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PARCEL STABILITY AND ITS RELATION TO SEMI-GEOSTROPHIC THEORY.

by

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Abstract

The classical concept of parcel (or hydrodynamic) stability is reviewed in the light of recent developments in semi-geostrophic theory. A certain Jacobian matrix is required to be positive-definite if solution of the semi-geostrophic equations is to be well-posed. Formally, this gives three conditions which must be satisfied, one of which corresponds to the positivity of the semi-geostrophic potential vorticity defined as the determinant of the matrix. The full implications of these stability criteria are sought here together with some simple expression of the positive-definite requirement.

The existence of a minimum energy principle, with respect to virtual parcel displacements which conserve absolute momentum and potential temperature, is shown to play a key role, as in the equivalent circular vortex problem studied by Eliassen and Kleinschmidt (1957).

1. Introduction

In recent years there has been renewed interest in parcel stability concepts following the work of Bennetts and Hoskins (1979) on conditional symmetric instability as a mechanism for frontal rainbands and that of Emanuel (1979) on inertial instability in viscous, rotating fluids. A parcel formulation was subsequently exploited by Emanuel (1983, a, b) in order to quantify the potential instability of moist airstreams subject to slantwise displacements and has led to a more general concept of convective available potential energy. A short history of inertial stability theory is provided by Emanuel (1979). In addition to the papers he quotes we would like to draw attention to the studies made by Sawyer (1949), Godson (1950), Van Mieghem (1952) and Eliassen and Kleinschmidt (1957) which are particularly relevant to our analysis. These studies usually dealt with zonal or circular flows which are solutions of the time-independent equations of motion. Godson (1950) suggested that the parcel stability concept was still relevant in non-stationary, three-dimensional flows and put forward generalized criteria for stability. Eliassen and Kleinschmidt show that balanced circular motion corresponds to a minimum energy state wiht respect to arbitrary axisymmetric displacements provided that a certain constraint on the circulation is satisfied.

In the context of semi-geostrophic theory, Cullen and Purser (1984) (hereafter referred to as CP) show that a particular Hessian matrix must be positive - definite for stability (convective or inertial) and prove that a modified pressure variable must be a convex function. The simplest physical example of the relevance of convexity to stability is that of a rotating liquid with a free surface. Viewed from below the free surface must always appear convex to guarantee stability. The theorems of CP allow mathematical generalization of this idea without restricting motion fields to be differentiable so that discontinuous structures (eg. the Margules front) are admissible as a class of balanced motion. Since the formation of frontal discontinuities intruding into fluid interiors is a natural consequence of integrating the Lagrangian conservation form of the semi-geostrophic equations (Cullen, 1983) this is of more than academic significance. Furthermore, Cullen et al (1985) show that the semi-geostrophic equations for dry, adiabatic flow correspond to a global minimum energy principle subject to arbitrary parcel displacements which conserve absolute momentum and potential temperature. The result is consistent with the parcel approach described here. In this paper we attempt to clarify the relationship of the three-dimensional stability matrix to semi-geostrophic theory and show how it is central to the minimum energy principle.

2. The Stability matrix

It is instructive to recall how the stability matrix of CP arises in semi-geostrophic theory. Following Hoskins (1975), the inviscid adiabatic equations of motion under the semi-geostrophic approximation may be written as:

$$\frac{Du_g}{Dt} - fv = - fv_g \qquad 2.1$$

$$\frac{Dv_g}{Dt} + fu = fu_g \qquad 2.2$$

$$\frac{D\theta}{Dt} = 0 \qquad 2.3$$

and $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad 2.4$$

(noting that $(fv_g, -fu_g, g\theta/\theta_o) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right)$)

where z is a pseudo-height coordinate; ϕ is the geopotential, θ is the potential temperature with basic state value, θ_o; f and g are the Coriolis parameter and acceleration due to gravity; $(u_g, v_g, 0)$ and (u, v, w) are the geostrophic and full vector wind velocities respectively and:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Using f^{-1}, L (an unspecified length scale), f^2L/g, f^2L^2 and θ_o as scaling parameters for t, (x,y), z, ϕ and θ respectively, eqns. (2.1 - 2.4) may be non-dimensionalized to give:

$$\frac{DM}{Dt} = y-N \qquad 2.5$$

$$\frac{DN}{Dt} = M-x \qquad 2.6$$

$$\frac{D\theta}{Dt} = 0 \qquad 2.7$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad 2.8$$

where $M = x + v_g$ and $N = y - u_g$.

CP state that $\underline{\underline{Q}}$ defined by:

$$\underline{\underline{Q}} = \begin{pmatrix} M_x & M_y & M_z \\ N_x & N_y & N_z \\ \theta_x & \theta_y & \theta_z \end{pmatrix} = \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix}$$

where $P(x,y,z) = \phi + {}^1/_2\,(x^2 + y^2)$ (non-dimensional), is a Hessian matrix governing the stability of the flow and following Hoskins (1975) that:

$$q = \det(\underline{\underline{Q}}) \qquad \text{satisfies} \qquad \frac{Dq}{Dt} = 0$$

q, therefore is the semi-geostrophic analogue of potential vorticity. We digress for the moment to consider this result.

If the forcing terms in eqns. 2.5 and 2.6 were zero, conservation of q would follow immediately since fluid defined initially to lie within a certain volume $\Delta M \Delta N \Delta \theta$ would remain bounded by the same M, N and θ surfaces and so, denoting $\Delta \tau$ as the corresponding volume in physical space, the incompressibility condition (2.8) gives:

$$\Delta \tau = J^{-1}\left(\frac{M,\ N,\ \theta}{x,\ y,\ z}\right). \Delta M \Delta N \Delta \theta = q^{-1} \Delta M \Delta N \Delta \theta = \text{constant}$$

where J denotes the Jacobian of transformation, so that q is a Lagrangian conservation property. However, M and N are not conserved in three-dimensional flow but it is easy to show that $\left(\frac{DM}{Dt}, \frac{DN}{Dt}, \frac{D\theta}{Dt}\right)$ represents a non-divergent vector field in (M, N, θ) space since:

$$\frac{\partial y}{\partial M} - \frac{\partial x}{\partial N} = - J\left(\frac{N,\ y,\ \theta}{M,\ N,\ \theta}\right) + J\left(\frac{x,\ M,\ \theta}{M,\ N,\ \theta}\right)$$

$$= q^{-1}\left[J\left(\frac{x,\ M,\ \theta}{x,\ y,\ z}\right) - J\left(\frac{N,\ y,\ \theta}{x,\ y,\ z}\right)\right]$$

(by the Jacobian property, $J\left(\frac{a,\ b,\ c}{x,\ y,\ z}\right) \times J\left(\frac{x,\ y,\ z}{d,\ e,\ f}\right) = J\left(\frac{a,\ b,\ c}{d,\ e,\ f}\right)$

$$= q^{-1}[M_y\theta_z - \theta_y M_z - N_x\theta_z + \theta_x N_z] = 0$$

provided that $q \neq 0$

using $M_y = N_x$, $M_z = \theta_x$ and $N_z = \theta_y$. Volumes in physical space and in (M, N, θ) space associated with a given set of fluid particles remain constant in time and so the Jacobian of transformation remains constant following the motion. A direct derivation of this conservation law for the compressible case is given in the Appendix.

CP also show that the matrix $\underline{\underline{Q}}$ appears in the diagnostic equation for the geopotential tendency in a two-dimensional deformation model of a front. The three-dimensional geopotential tendency equation may be derived as follows. Writing eqns. (2.5 - 2.7) as:

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial x}\right) + u_i \frac{\partial M}{\partial x_i} = y - N \qquad 2.9$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial y}\right) + u_i \frac{\partial N}{\partial x_i} = M - x \qquad 2.10$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial z}\right) + u_i \frac{\partial \theta}{\partial x_i} = 0 \qquad 2.11$$

(repeated summation on i)

using $(u_g, v_g, \theta) = (-\frac{\partial \phi}{\partial y} \quad \frac{\partial \phi}{\partial x} \quad \frac{\partial \phi}{\partial z})$ and introducing the geopotential tendency, $\Gamma = \frac{\partial \phi}{\partial t}$ then these equations may be expressed vectorially as:

$$\underline{\nabla}\,\Gamma + \underline{\underline{Q}}.\underline{V} = \underline{b} \qquad 2.12$$

where $\underline{V} \equiv u_i$ and $\underline{b} = (y-N, M-x, 0)$. Forming the scalar product of 2.12 with $\underline{\underline{Q}}^{-1}$ gives:

$$\underline{V} = \underline{\underline{Q}}^{-1}.\underline{b} - \underline{\underline{Q}}^{-1}.\underline{\nabla}\,\Gamma$$

so that the incompressibility condition $\underline{\nabla}.\underline{V}$ implies:

$$\underline{\nabla}.[\underline{\underline{Q}}^{-1}.\underline{b}] - \underline{\nabla}.[\underline{\underline{Q}}^{-1}.\underline{\nabla}\Gamma] = 0 \qquad 2.13$$

which is an elliptic equation for Γ provided that $\underline{\underline{Q}}$, and hence $\underline{\underline{Q}}^{-1}$, is positive-definite. Stability of flow is apparently lost when $\underline{\underline{Q}}$ has a negative eigenvalue rendering eqn. (2.13) hyperbolic. In the next section it will be demonstrated how the $\underline{\underline{Q}}$ matrix arises in the classical parcel approach to stability.

3. Parcel stability in the Boussinesq system

Before proceeding to the three-dimensional case, briefly consider the standard parcel stability analysis for unidirectional flow in geostrophic and hydrostatic balance. The non-dimensional equation of motion under the form of Boussinesq approximation used in equations (2.1-2.4) is:

$$\frac{D\underline{V}}{Dt} + \underline{k} \wedge \underline{V} + \underline{\nabla}\phi = \theta\underline{k} \qquad 3.1$$

where the same scaling parameters have been adopted. Without loss of generality assume that the basic flow is in the y-direction and independent of y so that:

$\underline{V} = (o,\ v(x,z),\ o)$ with

$$\frac{\partial\phi}{\partial x} = v \quad \text{and} \quad \frac{\partial\phi}{\partial z} = \theta$$

then (3.1) is satisfied exactly. Parcel stability analysis involves giving an impulsive velocity to a parcel without affecting the basic state pressure field. Because this is a two-dimensional analysis, the parcel extends arbitrarily far in the y-direction. The resulting motion then obeys the equations:

$$\frac{Du}{Dt} - v + \frac{\partial\phi}{\partial x} = 0 \qquad 3.2$$

$$\frac{Dv}{Dt} + u = 0 \qquad 3.3$$

$$\frac{Dw}{Dt} + \frac{\partial\phi}{\partial z} - \theta = 0 \qquad 3.4$$

and on defining the absolute momentum, M = v+x, eqn. (3.3) becomes:

$$\frac{DM}{Dt} = 0 \qquad 3.5$$

Now consider a Lagrangian view of the perturbed motion. Let the position vector of a parcel moving with the basic state flow be $\underline{R}$ and the position vector of a perturbed parcel be $\underline{r}$. The basic state is an exact solution of the unapproximated equation of motion so that:

$$\frac{d^2\underline{R}}{dt^2} + \underline{k} \wedge \frac{d\underline{R}}{dt} + (\underline{\nabla}\phi)_{\underline{R}} = \theta_{\underline{R}}\, \underline{k}$$

where the suffix $\underline{R}$ denotes evaluation at the position $\underline{R}$. The equation for the perturbed parcel is the same except with $\underline{r}$ replacing $\underline{R}$ and so the relative displacement of the parcel about the unperturbed trajectory $\underline{r}'(=\underline{r}-\underline{R})$ is given by:

$$\frac{d^2\underline{r}'}{dt^2} + \underline{k}\wedge \frac{d\underline{r}'}{dt} + (\underline{\nabla}\phi)_{\underline{r}} - (\underline{\nabla}\phi)_{\underline{R}} = (\theta_{\underline{r}} - \theta_{\underline{R}})\underline{k} \qquad 3.7$$

For adiabatic motion $\theta_{\underline{r}} = \theta_{\underline{R}}$ and eqn. (3.5) implies that:

$$(M)_{\underline{r}} = (M)_{\underline{R}}$$

or $\quad (v)_{\underline{r}} = (V)_{\underline{R}} - fx'$

where $\underline{r} = (x',z')$. Taylor expansion of $\underline{\nabla}\phi$ gives

$$(\underline{\nabla}\phi)_{\underline{r}} - (\underline{\nabla}\phi)_{\underline{R}} = \underline{r}'\cdot(\underline{\nabla}\phi)_{\underline{R}} + O(|\underline{r}'|^2)$$

so that eqn. (3.7) may be written as:

$$\frac{d^2x'}{dt^2} + f^2x' + x'\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)_{\underline{R}} + z'\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right)_{\underline{R}} + O(|\underline{r}'|^2) = 0 \qquad 3.8$$

and $$\frac{d^2z'}{dt^2} + x'\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)_{\underline{R}} + z'\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)_{\underline{R}} + O(|\underline{r}'|^2) = 0 \qquad 3.9$$

Seeking solutions proportional to $e^{i\omega t}$ and making the substitution $P = \phi + x^2/_2$ gives:

$$\omega^2 \begin{pmatrix} x' \\ z' \end{pmatrix} + \begin{pmatrix} P_{xx} & P_{xz} \\ P_{zx} & P_{zz} \end{pmatrix} \begin{pmatrix} x' \\ z' \end{pmatrix} = 0 \qquad 3.10$$

The stability condition is that the Hessian matrix $Q_{ij} = \frac{\partial^2 P}{\partial x_i \partial x_j}$ has no negative eigenvalues, which is exactly the form derived in section 2 and is the well-known condition for symmetric stability (Hoskins, 1974). Consider next a two-dimensional disturbance to a basic flow with arbitrary orientation in the (x,y) plane. The equation of motion (3.1) is satisfied by the basic state $(\beta V(\sigma,z), \alpha V(\sigma,z), 0)$ where $\sigma = \alpha x + \beta y$ and α, β are direction cosines. The same analysis as above leads to the condition that:

$$\begin{pmatrix} P_{\sigma\sigma} & P_{\sigma z} \\ P_{z\sigma} & P_{zz} \end{pmatrix} \text{ has no negative eigenvalues}$$

This matrix can be rewritten in terms of x and y as:

$$\begin{pmatrix} P_{xx} \cdot \frac{\alpha^2}{\delta^2} + P_{xy} \cdot \frac{2\alpha\beta}{\delta^2} + P_{yy} \frac{\beta^2}{\delta^2}, & P_{xz}\frac{\alpha}{\delta} + P_{yz} \frac{\beta}{\delta} \\ P_{zx}\frac{\alpha}{\delta} + P_{zy} \frac{\beta}{\delta}, & P_{zz} \end{pmatrix}$$

where $\delta = \alpha^2 + \beta^2$. The two dimensionality condition means that $\beta/_\delta \, P_x - \alpha/_\delta \, P_y = 0$ so that the matrix reduces to:

$$\begin{pmatrix} \frac{1}{\alpha^2} \cdot P_{xx} & \frac{1}{\alpha} P_{xz} \\ \frac{1}{\alpha} \cdot P_{zx} & P_{zz} \end{pmatrix} \qquad 3.11$$

It is easy to show that the two dimensionality condition also reduces the three-dimensional Hessian matrix Q_{ij} to the same form as 3.11, since the x and y rows become proportional and one variable can be eliminated (corresponding to a zero eigenvalue).

Thus the three-dimensional Hessian matrix includes the symmetric stability condition with arbitrary orientation. The condition (3.5) generalizes to conservation of M and N in the displacement.

It is now natural to generalize the analysis to a case where the basic state is not exactly two-dimensional but has a large length scale L in one direction, say τ. The foregoing analysis is valid as long as the scale of the perturbation is much smaller than L in the τ direction. Such a basic state will not satisfy the time-independent equations of motion exactly but will evolve on some slow time scale T proportional to L so that it would be appropriate to look for solutions with $\omega \gg T^{-1}$. These conditions are exactly those required for the geostrophic momentum approximation to be accurate.

Assume, therefore, a basic state satisfying the geostrophic momentum approximation as a solution of (3.1). The perturbed parcel trajectory is given by (3.6) with $\underline{R}$ replaced by $\underline{r}$. Again $(\theta)_{\underline{r}} = (\theta)_{\underline{R}}$ and (3.5) becomes the conservation of absolute momentum $\frac{d\underline{r}}{dt} + \underline{k} \wedge \underline{r}$ so that 3.7 reduces to:

$$\frac{d^2\underline{r}'}{dt^2} + \underline{r}'.\underline{\nabla}(x,y,o) + (\underline{r}'.\underline{\nabla})\,(\underline{\nabla}\phi)_{\underline{R}} + 0(|\underline{r}'|^2) = 0$$

which may also be written as:

$$\frac{d^2r}{dt2} + (\underline{r}'.\underline{\nabla})\,[x+v_g,\ y-u_g,\ \theta] = 0$$

or $$\frac{d^2\underline{r}'}{dt^2} + (\underline{r}'.\underline{\nabla})\,(M,\ N,\ \theta) = 0$$

Now if $\underline{r}' = (x',y',z') \propto \exp(i\omega t)$ then:

$$\begin{pmatrix} M_x & M_y & M_z \\ N_x & N_y & N_z \\ \theta_x & \theta_y & \theta_z \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \omega^2 \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \qquad 3.12$$

For perturbations to result in stable oscillations $\omega^2 > 0$ and the stability matrix must be positive-definite. Alternatively, the quadratic form:

$$(x' \; y' \; z') \begin{pmatrix} M_x & M_y & M_z \\ N_x & N_y & N_z \\ \theta_x & \theta_y & \theta_z \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

may be equated to the work required to move an air parcel from $\underline{R}$ to $\underline{r}$ whilst conserving absolute momentum and potential temperature. By implication, the unperturbed parcel trajectory corresponds to a sequence of minimum energy positions dictated by the evolving environmental field of motion. This interpretation can only make sense if the smallest eigenfrequency ω_{min} obtained from 3.12 satisfies the consistency condition:

$\omega_{min}^{-1} << T$

In the two-dimensional limit $L,T = \infty$ and the condition is trivially satisfied. In this case, a class of perturbation exists which exactly conserves absolute momentum in accordance with the classical approach. Otherwise, the eigenfrequencies of the semi-geostrophic stability matrix can only be consistent with parcel theory provided there exists a large length scale along the direction of the flow (ie. the radius of curvature of trajectories is large). But this is also a requirement for the validity of the semi-geostrophic equations themselves. The whole concept of parcel stability in rapidly evolving flows ($T \sim \omega^{-1}_{min} \sim f$) is thwarted by the impossibility of absolute momentum conserving perturbations (due to the absence of near two-dimensionality) and the loss of distinction between parcel instability and general flow transience. Although it is possible to obtain parcel oscillation frequencies directly from 3.7 without assuming absolute momentum conservation (Godson, 1950) ($\underline{\nabla}\phi$) and θ are assumed to be slowly varying during the time scale of parcel oscillations. Furthermore, exponential growth of $|\underline{r}'|$ is possible in perfectly stable flow configurations e.g. pure deformation flow.

The Jacobian matrix $\underline{\underline{Q}}$ also arises naturally from a variational approach to global stability with respect to absolute momentum preserving virtual displacements. (c.f. Eliassen and Kleinschmidt, 1957 pg.66).

The total energy E appropriate to the equation of motion 3.1 is given by:

$$E = \int_\tau \left[\frac{1}{2}(u^2 + v^2) - z\theta\right] d\tau$$

where τ denotes some arbitrary volume fixed in physical space. Without making the geostrophic or hydrostatic assumptions we define:

$$\underline{M} = M'\underline{i} + N'\underline{j} + \theta\underline{k}$$

where $M' = v+x$ and $N' = y-u$

and so:

$$E = \int_\tau \frac{1}{2}\left[(M'^2 + N'^2) - \underline{M}.(x,y,z)\right] d\tau + \text{constant}$$

Now consider the change in energy δE resulting from an infinitesmal rearrangement of the system such that $\underline{M}$ is conserved for individual parcels. Let $\underline{\xi}$ (x,y,z) represent the vector displacement field then:

$$\delta E = -\int_\tau \underline{M}.\underline{\xi}\, d\tau \qquad 3.13$$

Now, under geostrophic and hydrostatic balance:

$$\underline{M} = M\underline{i} + N\underline{j} + \theta\underline{k} = \underline{\nabla}P$$

so that gives:

$$\delta E = - \int_\tau \underline{\nabla}P.\underline{\xi}\, d\tau = - \int_\tau \underline{\nabla}.(P\underline{\xi})\, d\tau = 0$$

since the incompressibility assumption (2.4) implies $\underline{\nabla}.\underline{\xi} = 0$, and $\underline{\xi}.d\underline{S} = 0$ at the boundary (where $d\underline{S}$ is a boundary element). Therefore, geostrophic and hydrostatic balance is a minimum energy state with respect to adiabatic and absolute momentum preserving rearrangements. (see also Cullen et al, 1985).

Writing eqn. (3.13) in the form:

$$\delta E = - \int_\tau (\underline{M} - \underline{\nabla}P).\underline{\xi}\, d\tau$$

then the second variation with respect to parcel rearrangements is given by:

$$\delta^2 E = - \frac{1}{2} \int_\tau \left[\delta\, (\underline{M} - \underline{\nabla}P).\underline{\xi} + (\underline{M} - \underline{\nabla}P).\delta\underline{\xi} \right] d\tau$$

$$= \frac{1}{2} \int_\tau \delta\, (\underline{\nabla}P).\underline{\xi}\, d\tau \qquad 3.14$$

where δ indicates the change measured following a parcel (eg. $\delta\underline{M}=0$) and the equilibrium condition $(M, N, \theta) = \underline{\nabla}P$ has been used.

But $\delta(\underline{\nabla}P) = (\underline{\xi}.\underline{\nabla})\nabla P$ and so eqn (3.14) may be written as:

$$\delta^2 E = \frac{1}{2} \int_\tau \underline{\xi}.\ (\underline{\nabla\nabla P}).\underline{\xi}\, d\tau$$

or $$\delta^2 E = \frac{1}{2} \int_\tau \underline{\xi}.\underline{\underline{Q}}.\underline{\xi}\, d\tau \qquad 3.15$$

so that if $\underline{\underline{Q}}$ is positive definite, $\delta^2 E > 0$ and the equilibrium is stable.

The semi-geostrophic equations therefore describe flow evolution as a sequence of minimum energy states governed by eqns. (2.5-2.8). Given parcel values of M, N and θ at any instant, the stability requirement that $\underline{\underline{Q}}$ be positive definite is sufficient to uniquely determine the position of parcels in physical space subject to the convexity of the domain (CP). These ideas will now be generalized to the unapproximated equations of motion.

4. A minimum energy principle governing parcel stability in the unapproximated equations of motion

Eliassen and Kleinschmidt (1957) showed, using the fully compressible equations for circular flow, that the stability of a baroclinic vortex to toroidal displacements was governed by a quadratic form derived from the second variation of the energy integral as in the previous section. In the spirit of the three-dimensional stability matrix arising in semi-geostrophic theory we extend their result assuming absolute momentum conserving displacements.

The equations of motion for a rotating system characterized by a rotation vector $\underline{\Omega}$ and a geopotential force $\underline{\nabla}\Phi$ (representing the combined effects of gravitation and centripedal forces) are:

$$\frac{D\underline{v}}{Dt} + 2\underline{\Omega}_{\Lambda}\,\underline{V} + \underline{\nabla}\Phi + \alpha\underline{\nabla}p = 0 \qquad 4.1$$

$$\frac{D\alpha}{Dt} = \alpha\,\underline{\nabla}.\,\underline{V} \quad \text{(continuity)} \qquad 4.2$$

$$C_V\frac{DT}{Dt} + p\,\frac{D\alpha}{Dt} = 0 \quad \text{(adiabatic assumption)} \qquad 4.3$$

where $\underline{V}$ is the wind vector, α is the specific volume, p is the pressure, C_V is the specific heat at constant volume and T is the temperature. In a closed system, global energy conservation consistent with these equations demands that:

$$E = \int_{\tau}\left[\frac{1}{2}\,|\underline{v}|^2 + \Phi + CvT\right]\alpha^{-1}\,d\tau = \text{a constant} \qquad 4.4$$

where $d\tau$ denotes a volume increment.

We require the change in global energy (E) on subjecting the fluid to an infinitesmal displacement field $\xi(x,y,z)$. The integral 4.4 may be regarded from the Lagrangian viewpoint whereby the increment $\alpha^{-1}d\tau$ is identified with a fixed mass of fluid so that the first variation of E with respect to parcel rearrangements is:

$$\delta E = \int_\tau \left[\delta\left(\frac{1}{2}\,|\underline{v}|^2\right) + \delta\Phi + C_v\,\delta T \right] \alpha^{-1}\,d\tau \qquad 4.5$$

Now, for instance, $\delta\Phi$ represents the change in geopotential energy experienced by a parcel during its displacement and so:

$$\delta\Phi = \underline{\xi}.\underline{\nabla}\Phi \qquad 4.6$$

To be consistent with the continuity equation 4.2 the displacement field must satisfy:

$$\delta\alpha = \alpha\ \underline{\nabla}.\underline{\xi}$$

and using the adiabatic expression of the first law of thermodynamics (4.3) gives:

$$C_v\ \delta T = -\,p\delta\alpha = -\,p\alpha\underline{\nabla}.\underline{\xi} \qquad 4.7$$

In order to evaluate $\delta(^1/_2|\underline{v}|^2)$, absolute momentum conservation must be imposed, ie.

$$\delta(\underline{V} + 2\underline{\Omega}\Lambda\underline{r}) = 0 \qquad (\underline{r}\text{ is the position vector})$$

or $$\delta\underline{V} = -\,2\underline{\Omega}\Lambda\underline{\xi}$$

But $\delta(^1/_2|\underline{v}|^2) = \underline{V}.\delta\underline{V} = \underline{\xi}.\ (2\underline{\Omega}\Lambda\underline{V})$ (using the above relation) and so eqn. 4.5 may be written as:

$$\delta E = \int_\tau \left\{ \underline{\xi}.\ [\alpha^{-1}(2\underline{\Omega}\Lambda\underline{V} + \underline{\nabla}\Phi)] - p\underline{\nabla}.\underline{\xi} \right\} d\tau \qquad 4.8$$

Now $p\underline{\nabla}.\xi = \underline{\nabla}.(p\underline{\xi}) - \underline{\xi}.\nabla p$ and since the component of $\underline{\xi}$ normal to all rigid boundaries vanishes and $p\underline{\xi} \to o$ at the upper limit of a compressible atmosphere, Gauss' theorem may be used to re-express 4.8 as:

$$\delta E = \int_\tau \underline{\xi}.\ [\alpha^{-1}\ (2\underline{\Omega}\wedge\underline{V} + \underline{\nabla}\Phi) + \underline{\nabla}p]\ d\tau \qquad 4.9$$

The condition for an extremum, $\delta E = 0$ then implies that:

$$2\underline{\Omega}\wedge\underline{V} + \underline{\nabla}\Phi + \alpha\underline{\nabla}p = 0 \qquad 4.10$$

which is a general statement of geostrophic and hydrostatic balance. The second variation is formed by taking a factor α^{-1} outside the square brackets in eqn. (4.9) and using the definition for the absolute momentum:

$$\underline{M} = 2\Omega\underline{r}_* - \underline{\hat{\Omega}}\wedge\underline{V}$$

where $\underline{r}_* = \underline{r} - (\underline{\hat{\Omega}}.\underline{r})\underline{\hat{\Omega}}$, $\Omega = |\underline{\Omega}|$ and $\underline{\Omega} = \Omega\ \underline{\hat{\Omega}}$

so that:

$$\delta^2 E = \frac{1}{2}\int_\tau \underline{\xi}.\ [\delta(2\underline{\Omega}\wedge\underline{V}) + \delta(\underline{\nabla}\Phi) + \delta(\alpha\underline{\nabla}p)]\ \alpha^{-1}d\tau$$

$$\delta^2 E = \frac{1}{2}\int_\tau \underline{\xi}.\ [4\Omega^2\delta\underline{r}_* + \underline{\xi}.\underline{\nabla}(\underline{\nabla}\Phi) + \delta(\alpha\underline{\nabla}p)]\ \alpha^{-1}\ d\tau \qquad 4.11$$

Now for adiabatic and mass conserving rearrangements of a perfect gas:

$$\frac{\delta\alpha}{\alpha} = -\ \gamma\frac{\delta p}{P} = \underline{\nabla}.\underline{\xi} \qquad 4.12$$

where $\gamma = C_v/C_p$ and C_p is the specific heat at constant pressure. If ∂ is used to denote the change measured at a fixed point in space after rearrangement then from eqn. (4.12)

$$\delta p = \partial p + \xi_i\frac{\partial p}{\partial x_i} = -\ p\,\gamma^{-1}\frac{\partial\xi_k}{\partial x_k} \qquad \text{(repeated summation on indices).}$$

and it can readily be shown that:

$$\delta\left(\frac{\partial p}{\partial x_j}\right) = \partial\left(\frac{\partial p}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\xi_i \frac{\partial p}{\partial x_i}\right) = \xi_i \frac{\partial^2 p}{\partial x_j \partial x_i} - \frac{\partial}{\partial x_j}\left[\xi_i \frac{\partial p}{\partial x_i} + p\gamma^{-1}\underline{\nabla}.\underline{\xi}\right] \quad 4.13$$

Using eqns. 4.12 and 4.13 it can then be shown that:

$$\alpha^{-1}\delta(\alpha\underline{\nabla}p) = \frac{\partial}{\partial x_i}\left[\frac{\partial p}{\partial x_i}\xi_i\right] - \frac{\partial}{\partial x_j}\left[\xi_i \frac{\partial p}{\partial x_i} + p\gamma^{-1}\,\underline{\nabla}.\underline{\xi}\right] \quad 4.14$$

which is required for the evaluation of the second variation. If S is the gas entropy defined as the logarithm of potential temperature then:

$$\partial S = \frac{\gamma \partial p}{p} - \frac{\partial \rho}{\rho}$$

so that using the definition of $\underline{M}$ and the balance condition 4.10:

$$(\underline{\xi}.\underline{\nabla})\underline{\nabla}\Phi = \xi_i\left[\xi_j \frac{\partial}{\partial x_j}(2\Omega\underline{M} - 4\Omega^2 \underline{r}_*)\right] + \alpha\left[\xi_j \frac{\partial p}{\partial x_j}\,\xi_i\left(\frac{\gamma}{p}\frac{\partial p}{\partial x_i} - \frac{\partial S}{\partial x_i}\right)\right]$$

$$- \alpha\xi_i\xi_j \frac{\partial^2 p}{\partial x_i \partial x_j} \quad 4.15$$

Combining eqns. (4.11, 4.14 and 4.15) and using Gauss's theorem it can be shown that:

$$\delta^2 E = \frac{1}{2}\int_\tau \left[\underline{\xi}.[2\Omega\rho\underline{\nabla M} - \underline{\nabla p}\underline{\nabla S}].\underline{\xi} + p\gamma^{-1}\,[\underline{\nabla}.\underline{\xi} + \frac{\gamma}{p}\,(\underline{\xi}.\underline{\nabla}p)]^2\right] d\tau$$

where, as in the Boussinesq case, we assume no boundary fluxes. The extremum corresponding to geostrophic and hydrostatic balance (4.10) is a minimum energy state for the system

$$\underline{\underline{\Lambda}} = 2\Omega\rho\underline{\nabla M} - \underline{\nabla p}\underline{\nabla s}$$

provided that the quadratic form:

$$\underline{\xi}.\underline{\underline{\Lambda}}.\underline{\xi} \geq 0$$

where $\underline{\underline{\Lambda}}$ is the unapproximated stability tensor corresponding to $\underline{\underline{Q}}$ in the Boussinesq analysis. The latter may be retrieved (in dimensional form) by setting $\underline{\Omega} = \underline{k}$, $2\Omega = f$, $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and by replacing $\underline{\nabla}p$ with $-\rho g\underline{k}$ and S with $g\theta/\theta_o$.

5. Interpretation of the Stability Criteria

Rather than solve eqn. (3.12) for the eigenvalues ω^2 of the stability matrix the following three conditions can be shown to be necessary and sufficient for $\underline{Q}$ to be positive definite (eg. Stephenson, 1971):

(i) $M_x > 0$ (ii) $\begin{vmatrix} M_x & M_y \\ N_x & N_y \end{vmatrix} > 0$ and

(iii) $\det(\underline{\underline{Q}}) > 0$ (non-dimensional expressions)

Now condition (iii) can be written as:

$$\det(\underline{\underline{Q}}) = J\left(\frac{M,\ N,\ \theta}{x,\ y,\ z}\right) = J\left(\frac{M,\ N,\ \theta}{M,\ N,\ z}\right) \cdot J\left(\frac{M,\ N,\ z}{x,\ y,\ z}\right)$$

$$= \left(\frac{\partial\theta}{\partial z}\right)_{M,N} \cdot J\left(\frac{M,\ N}{x,\ y}\right)$$

(subscripts denote which variables are to be held constant if different from x, y and z).

so that $\left(\frac{\partial\theta}{\partial z}\right)_{M,N} > 0$ guarantees the satisfaction of condition (iii) provided that (ii) is satisfied. Furthermore, condition (ii) may be written as:

$$J\left(\frac{M,\ N}{x,\ y}\right) = J\left(\frac{M,\ N}{M,\ y}\right) \cdot J\left(\frac{M,\ y}{x,\ y}\right) = \left(\frac{\partial N}{\partial y}\right)_{M,z} . M_x$$

so that if condition (i) is satisfied then:

$$\left(\frac{\partial N}{\partial y}\right)_{M,z} > 0 \quad \text{guarantees condition (ii).}$$

Therefore the stability criteria (i) - (iii) may be expressed in the concise form:

$$\text{(i)}\quad M_x > 0 \qquad \text{(ii)}\quad \left(\frac{\partial N}{\partial y}\right)_{M,z} > 0 \qquad \text{(iii)}\quad \left(\frac{\partial \theta}{\partial z}\right)_{M,N} > 0$$

To clarify their physical significance, consider firstly the two-dimensional problem in (x,z). In this case the stability conditions reduce to:

$$M_x > 0 \quad \text{and} \quad \left(\frac{\partial \theta}{\partial z}\right)_M > 0$$

which are the well-known conditions for symmetric stability (eg Bennetts and Hoskins, 1979). Notice that det. $\begin{pmatrix} M_x & M_z \\ \theta_x & \theta_z \end{pmatrix}$ is equal to the Ertel potential vorticity since a two-dimensional airstream in geostrophic and hydrostatic balance on an f-plane is an exact solution of the inviscid equations of motion. The sign of the potential vorticity is however, not a sufficient condition for stability since:

$$q = M_x . \left(\frac{\partial \theta}{\partial z}\right)_M = \left(\frac{\partial M}{\partial x}\right)_\theta . \theta_z$$

and it is possible (though unlikely) for both M_x and $\left(\frac{\partial \theta}{\partial z}\right)_M < 0$. Note that the stability criteria may be re-expressed as:

$$\theta_z > 0 \quad \text{and} \quad \left(\frac{\partial M}{\partial x}\right)_\theta > 0$$

ie. for a gravitationally stable two-dimensional atmosphere ($\theta_z > 0$) the isentropic absolute vorticity must be positive for stability.

The complementary two-dimensional barotropic problem in the x-y plane gives criteria for inertial stability expressible as:

$$M_x > 0 \quad \text{and} \quad \left(\frac{\partial N}{\partial y}\right)_M > 0$$

or $\quad N_y > 0 \quad$ and $\quad \left(\frac{\partial M}{\partial x}\right)_N > 0$

ie. M must increase monotonically eastwards and N must increase northwards along lines of constant M. Clearly if $u_g = 0$, the second criterion is automatically satisfied and the first condition is the classical stability requirement for parallel flow - that the absolute vorticity should be positive.

Consider now the implications of the above conditions for barotropic, circular flow with azimuthal flow speed V(r). Then it is straightforward to show that:

$$M_x = f + \frac{V}{r} + r \cos^2\lambda \frac{d}{dr}\left[\frac{V}{r}\right]$$

and $\quad$ det. $\begin{pmatrix} M_x & M_y \\ N_x & N_y \end{pmatrix} = r^{-1}(f+\frac{dV}{dr})\ (fr+V)$

where (r, λ) are the radius and azimuthal angle, so that stability in the semi-geostrophic system requires:

$$f + \frac{V}{r} + r\frac{d}{dr}\left[\frac{V}{r}\right] > 0$$

$$\text{and} \quad (f + \frac{dV}{dr})\ (fr + V) > 0 \qquad \text{(iv)}$$

These two conditions are to be contrasted with the single condition for balanced cyclostrophic flow (eg. Van Mieghem 1952):

$$\frac{d}{dr}\left[\frac{1}{2} f\ r^2 + V(r).r\right]^2 > 0 \qquad \text{(v)}$$

Consider now the special case of the Rankine vortex for which:

$$V(r) = \begin{cases} \mu r & 0 \leq r \leq r_o \\ \mu \frac{r_o^2}{r} & r \geq r_o \end{cases}$$

where μ is the angular velocity of an inner solid-rotating core of radius r_o. It is easily seen that condition (v) is satisfied provided that $r^2 > - 2\mu r_o^2/f$ or $- \mu < f/2$ and the vortex is stable to axisymmetric perturbations. On the other hand, conditions (iv) are satisfied when $- f < \mu < f$. The onset of instability when $\mu = \pm f$ coincides with the point when the centrifugal force based on the geostrophic wind becomes equal to the coriolis force at $r = r_o$. Clearly, the apparent instability of the Rankine vortex in semi-geostrophic theory coincides with the breakdown of the semi-geostrophic approximation and does not therefore indicate the physical instability of the vortex. This need not always follow as the case of solid 'geostrophic' rotation demonstrates.

Similarly for the case of pure deformation flow given by $u_g = - \varepsilon x$ and $v_g = \varepsilon y$, the stability conditions become:

$$M_x = f > 0$$

and det. $\begin{pmatrix} M_x & M_y \\ N_x & N_y \end{pmatrix} = f^2 - \varepsilon^2 > 0$

or $\varepsilon < f$

Here again the requirement for instability occurs in the parameter range for which the semi-geostrophic equations are invalid i.e. the Lagrangian time scale $< f^{-1}$. In the comparison of the stability results for balanced circular flow and the barotropic, semi-geostrophic system it should also be noted that the former are based on axisymmetric displacements of the field of motion in contrast to the semi-geostrophic analysis and are not therefore a condition for physical instability. It is an observed fact, however, that real vortices can be stable.

In general, if det($\underline{\underline{Q}}$) is initially positive everywhere then for the adiabatic, inviscid semi-geostrophic system it will remain positive and at least one of the stability conditions is satisfied.

CP provide an existence proof which shows that a unique solution satisfying <u>all</u> the criteria exists even in the presence of discontinuities.

6. Conclusion

The relationship of the three-dimensional positive-definite matrix identified by Cullen and Purser (1984) to the classical theory of parcel stability has been explored. The determinant of this matrix is the Jacobian of transformation between physical space and the space formed by the two components of geostrophic absolute momentum and potential temperature. Only for strictly uni-directional flow does this correspond exactly to the Ertel potential vorticity, otherwise the link between stability and the sign of the potential vorticity is rather tenuous. Moreover the sign of the determinent is just one of three stability conditions which must be satisfied if solution of the semi-geostrophic equations is to be meaningful. These embody the well-known hydrodynamic stability condition for a zonal current that the isentropic absolute vorticity must be positive.

Real eigenfrequencies associated with the stability matrix exactly correspond with those of classical parcel theory only for two-dimensional flow with absolute momentum conserving perturbations. Parcel theory is still relevant under the conditions of validity of semi-geostrophic theory (near two-dimensionality).

It is shown using the unapproximated equations of motion that at any instant, geostrophic and hydrostatic balance corresponds to an extremum in the total energy with respect to infinitesmal virtual displacements which preserve absolute momentum and potential temperature. The method is an extension of the analysis carried out by Eliassen and Kleinschmidt (1957) for the case of a circular vortex. Using the calculus of variations, it is further shown that the sign of the second variation is guaranteed to be positive if a certain matrix (more general than that of CP) is positive definite. The extremum is then a minimum energy state corresponding to stable equilibrium. The semi-geostrophic stability theorems of CP constitute a finite amplitude generalization of the parcel method for which the requirement of a positive-definite Hessian matrix $\underline{\underline{Q}}$ is replaced by the more general concept of convexity (of the modified pressure $P(x,y,z)$). The semi-geostrophic solution represented at any time by P is continous though its first derivatives need not be. In this way for example a frontal discontinuity intruding into the fluid from a boundary is an admissible, stable class of solution even though the stability conditions of section 4 become meaningless at the discontinuity.

The inclusion of moist thermodynamics into the foregoing analysis is straightforward if we assume a pseudo-adiabatic process and replace θ by wet-bulb or equivalent potential temperature. Conditional slantwise instability is then implied by the same positive-definite matrix requirement though it should be noted that the corresponding determinant is not a Lagrangian conservation property.

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<u>Appendix</u> - The three-dimensional semi-geostrophic 'potential vorticity' equation.

Using repeating suffix notion, eqns. (2.5 - 2.7) may be written as:

$$\frac{\partial M}{\partial t} + u_p \frac{\partial M}{\partial x_p} = x_2 - N \qquad \text{(A)}$$

$$\frac{\partial N}{\partial t} + u_p \frac{\partial N}{\partial x_p} = M - x_1 \qquad \text{(B)}$$

$$\frac{\partial \theta}{\partial t} + u_p \frac{\partial \theta}{\partial x_p} = 0 \qquad \text{(C)}$$

where $(x_1, x_2, x_3) \equiv (x, y, z)$

It is required to show that $q = \varepsilon_{ijk} \frac{\partial \theta}{\partial x_i} \frac{\partial M}{\partial x_j} \frac{\partial N}{\partial x_k}$

$$\text{where } \varepsilon_{ijk} = \begin{cases} 0 \text{ if any two of } i,\ j \text{ or } k \text{ are equal.} \\ +1 \text{ if } i,\ j \text{ and } k \text{ are a cyclic permutation of } 1, 2, 3. \\ -1 \text{ if } i,\ j,\ k \text{ are a cyclic permutation of } 2, 1, 3. \end{cases}$$

is conserved following the motion.

Now $\frac{\partial N}{\partial x_k} \frac{\partial}{\partial x_j}$ (A) $+ \frac{\partial M}{\partial x_j} \frac{\partial}{\partial x_k}$ (B) can be shown to

give:

$$\left(\frac{\partial}{\partial t} + u_p \frac{\partial}{\partial x_p}\right) \frac{\partial M}{\partial x_j} \frac{\partial N}{\partial x_k} = -\left(\underbrace{\frac{\partial u_p}{\partial x_j} \frac{\partial M}{\partial x_p} \frac{\partial N}{\partial x_k} + \frac{\partial u_p}{\partial x_k} \frac{\partial M}{\partial x_j} \frac{\partial N}{\partial x_p}}_{\mathrm{I}}\right) + \left(\underbrace{\frac{\partial M}{\partial x_j} \frac{\partial M}{\partial x_k} - \frac{\partial N}{\partial x_j} \frac{\partial N}{\partial x_k}}_{\mathrm{II}}\right)$$

$$+ \underbrace{\delta_{2j} \frac{\partial N}{\partial x_k} \qquad \delta_{1k} \frac{\partial M}{\partial x_j}}_{\mathrm{III}}$$

$$\varepsilon_{ijk} \times \text{term(I)} = \varepsilon_{ijk} \frac{\partial u_p}{\partial x_j} \frac{\partial M}{\partial x_p} \frac{\partial N}{\partial x_k} + \varepsilon_{ikj} \frac{\partial u_p}{\partial x_j} \frac{\partial M}{\partial x_k} \frac{\partial N}{\partial x_p}$$

$$= \varepsilon_{ijk} \frac{\partial u_p}{\partial x_j} \left(\frac{\partial M}{\partial x_p} \frac{\partial N}{\partial x_k} - \frac{\partial M}{\partial x_k} \frac{\partial N}{\partial x_p}\right)$$

Equation (D) x $\frac{\partial \theta}{\partial x_i}$ may then be written as:

$$\frac{\partial\theta}{\partial x_i}\left(\frac{\partial}{\partial t} + u_p\frac{\partial}{\partial x_p}\right)\varepsilon_{ijk}\frac{\partial M}{\partial x_j}\frac{\partial N}{\partial x_k} = - \varepsilon_{ijk}\frac{\partial\theta}{\partial x_i}\frac{\partial u_p}{\partial x_j}\left(\frac{\partial M}{\partial x_p}\frac{\partial N}{\partial x_k} - \frac{\partial M}{\partial x_k}\frac{\partial N}{\partial x_p}\right)$$

$$+ \varepsilon_{ijk}\frac{\partial\theta}{\partial x_i}\left(\frac{\partial M}{\partial x_j}\frac{\partial M}{\partial x_k} - \frac{\partial N}{\partial x_j}\frac{\partial N}{\partial x_k}\right) + \varepsilon_{ijk} \cdot \frac{\partial\theta}{\partial x_i}\left(\delta_{2j}\frac{\partial N}{\partial x_k} - \delta_{1k}\frac{\partial M}{\partial x_j}\right) \quad - \text{(E)}$$

The second term on the right hand side (RHS) of eqn. (E) is equal to zero in the sum over repeated indices since the expression in brackets is symmetric with respect to j and k.

Consider now $\frac{\partial}{\partial x_i}$ (C) . $\varepsilon_{ijk}\frac{\partial M}{\partial x_j}\frac{\partial N}{\partial x_k}$

$$\varepsilon_{ijk}\frac{\partial M}{\partial x_j}\frac{\partial N}{\partial x_k}\left(\frac{\partial}{\partial t} + u_p\frac{\partial}{\partial x_p}\right)\frac{\partial\theta}{\partial x_i} = - \varepsilon_{ijk}\frac{\partial M}{\partial x_j}\frac{\partial N}{\partial x_k}\frac{\partial U_p}{\partial x_i}\frac{\partial\theta}{\partial x_p}$$

$$= \varepsilon_{ijk}\frac{\partial M}{\partial x_i}\frac{\partial N}{\partial x_k}\frac{\partial u_p}{\partial x_j}\frac{\partial\theta}{\partial x_p} \quad - \text{(F)}$$

The semi-geostrophic potential vorticity equation is obtained by adding eqns. (E) and (F) giving:

$$\frac{Dg}{Dt} = - \varepsilon_{ijk} \cdot \frac{\partial u_p}{\partial x_j}\left(\frac{\partial M}{\partial x_p}\frac{\partial N}{\partial x_k}\frac{\partial\theta}{\partial x_i} - \frac{\partial M}{\partial x_k}\frac{\partial N}{\partial x_p}\frac{\partial\theta}{\partial x_i} - \frac{\partial M}{\partial x_i}\frac{\partial N}{\partial x_k}\frac{\partial\theta}{\partial x_p}\right)$$

$$+ \varepsilon_{ijk}\frac{\partial\theta}{\partial x_i}\left(\delta_{2j}\frac{\partial N}{\partial x_k} - \delta_{1k}\frac{\partial M}{\partial x_j}\right) \quad - \text{(G)}$$

Now the first term on the RHS can, by swopping dummy indices i and k, be written as:

$$\varepsilon_{ijk}\frac{\partial u_p}{\partial x_j}\left(\frac{\partial M}{\partial x_p}\frac{\partial N}{\partial x_i}\frac{\partial\theta}{\partial x_k} - \frac{\partial M}{\partial x_i}\frac{\partial N}{\partial x_p}\frac{\partial\theta}{\partial x_k} - \frac{\partial M}{\partial x_k}\frac{\partial N}{\partial x_i}\frac{\partial\theta}{\partial x_p}\right)$$

or, by forming the mean of the two expressions,

$$\frac{1}{2} \cdot \varepsilon_{ijk}\frac{\partial u_p}{\partial x_j}\left(S_{pi}\frac{\partial\theta}{\partial x_k} + S_{ik}\frac{\partial\theta}{\partial x_p} + S_{kp}\frac{\partial\theta}{\partial x_i}\right)$$

where $S_{ij} = \frac{\partial M}{\partial x_i}\frac{\partial N}{\partial x_j} - \frac{\partial M}{\partial x_j}\frac{\partial N}{\partial x_i} = - S_{ji}$

Also if we let Λ_{ikp} denote the expression in brackets above then:

$$\Lambda_{ikp} = S_{pi} \frac{\partial \theta}{\partial x_k} + S_{ik} \frac{\partial \theta}{\partial x_p} + S_{kp} \frac{\partial \theta}{\partial x_i} = \varepsilon_{ikp}\, q$$

and the first term on the RHS of (g) may be simplified to:

$$- \frac{g}{2} \varepsilon_{ijk} \varepsilon_{ipk} \frac{\partial u_p}{\partial x_j}$$

Using the identity:

$\varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$ it is readily shown that this term further reduces to $- q \frac{\partial u_j}{\partial x_j}$

Finally, we use the balance relation

$$(M, N, \theta) = \left(\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, \frac{\partial P}{\partial x_3} \right)$$

where $P = \phi + {}^1/_2 . (x_1^2 + x_2^2)$ to show that the second term on the RHS of (G) vanishes, so that:

$$\frac{Dg}{Dt} = \varepsilon_{ijk} \frac{\partial^2 P}{\partial x_i \partial x_3} \left(\delta_{2j} \frac{\partial^2 P}{\partial x_k \partial x_2} - \delta_{1k} \frac{\partial^2 P}{\partial x_j \partial x_1} \right) - q \frac{\partial u_j}{\partial x_j}$$

which on expansion can readily be shown to be zero, leading to:

$$\frac{Dg}{Dt} = - q \frac{\partial u_j}{\partial x_j} \qquad - (H)$$

Using the compressible anelastic form of the continuity equation quoted in HB ie.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{r_*(z)} \frac{\partial (r_* w)}{\partial z} = 0$$

where $r_*(z)$ is a basic state pseudo-density dependent of z alone then eqn. (H) may be written as:

$$\frac{D}{Dt} \left({}^{q}/_{r*(z)} \right) = 0$$