but it is clearly not satisfied as $N \to 0$.]

White and Bromley (1995) proposed a model based on a pseudo-radius defined as

$$r_s(p) = a + \int_p^{p_0}\frac{RT_s(p')}{p'}dp' \qquad (11.13)$$

in which p' is a dummy variable and p_0 a mean sea-level pressure. Use of $r_s(p)$ as a vertical coordinate entails no approximation; interpreting r_s as distance from the centre of the Earth does.

From (11.13),
$$\frac{Dr_s}{Dt} = -\frac{RT_s(p)}{gp}\omega \equiv \widetilde{w} \qquad (11.14)$$

This is the approximation to the vertical velocity used in the Miller-Pearce model (section 11.2). The material derivative is written as
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}.\widetilde{\nabla} \qquad (11.15)$$

with $\mathbf{u} = (u, v, \widetilde{w})$ and $\widetilde{\nabla} = \left(\frac{1}{r_s\cos\phi}\frac{\partial}{\partial\lambda} \; , \; \frac{1}{r_s}\frac{\partial}{\partial\phi} \; , \; \frac{\partial}{\partial r_s}\right)$ (11.16)

The following pressure-coordinate equations were proposed:
$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r_s\cos\phi}\right)(v\sin\phi - \widetilde{w}\cos\phi) + \frac{g}{r_s\cos\phi}\frac{\partial z'}{\partial\lambda} = F_\lambda \qquad (11.17)$$
$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r_s\cos\phi}\right)u\sin\phi + \frac{v\widetilde{w}}{r_s} + \frac{g}{r_s}\frac{\partial z'}{\partial\phi} = F_\phi \qquad (11.18)$$
$$-2\Omega u\cos\phi - \left(\frac{u^2+v^2}{r_s}\right) - g\frac{T'}{T_s} + g\frac{\partial z'}{\partial r_s} = 0 \qquad (11.19)$$
$$\widetilde{\nabla}_p.\mathbf{v} + \frac{1}{r_s^2}\frac{\partial}{\partial p}\left(r_s^2\omega\right) = 0 \qquad (11.20)$$
$$\frac{D\theta}{Dt} = \left(\frac{\theta}{Tc_p}\right)Q \qquad (11.21)$$

In the continuity equation (11.20),
$$\widetilde{\nabla}_p.\mathbf{v} = \frac{1}{r_s\cos\phi}\left\{\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right\} \qquad (11.22)$$

Some of the terms retained in these equations are typically very small, but they are needed for the delivery of the following conservation properties:
$$\frac{D}{Dt}\{(u + \Omega r_s\cos\phi)r_s\cos\phi\} = F_\lambda r_s\cos\phi - g\frac{\partial z'}{\partial\lambda} \qquad (11.23)$$
$$\frac{D}{Dt}\left(\frac{1}{2}\mathbf{v}^2 + c_pT\right) + \widetilde{\nabla}_p.(\mathbf{v}gz) + \frac{1}{r_s^2}\frac{\partial}{\partial p}\left(r_s^2\omega gz\right) = Q + \mathbf{v}.\mathbf{F}_h \qquad (11.24)$$
$$\rho_s\frac{D}{Dt}\left(\frac{\widetilde{\mathbf{Z}}.\widetilde{\nabla}\theta}{\rho_s}\right) = \widetilde{\nabla}.\left[\theta\widetilde{\nabla}\times\mathbf{F}_h + \widetilde{\mathbf{Z}}\frac{D\theta}{Dt}\right] \qquad (11.25)$$

In (11.25),
$$\widetilde{\mathbf{Z}} \equiv \left[-\frac{1}{r_s}\frac{\partial}{\partial r}(vr_s)\, , \, 2\Omega\cos\phi + \frac{1}{r_s}\frac{\partial}{\partial r_s}(ur_s)\, , \, 2\Omega\sin\phi + \frac{1}{r_s\cos\phi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\phi}(u\cos\phi)\right)\right] \qquad (11.26)$$
$$= 2\mathbf{\Omega} + \widetilde{\nabla}\times\mathbf{u}$$

and, for any vector $\mathbf{A} = (\mathbf{A}_h, A_r)$,

$$\widetilde{\nabla}.\mathbf{A} \;=\; \widetilde{\nabla}_p.\mathbf{A}_h + \frac{1}{r_s^2}\frac{\partial}{\partial r_s}\left(r_s^2 A_r\right) \tag{11.27}$$

Eqs (11.23) - (11.25) are axial angular momentum, energy and potential vorticity conservation laws; their derivation is eased by noting an isomorphism with corresponding equations for the motion of an incompressible fluid. Only the term Dw/Dt (in the vertical component of the momentum equation) is unrepresented in (11.18) - (11.22). Its inclusion could be achieved by using the Miller-Pearce technique directly, but at the expense of having to solve an elliptic 3-D p.d.e. for z at each timestep; in the global model as set out above this is not necessary.

Roulstone and Brice (1995) demonstrated the Hamiltonian structure of isomorphs of (11.17) - (11.21), and White and Bromley (1995) derived σ-coordinate versions by direct transformation and described an integration strategy. Versions using another vertical coordinate system form the dynamical basis of the UK Met. Office's Unified Model (see Cullen (1993)) which is a gridpoint numerical model. The presence of the pseudo-radius $r_s(p)$ would complicate implementation of these equations in current spectral numerical models (see section 12).

11.4 A nonhydrostatic, global, shallow atmosphere model

As we noted in section 5.4, the HPEs omit the $\cos\phi$ Coriolis terms and various metric terms from the components of the momentum equation, and adopt the shallow-atmosphere approximation throughout. The nonhydrostatic, global, shallow-atmosphere model used by Tanguay *et al.* (1990) [see also Müller (1989)] consists of the HPEs augmented only by the term Dw/Dt in the vertical component of the momentum equation. In place of (5.19), the model therefore has

$$\frac{Dw}{Dt} + g + \frac{1}{\rho}\frac{\partial p}{\partial z} \;=\; 0 \tag{11.28}$$

with D/Dt given by the shallow atmosphere form (5.16). The retention of Dw/Dt in (11.28) appears unjustifiable for a range of mesoscale motion given that $-2\Omega u\cos\phi$ has been neglected (Draghici 1989), but the model is of theoretical interest because of its good conservation properties. The axial angular momentum conservation law is the HPE form (5.29), the energy conservation law is the HPE form (5.24) but with specific kinetic energy $\frac{1}{2}\left(\mathbf{v}^2 + w^2\right)$, and the PV law is of the HPE form (5.25) but with absolute vorticity ξ defined by

$$\xi \;\equiv\; \left(\frac{1}{a}\frac{\partial w}{\partial\phi} - \frac{\partial v}{\partial z}\;,\;\frac{\partial u}{\partial z} - \frac{1}{a\cos\phi}\frac{\partial w}{\partial\lambda}\;,\;2\Omega\sin\phi + \frac{1}{a\cos\phi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\phi}(u\cos\phi)\right]\right) \tag{11.29}$$

Roulstone and Brice (1995) showed that isomorphs of (11.17) – (11.21) arise when the functional form of the Hamiltonian is modified to exclude the contribution of the vertical motion to the kinetic energy. They showed too that the HPEs arise if the geometric factors in the Hamiltonian integral (its "phase space") are also modified. It seems likely that the model of Tanguay *et al.* (1990) arises when the geometric factors in the Hamiltonian are modified but its functional form is left unchanged, and thus that there are two dynamically-consistent models intermediate in accuracy between the HPEs and the unapproximated equations – the model of Tanguay *et al.* (1990) and the model of White and Bromley (1995). These suggestions deserve further study.

12. DISCUSSION: DYNAMICAL MODELS, NUMERICAL WEATHER PREDICTION AND CLIMATE SIMULATION

This article has given an account of the basis and nature of many of the approximate models of meteorological dynamics. Here some remarks are offered on the approximation problem, on the applications of the approximate models, and on basic issues in the design of numerical models.

A theory of approximation for the equations governing meteorological flows is not yet fully developed or its rationale agreed upon. The retention of conservation properties during the approximation process is an attractive guiding principle, and if it is given exclusive priority, then the Hamiltonian technique pioneered by Salmon (1983) offers the best way forward. If a dynamical model having the desired conservation properties has been derived by other means, then the demonstration of Hamiltonian structure lends further credence. Prompted by evidence that Hamiltonian structure does not ensure superior performance in numerical practice (see, for example, Barth *et al.* (1990) and Allen and Newberger (1993)) some researchers consider that retention of all conservation properties should not be the priority. Few consider that conservation properties should be disregarded, but opinions differ on which should be favoured. Traditionally, global energy conservation has received most emphasis, but Lagrangian potential vorticity conservation is increasingly seen as paramount; at the time of writing, some striking results are emerging from studies of balanced, PV-conserving versions of the SWEs (McIntyre and Norton 2000).

What are the approximate models used for in meteorology? As we have noted, the HPEs are the foundation of most of the numerical weather prediction and climate simulation models run by operational and research centres worldwide, but the use of more accurate models is becoming more widespread. For example, the U.K. Met. Office's Unified Model is based on the acoustically-filtered equations discussed in section 11.3, and the use of virtually unapproximated forms is planned – the strategy being to use semi-implicit integration schemes to overcome the restriction to very short time steps which the presence of acoustic modes would otherwise impose; see Staniforth (2000). A trend towards the use of formulations more accurate than the HPEs is evident also in ocean modelling (Marshall *et al.* 1997). The utility of approximate models – especially those more heavily approximated than the HPEs – also lies in the development of a conceptual framework for the analysis of numerically-generated and observational data. Such a framework is necessary both in general scientific terms and to guide the development of better techniques for assimilating data into numerical models and effecting their time integration. We shall briefly discuss these aspects.

Analysing and understanding a simplified model is clearly easier than analysing and understanding a complicated one. Some uses of the barotropic vorticity equation in this respect were noted in section 5.6. In so far as it embodies notions of vorticity and temperature evolution and advection, the QG1 model systematises these concepts of the synoptic meteorologist and weather forecaster. A more modern view – not necessarily a competing view – is the PV perspective (Hoskins *et al.* 1985) which is embodied in QG1 and in some of the other balanced models discussed in section 9. The articles in Meteorological Applications (1997) elucidate the current interplay of ideas in this area.

Approximate models also play a major role as apparatus for thought experiments (which may be carried out either analytically or numerically). A particularly influential type of thought experiment is the stability analysis: a steady flow is subject to perturbations at $t = 0$ and the subsequent evolution determined by solving linearised forms of the governing equations. Eady's baroclinic stability problem (Eady 1949) can be solved analytically in the QG1 case, and its dominant eigenmodes resemble structures seen in developing mid-latitude weather systems. A large literature has grown up which extends Eady's analysis to more realistic initial steady flows and assumed external conditions, and explores development into the nonlinear stages using either analytical or numerical methods (in many cases using less heavily approximated dynamics); see Hart (1979) and Held and Hoskins (1985) for reviews. It could be argued that such stability problems, though illuminating, have been somewhat overemphasised, since one may reasonably enquire how the real atmosphere could ever reach the supercritical states which may be chosen for investigation. [Some recent work in this area has focussed on influences that stabilise flows, and on the hypothesis that the real atmosphere evolves close to a stability threshold; see Mole and James (1990), Stone and Nemet (1996), Dong and James (1997), Harnik and Lindzen (1998) and Nakamura (1999)]. Also, the complete initial value problem is complicated by the presence of continuous spectrum instabilities as well as normal mode growth (or decay): many early analyses emphasised the latter at the expense of the former – see Farrell (1989). This important aspect of the stability problems reflects the non-self-adjointness of the relevant operators; Held (1985) gives a lucid account.

As reviewed by Errico (1997), adjoint operator theory has recently found practical application in ensemble forecasting and data assimilation. Ensemble methods (see Farrell 1990, Buizza and Palmer 1995, Buizza *et al.* 1997) aim to determine the sensitivity of a numerical forecast to its initial conditions. Since numerical integrations are time-consuming and the number of degrees of freedom is vast, ways must be found to identify patterns which capture the main instabilities of the initial flow and hence the sensitivity of the forecast. One way (of several) is to calculate singular vectors, having defined a suitable norm to gauge differences between integrations with slightly different initial conditions. The assimilation of observed data is a key part of the process of numerical weather prediction; see Daley (1991). The 4-D variational technique (Talagrand and Courtier 1987) minimises a *cost function* that measures differences between evolving model values and observations over a chosen assimilation "window" (typically a few hours). The minimisation is carried out with respect to fields at the beginning of the window period, and may be subject to constraints whose nature reflects knowledge of atmospheric behaviour developed from more heavily approximated dynamical models such as the semi-geostrophic and quasi-geostrophic forms.

Further examples of the use of knowledge gained from approximate models are noted in the following brief discussion of numerical model design.

Because of the nature of the governing equations, any reliable numerical model of the atmophere must use a finite representation of its fields. That finite representation may involve field values at a number of chosen points (the gridpoint method) or fields specified in terms of amplitudes of a number of chosen functions (the Galerkin method). The Galerkin representation most frequently chosen for horizontal variations is a *spectral* representation in terms of surface spherical harmonics Y_n^m. (The viability of the technique depends on the use of finite Fourier transforms and Gaussian integration to handle product terms; see Hoskins and Simmons (1975), Côté and Staniforth (1988), Hortal and Simmons (1991) and Temperton *et al* (2000)). Almost all models use the gridpoint method for vertical variations; thus, in global spectral models, the fields are represented by finite spherical harmonic expansions at a number of levels (typically 30 or more). There is advantage in using a staggered arrangement in which different fields are held at different levels. Many models hold the relevant vertical velocities at levels between those at which the horizontal velocity components and the potential temperature are held. This "Lorenz" arrangement cannot give the most natural and accurate depiction of thermal wind balance (see section 7.3) and it also leads to the occurrence of spurious vertical modes (Schneider 1987). Thermal wind balance (Eq(7.11)) is better represented by holding potential temperature at the intermediate level – the "Charney-Phillips" arrangement. The practical advantage of the Charney-Phillips arrangement over the Lorenz arrangement has yet to be demonstrated conclusively, but its theoretical advantage (at least for geostrophically-balanced motion) is partly an implication of the QG1 model. Horizontal grid staggering in gridpoint models is an issue of even greater variety; see Adcroft *et al.* (1999) for a recent discussion.

The spectral method of representing horizontal field variations has the considerable advantage of satisfactorily treating field variations close to coordinate poles. (Indeed, the "triangular truncation" of the spherical harmonic series gives an isotropic representation which is independent of the location of the coordinate pole; see Hoskins and Simmons (1975).) Use of a gridpoint representation on points defined by the intersections of circles of latitude and longitude leads to numerous difficulties in the vicinity of the poles. Amongst various ways of coping with these, one of the most obvious and attractive is to use another distribution of points. Because of the existence of only 5 regular polyhedra in 3-D space, a regular distribution of more than 20 points over the surface of a sphere is not possible. However, a quasi-regular distribution may be achieved by triangulating an icosahedron and centrally projecting the triangle vertices onto the circumscribing sphere (Sadourny *et al.* 1968, Thuburn 1997, Majewski 1998). Alternatively, projections of points on an inscribed cube may be used (Rančić *et al.* 1995, McGregor 1996). Another way of mitigating the pole problem is to use a subsidiary grid in the vicinity of the geographical poles. This is a particularly attractive option when it is used in conjunction with the semi-Lagrangian representation of material derivatives; see Staniforth and Côté (1991) and references therein. Regional models can avoid the pole problem by using a rotated coordinate system whose poles are outside the domain; another strategy is to use a distribution of gridpoints that covers the sphere, but has their separation smoothly increasing away from the region of main interest – see Staniforth (2000).

The gridpoint method is reasonably expected to be better than the spectral method at representing near-discontinuities such as fronts in the atmosphere. It also has the advantage of allowing choice in the locations at which the various fields are held, and generally permitting more freedom – and thus scope for improvement – via the finite differencing. At present, however, their superior treatment of the poles makes spectral models at least competitive with gridpoint models.

Global gridpoint models nowadays have a grid interval of 50 km or so in the horizontal; so systems having a wavelength of less than 100km are not resolved. Global spectral models are subject to broadly similar restrictions. Many important scales are therefore not explicitly represented, but their effects – in terms of heat, moisture and momentum transfers – must be allowed for. Especially in climate simulation, this problem of subgridscale *parametrization* is acute. An understanding of the fluxes carried by, for example, cumulonimbus systems, and their relation to the resolved flow is crucial for the development of appropriate parametrizations. Numerical simulation and theoretical analysis of motion on the relevant scales are the subjects of intense study, and the formulations described in sections 11.1 and 11.2 are frequently used for this purpose. Closely related is the problem of the scales that are barely resolved, and thus poorly treated, by the large-scale model, be it gridpoint or spectral. For an analysis of this key issue see Lander and Hoskins (1997).

In conclusion, it should be emphasised that meteorological dynamics is not solely concerned with the equations used for numerical weather forecasting and climate simulation. A glance at a text on satellite imagery (such as Bader *et al.* 1995) – or, indeed, out of a window during most daylight hours – serves to remind that the atmosphere is populated by flow structures and associated phenomena. These are naturally the concern of the users of weather forecasts, and could be said to be the weather itself. An appreciation of the structure of weather systems and phenomena, as well as of the structure of the governing equations, should guide the development of numerical models of the atmosphere and the appraisal of their performance.

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Figure captions

Figure 1

(a) Displacement in time Δt of fluid particles that are in the neighbourhood of the point $P=(x_0, y_0)$ at time $t=t_0$. To leading order, the fluid particle which is at P at $t=t_0$ is displaced to $(x_0+u\Delta t, y_0+v\Delta t)$ at $t=t_0+\Delta t$, where u and v (the components of the flow in the x and y directions) are evaluated at (x_0, y_0, t_0). Also to leading order, a fluid particle which is at $Q=(x_0+\delta x, y_0+\delta y)$ at $t=t_0$ is displaced to $(x_0+\Delta x, y_0+\Delta y)$ at $t=t_0+\Delta t$, where Δx and Δy are related to u, v and the spatial derivatives u_x, u_y, v_x, v_y at (x_0, y_0, t_0) according to (2.1). As well as the coordinate system Oxy relative to which u and v are measured, the diagram shows (at $t=t_0+\Delta t$) the coordinate system $O'x'y'$ which moves with the flow velocity at (x_0, y_0, t_0) and is coincident with the Oxy system at $t=t_0$.

(b) Illustrating that the evolution of an initial circle of fluid particles in a short time Δt is the sum of a translation, a rotation, a scaling and a deformation.

(c) Showing the effects of deformation on pre-existing gradients of a conserved scalar C when the stretching axis is respectively perpendicular to and parallel to the gradient of C.

Figure 2

The (λ, ϕ, r) spherical polar system whose origin O is at the centre of the Earth and which co-rotates with angular velocity $\mathbf{\Omega}$. The unit vector triad $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ at the generic point $P=(\lambda, \phi, r)$ is also indicated, as are the corresponding zonal, meridional and radial velocity components u, v and w.

Figure 3

(a) An element of air of mass $\rho\delta\tau = \rho r^2 \cos\phi\,\delta\lambda\,\delta\phi\,\delta r$ centred (at some time t) at longitude λ, latitude ϕ and distance r from the centre of the Earth. If the net zonal force acting on the element is X_λ, then the net torque about the polar axis is $X_\lambda r\cos\phi$. The axial component of the absolute angular momentum of the element is $\delta A = \rho\delta\tau(\Omega r\cos\phi + u)r\cos\phi$, where u is the zonal component of its velocity relative to the Earth. Since the mass $\rho\delta\tau$ of the element is by definition constant, equating the rate of change of δA to the net torque gives

$$\rho\delta\tau\frac{D}{Dt}\{(\Omega r\cos\phi + u)r\cos\phi\} \quad = \quad X_\lambda r\cos\phi$$

Eq (4.7) then results when X_λ is appropriately expressed as the sum of contributions from the pressure gradient force and the force $\mathbf{F}$ (see (3.11)).

(b) A small cylinder has bases δS which lie within isentropes θ_1 and $\theta_1 + \delta\theta$, and generators parallel to gradθ. In the case considered (see text) the motion is assumed adiabatic, and the cylinder is a material volume.

Figure 4

(a) Temperature variation with height to 90 Km in the U.S Standard Atmosphere. Profile consists of straight-line segments, as shown. Arrows span the lowest and highest mean monthly temperatures obtained for any location, and so indicate the spatial and temporal variability of monthly means about the standard profile. Scale at right gives pressure implied by the standard profile (assuming hydrostatic balance). After NOAA/NASA/USAF (1976) and Gill (1982).

(b) A broad-brush view of the Northern Hemisphere potential temperature field $\theta(\phi, z)$, temporally and longitudinally averaged. Isentropes (contours of constant θ) are marked every 30K from 270-390K by thin lines; the thick line indicates the tropopause. After Hoskins (1991)

Figure 5

A parcel of air displaced vertically a distance δz from its equilibrium position within a hydrostatic environment.

Figure 6

Transformation of horizontal and local time derivatives between height and pressure coordinates. $X = \lambda, \phi$ or t. Line AB (parallel to OX) has length δX, line BC (parallel to Oz) has length δz; pressure p is constant on AC. Let δQ_{RS} denote the difference in $Q = Q(\lambda, \phi, z, t)$ between any points R and S. Then

$$\delta Q_{AC} = \delta Q_{AB} + \delta Q_{BC}$$

Thus
$$\left.\frac{\partial Q}{\partial X}\right|_p \delta X = \left.\frac{\partial Q}{\partial X}\right|_z \delta X + \frac{\partial Q}{\partial z}\delta z$$

i.e.
$$\left.\frac{\partial Q}{\partial X}\right|_p = \left.\frac{\partial Q}{\partial X}\right|_z + \frac{\partial Q}{\partial z}\left.\frac{\partial z}{\partial X}\right|_p$$

Figure 7

(a) Illustrating the balance between the horizontal components of the Coriolis and pressure gradient forces acting on unit mass of air in (horizontal) geostrophic flow $\mathbf{v}_G$. The diagram is drawn assuming $f > 0$ (Northern Hemisphere). If $f < 0$ (Southern Hemisphere) $\mathbf{v}_G$ would be oppositely directed, given the horizontal pressure gradient $\nabla_z p$ shown. $\mathbf{k}$ is unit vector in the upward vertical direction (perpendicular to the plane of the diagram).

(b) A typical map of mean sea-level pressure; analysed locations of warm, cold and occluded fronts are also shown. Contour interval 4hPa. From The Met. Office's *Daily Weather Summary*. [In land regions the pressure at mean sea-level has been obtained by a standard extrapolation based on the hydrostatic approximation and knowledge of the atmosphere's temperature structure.]

(c) A typical 300hPa height map, valid 1200 UTC 11 May 1999. Contour interval 8 decametres. Data entries indicate the observation density (and use a standard code). From the *European Meteorological Bulletin* of the Deutscher Wetterdienst.

Figure 8

(a) Illustrating the relative orientation (in the horizontal plane) of the shear of the geostrophic wind in the vertical $(\partial \mathbf{v}_G / \partial z)$ and the temperature gradient ($\nabla_p T$) on pressure surfaces in the Northern Hemisphere. (In the Southern Hemisphere, the geostrophic wind shear vector $(\partial \mathbf{v}_G / \partial z)$ would be oppositely directed, given the temperature gradient shown).

(b) Two columns of air having different mean temperatures. Suppose that the hydrostatic approximation is applicable and that the pressure at level $z = 0$ is p_0 in each column, so that the component of geostrophic wind $\mathbf{v}_G$ perpendicular to the plane of the paper is zero there. Define the top of each column as the level at which pressure is p_1 $(< p_0)$. The top of the warm column, thus defined, is at a level z_{warm} greater than that of the top of the cool column z_{cool} (see (7.12)); the "thickness" of the warm column is greater than the "thickness" of the cool column. At height z_{cool}, the pressure in the warm column is therefore greater than p_1; hence, at this level, a horizontal pressure gradient exists and the geostrophic wind component perpendicular to the plane of the diagram is non-zero.

Figure 9

The geostrophic wind $\mathbf{v}_G$ plotted as the diameter of a circle (centre O). By definition, $\mathbf{v} = \mathbf{v}_G + \mathbf{v}_{AG}$, where $\mathbf{v}$ is the total (horizontal) wind and $\mathbf{v}_{AG}$ is the ageostrophic wind. If $\mathbf{v}$ and $\mathbf{v}_G$ are steady and spatially uniform, and a friction law $\mathbf{F} = -C\mathbf{v}$ holds, then $\mathbf{v}.\mathbf{v}_{AG} = 0$, and $\mathbf{v}$ and $\mathbf{v}_{AG}$ meet on the circle of which $\mathbf{v}_G$ is a diameter. Eq(7.17) specifies the angle α between $\mathbf{v}_G$ and $\mathbf{v}$. The diagram has been drawn assuming $f > 0$ (Northern Hemisphere) so that high pressure lies to the right of $\mathbf{v}_G$ and $\mathbf{v}$ to the left of $\mathbf{v}_G$. If $f < 0$ (Southern Hemisphere) high pressure lies to the left of $\mathbf{v}_G$, and $\mathbf{v}$ would lie to the right of $\mathbf{v}_G$.

Figure 10

Showing the four non-zero solutions of the acoustic/gravity wave dispersion equation (8.13). The radius of the circle is the approximate acoustic wave frequency Ω_a given by (8.16), and the angle ψ $(< \pi/4)$ is given in terms of Ω_a and the approximate gravity wave frequency Ω_g (see (8.17)) by (8.18). ω_0 and ω_2 are the exact gravity wave frequencies $\pm\omega_g$; ω_1 and ω_3 are the exact acoustic wave frequencies $\pm\omega_a$. The straight lines COA and DOB are perpendicular diameters of the circle, and OA subtends ψ with OX. The angle ψ may be constructed by plotting a chord $\mathrm{EF} = 4\Omega_g$ perpendicular to OX, joining OE, and bisecting the angle $\mathrm{EOX} = 2\psi$ (see (8.18)).

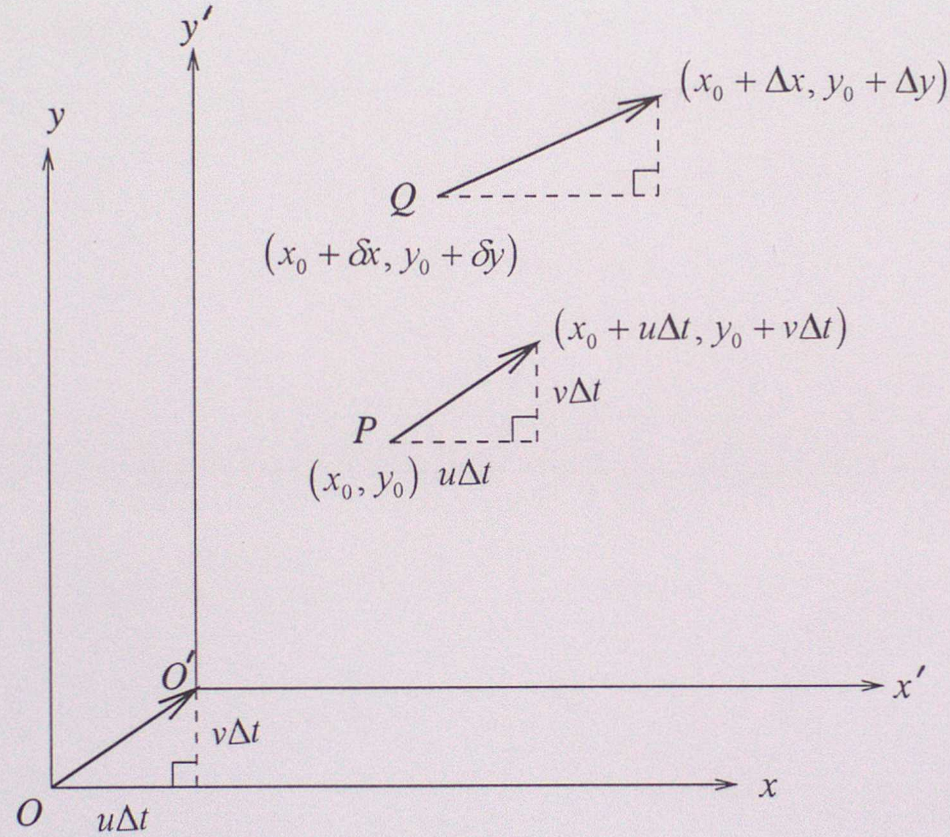

y′
y
$(x_0 + \Delta x, y_0 + \Delta y)$
Q
$(x_0 + \delta x, y_0 + \delta y)$
$(x_0 + u\Delta t, y_0 + v\Delta t)$
$v\Delta t$
P
(x_0, y_0)
$u\Delta t$
O′
x′
$v\Delta t$
x
O
$u\Delta t$

Figure 1(a)

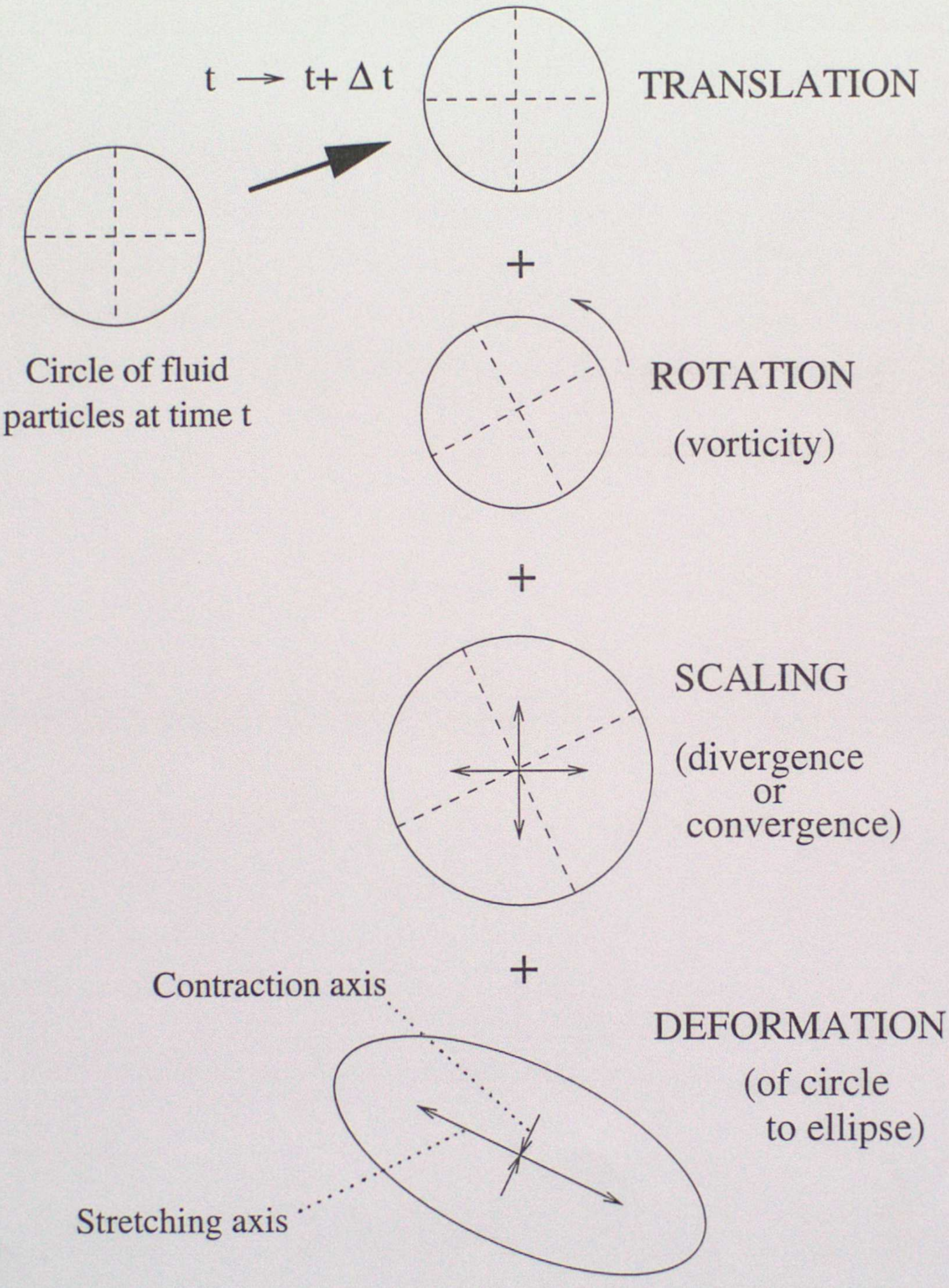

t → t+ Δ t
TRANSLATION
+
Circle of fluid
particles at time t
ROTATION
(vorticity)
+
SCALING
(divergence
or
convergence)
+
Contraction axis
DEFORMATION
(of circle
to ellipse)
Stretching axis

Figure 1(b)

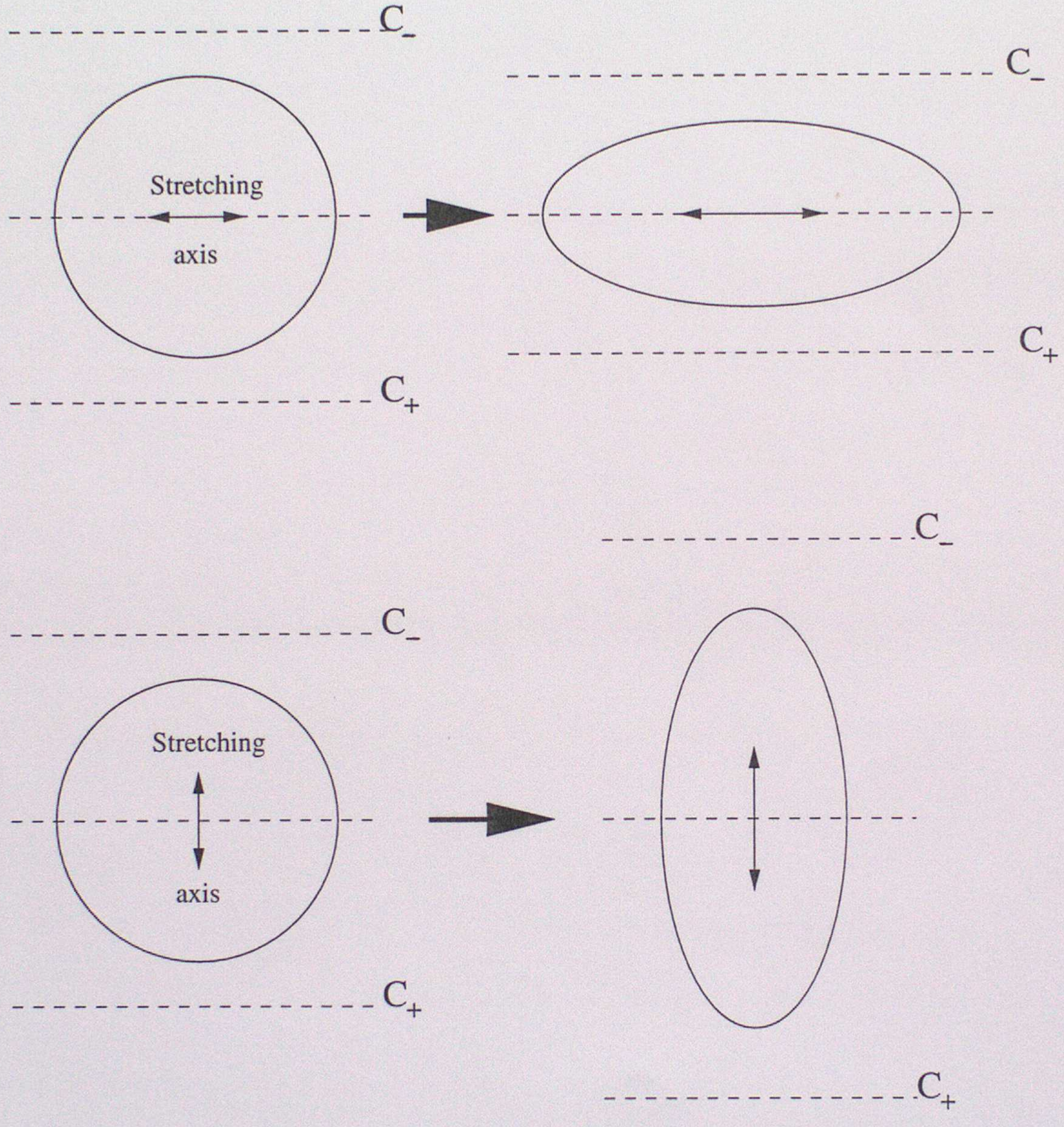
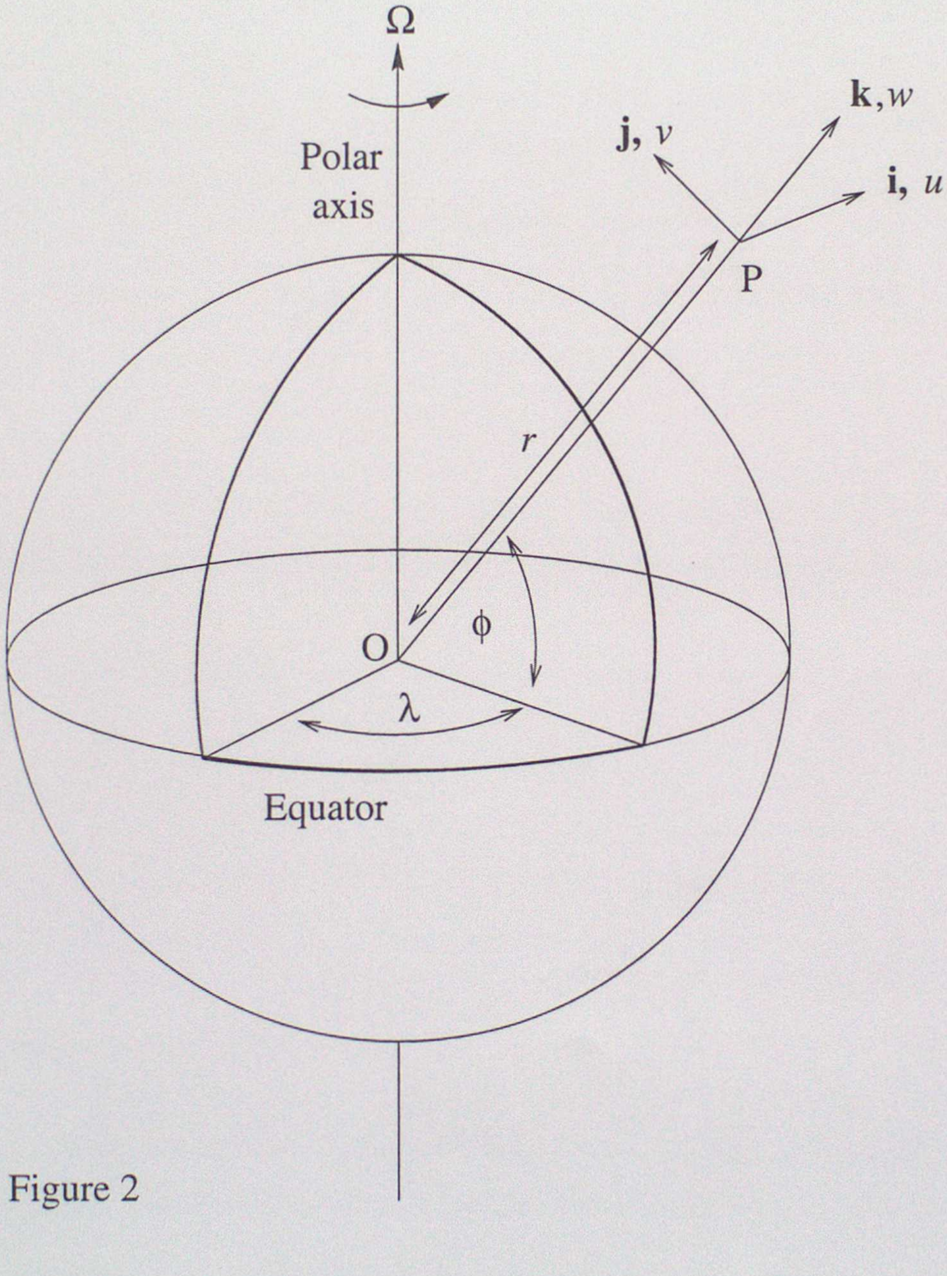


Figure 1(c)



Ω
Polar axis
$\mathbf{k}, w$
$\mathbf{j}, v$
$\mathbf{i}, u$
P
r
O
ϕ
λ
Equator

Figure 2

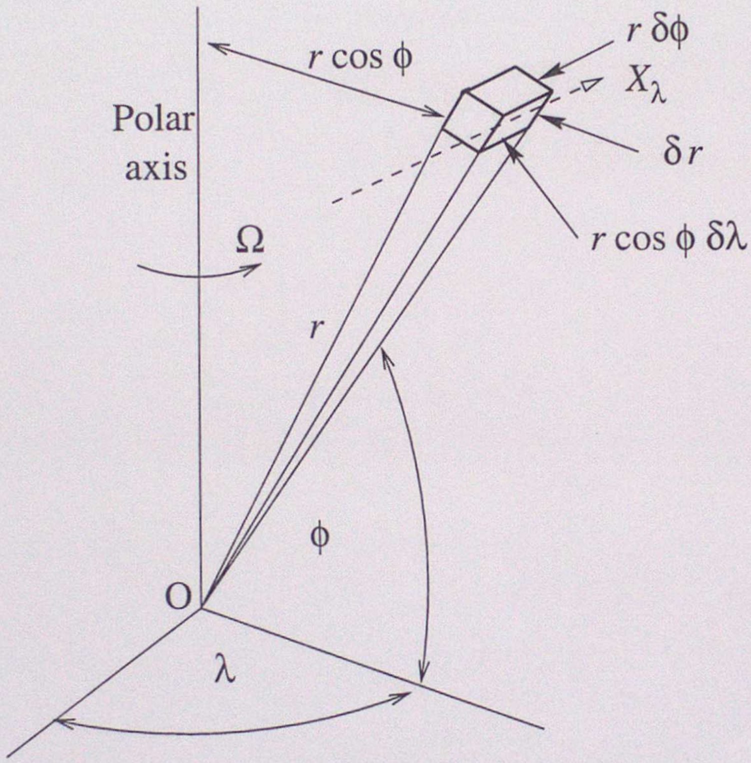

$r\,\delta\phi$
$r\cos\phi$
X_λ
Polar
axis
δr
$r\cos\phi\,\delta\lambda$
Ω
r
ϕ
O
λ

Figure 3(a)

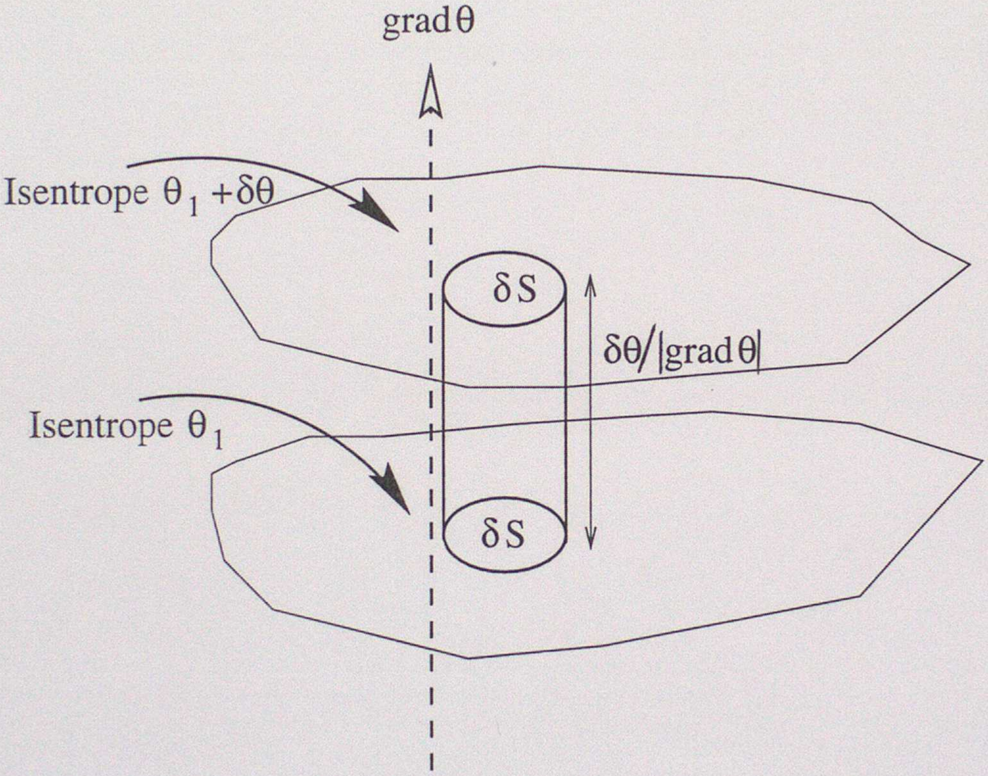
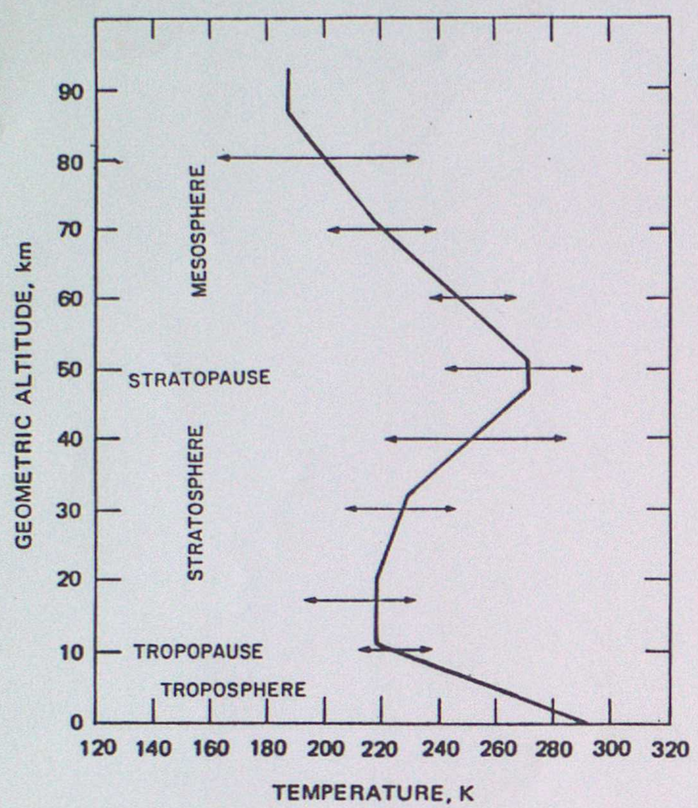
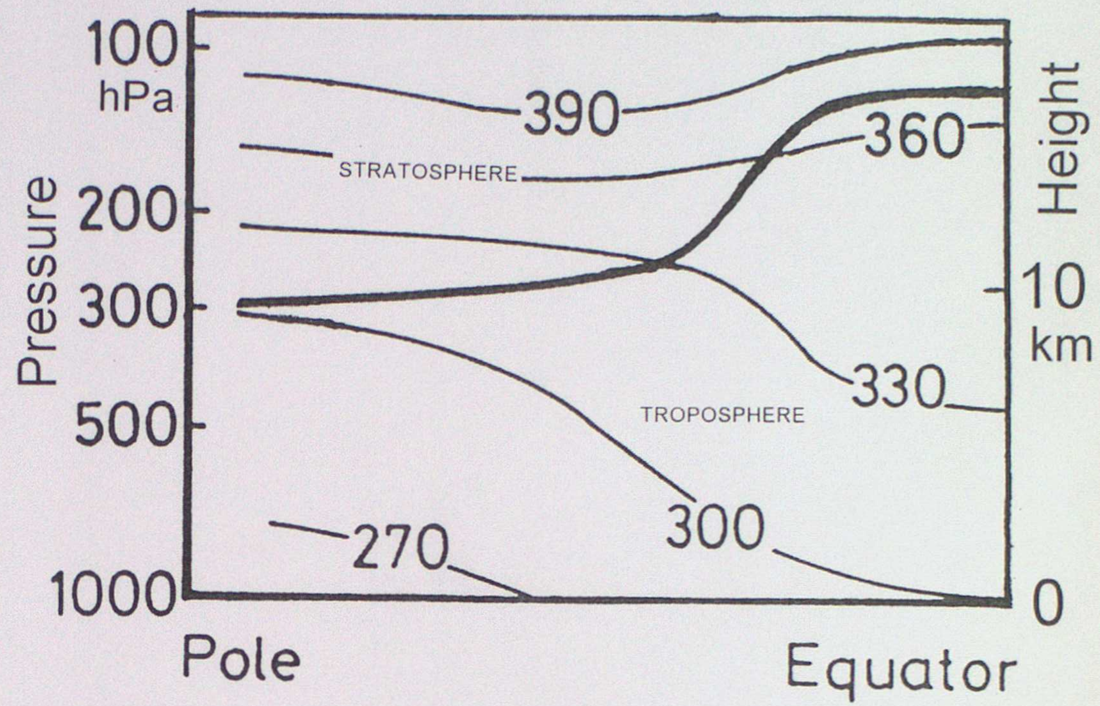


Figure 3(b)


GEOMETRIC ALTITUDE, km
90
80
70
60
50
40
30
20
10
0
MESOSPHERE
STRATOPAUSE
STRATOSPHERE
TROPOPAUSE
TROPOSPHERE
120 140 160 180 200 220 240 260 280 300 320
TEMPERATURE, K

(a)


Pressure
100
hPa
200
300
500
1000
390
360
330
300
270
STRATOSPHERE
TROPOSPHERE
Height
10
km
0
Pole
Equator

(b)

Figure 4

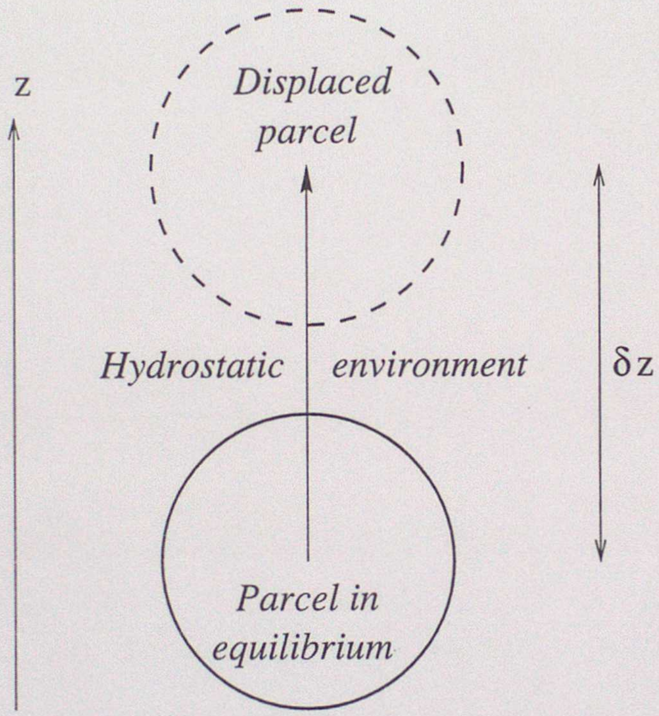

z
Displaced
parcel
Hydrostatic
environment
δz
Parcel in
equilibrium

Figure 5

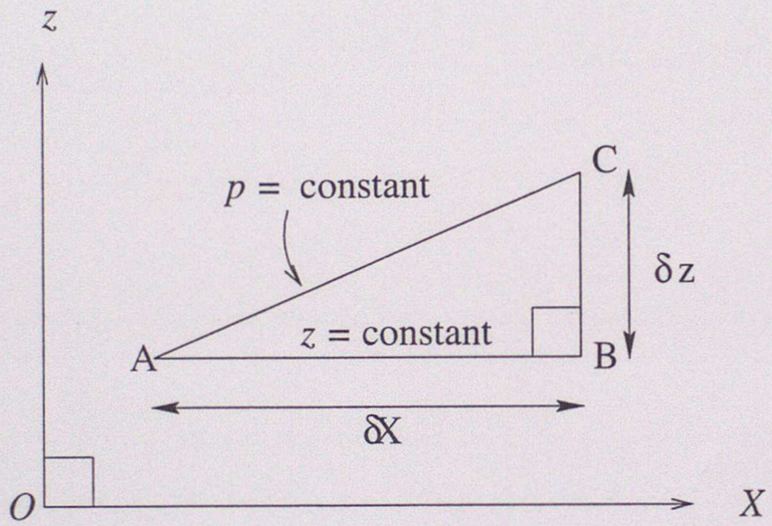


Figure 6

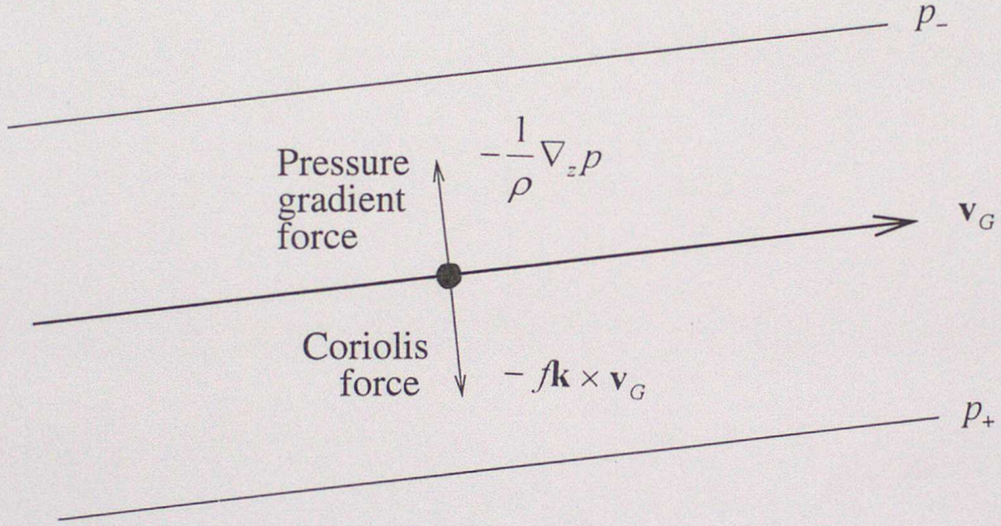
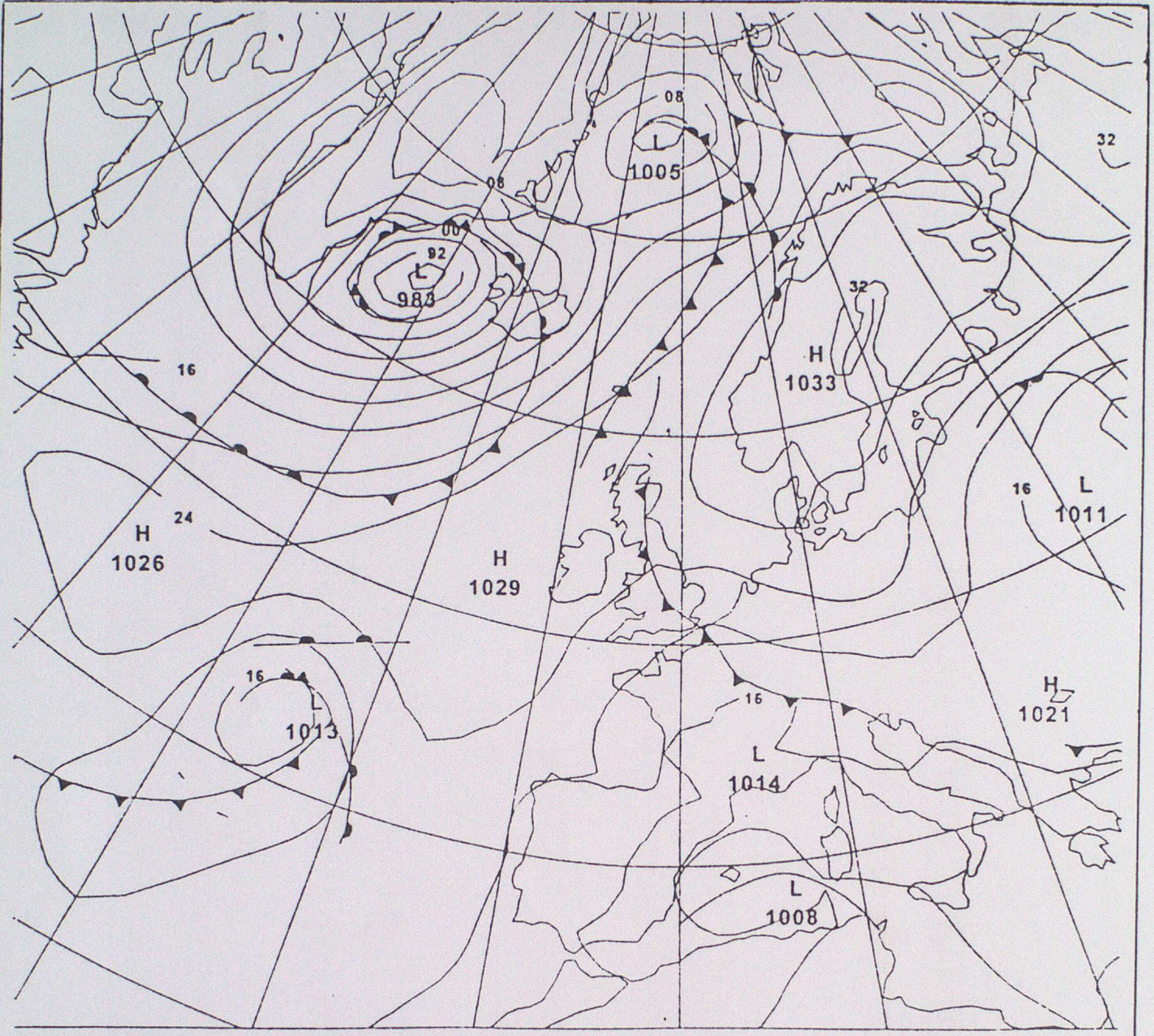

p_-
Pressure gradient force
$-\frac{1}{\rho}\nabla_z p$
$\mathbf{v}_G$
Coriolis force
$-f\mathbf{k}\times\mathbf{v}_G$
p_+

Figure 7(a)



VALID 0000 UTC FRI 15 MAY 1998

Figure 7(b)

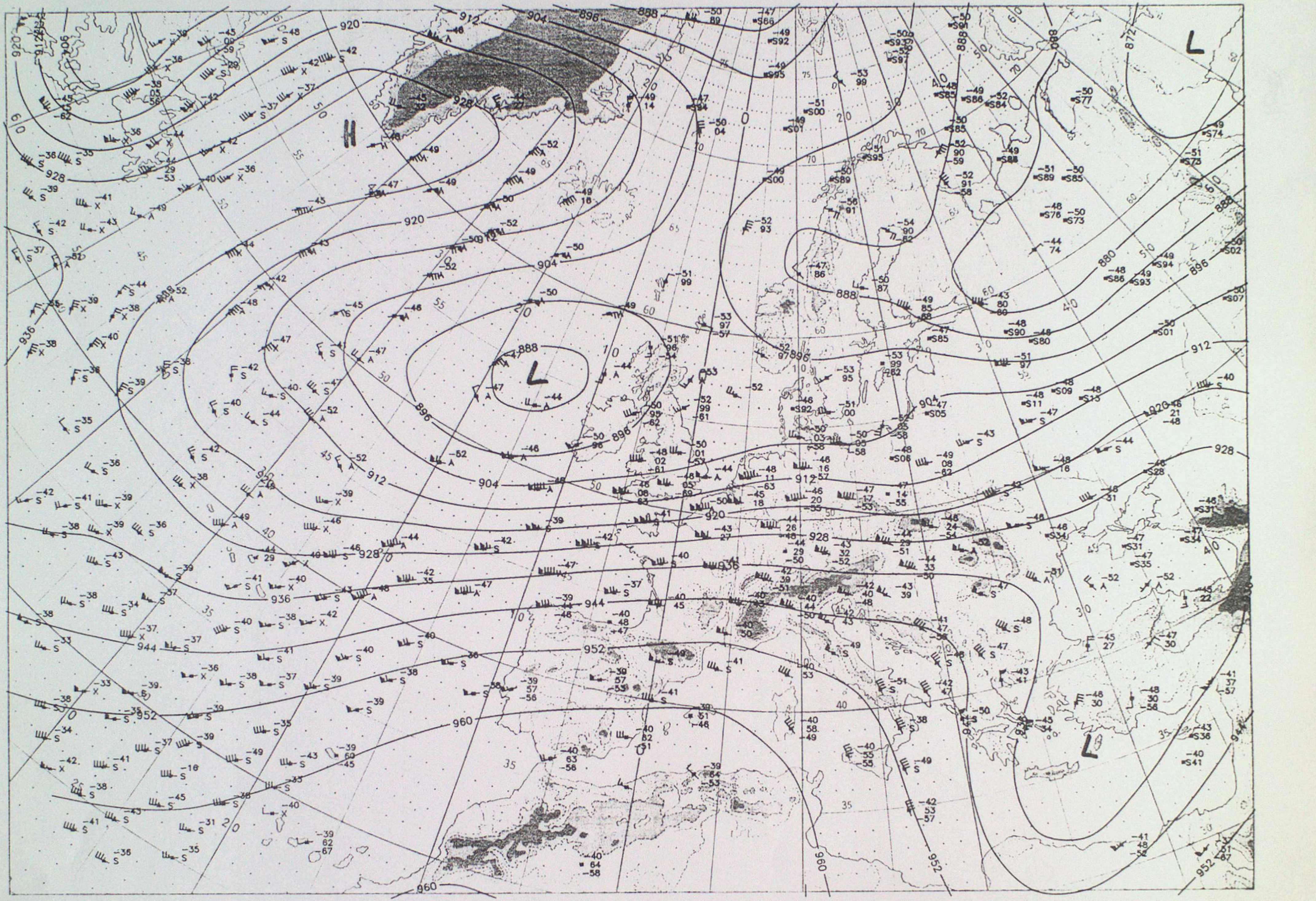


Figure 7(c)

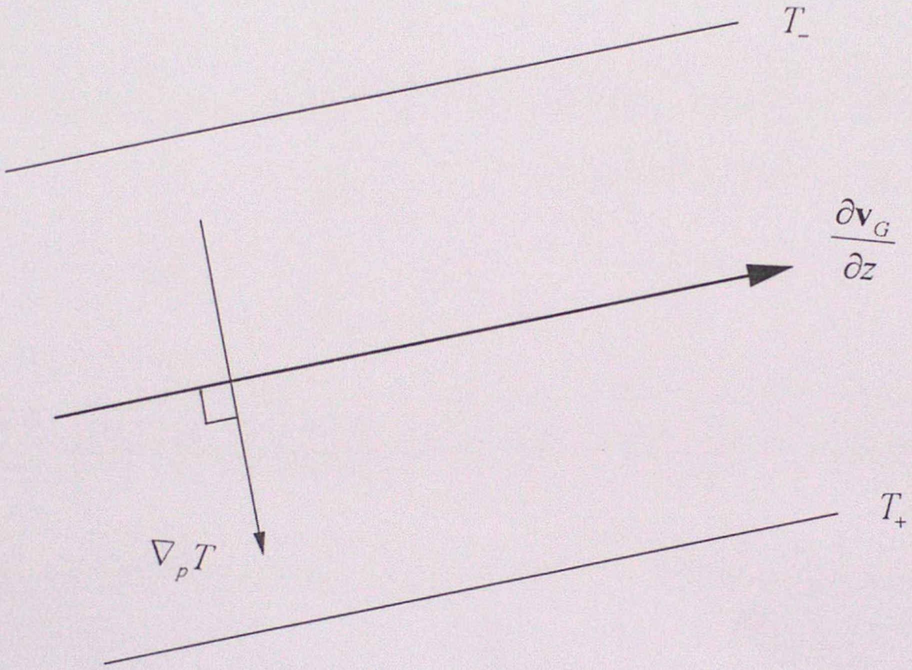
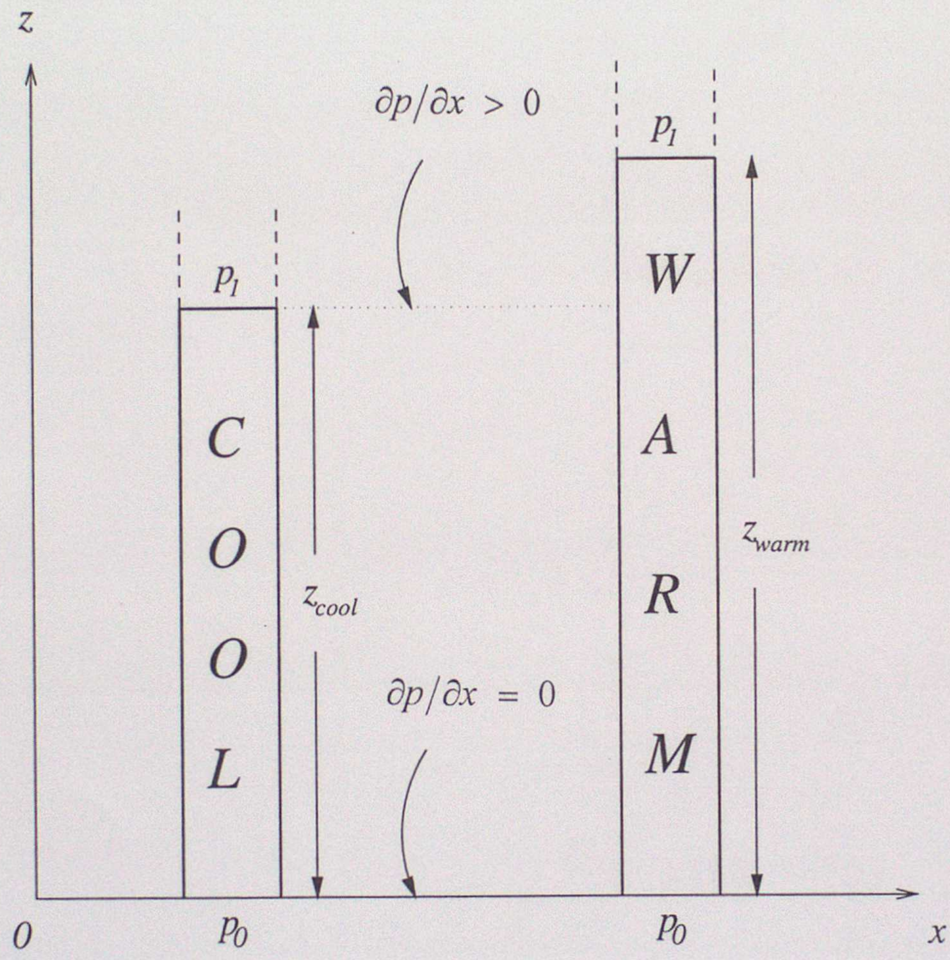


Figure 8(a)



z
$\partial p/\partial x > 0$
p_1
p_1
W
A
R
M
C
O
O
L
z_{cool}
z_{warm}
$\partial p/\partial x = 0$
0
p_0
p_0
x

Figure 8(b)

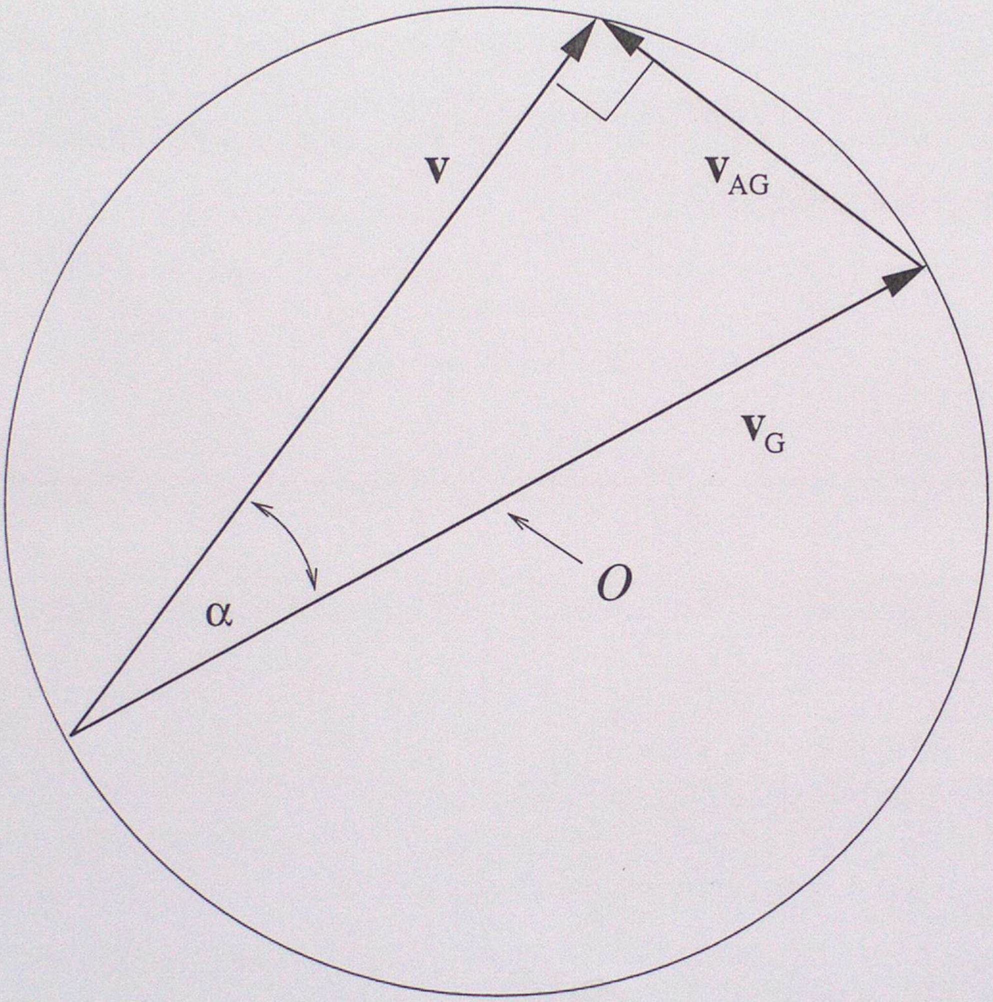
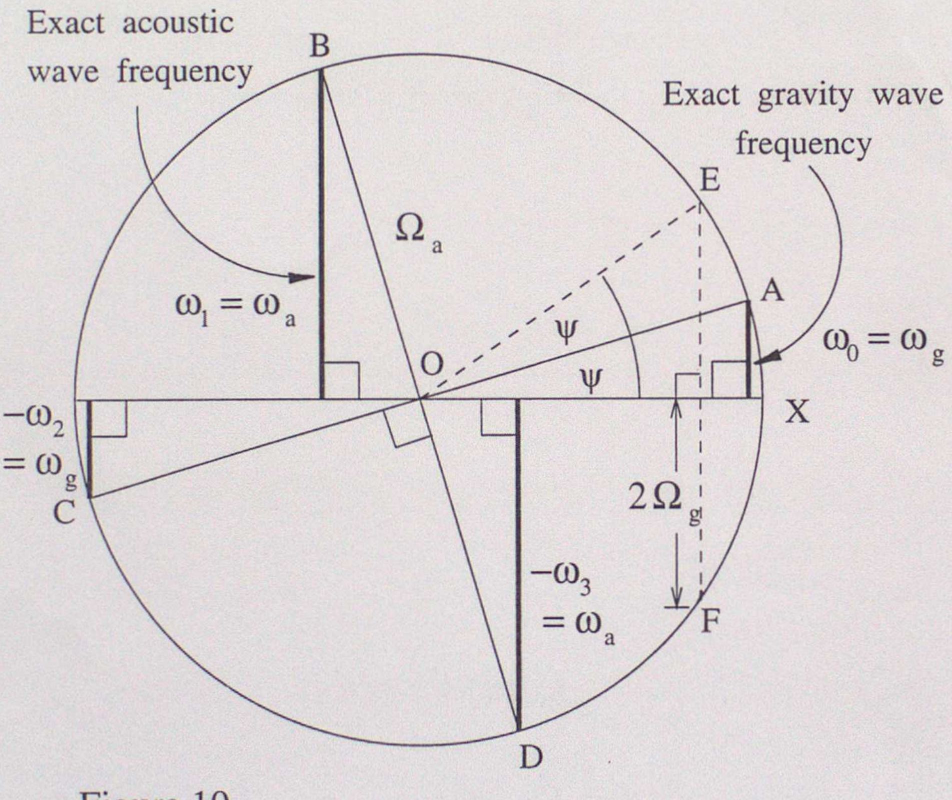


Figure 9



Exact acoustic wave frequency
B
Exact gravity wave frequency
E
Ω_a
A
$\omega_1 = \omega_a$
ψ
$\omega_0 = \omega_g$
O
ψ
X
$-\omega_2$
$= \omega_g$
C
$2\Omega_g$
$-\omega_3$
$= \omega_a$
F
D

Figure 10