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Abstract

A variational approach to the Euler-Lagrange equations for the Quasi-Hydrostatic model is presented together with an explicit Hamiltonian (Poisson bracket) formulation. The bracket acts on functionals of the zonal and meridional angular momentum. Features of the relationship between the Hamiltonian formulation of the equations for an ideal fluid and the Hamiltonian formulation of the primitive equations are discussed.

1 Introduction

The Quasi-Hydrostatic equations (henceforth QH) formulated by White & Bromley (1988, 1994; hereafter referred to collectively as WB), are a more accurate alternative to the traditional hydrostatic primitive equations (HPEs), and they are currently used in the U.K. Meteorological Office Unified Model. The QH model retains the terms in $2\Omega\cos\phi$ that occur in the zonal and vertical components of the momentum equation when motion is measured relative to a rotating frame. The simplest modifications to the HPEs that retain these terms conserve energy but do not imply exact angular momentum principles or analogues of Ertel's potential vorticity conservation law. These latter principles are satisfied if all the metric terms are retained and the shallow atmosphere approximation is relaxed. We shall be concerned with the height coordinate version of the QH equations here, but it is important to note that a pressure coordinate version of the equations exists (see WB). Indeed, the U.K. Met. Office Unified Model is based on the pressure coordinate version of these equations, which are not precisely equivalent (physically) to the height coordinate version.

The existence of these conservation laws invites an investigation of a possible Hamiltonian formulation. Within such a description one expects the conservation laws to be deduced from symmetries of the Lagrangian (via Noether's theorem - see *e.g.* Shepherd 1990), or equivalently, if a

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Poisson bracket operator exists, then the conserved quantities commute with the Hamiltonian. For an overview and discussion of Hamiltonian methods and Hamilton's principle see Salmon (1988a).

In this paper we give a derivation of the QH equations using Hamilton's Principle and an independent formulation of the same equations using a noncanonical Poisson bracket. In both instances we use Eulerian kinematics. This is a departure from the philosophy of the fundamental role of the Lagrangian description, however the derivation of these equations (WB) is entirely within the Eulerian framework and a number of subtleties arising from this description are of importance to the Hamiltonian approach. It had long been known how to derive the HPEs from Hamilton's Principle using Lagrangian particle labels (see, *e.g.*, Shutts 1989), and Salmon (1988b) showed how they could be written down using a Poisson bracket. This latter form uses the theory of Generalized Hamiltonian systems also known as the Dirac method of constraints, and the HPEs are derived from the three-dimensional Euler equations by imposing the constraint that the vertical velocity w, is equal to zero.

The noncanonical Hamiltonian (Poisson bracket) formulation of the HPEs was discovered by Holm & Long (1989), using isopycnal coordinates. The time derivative on a particle, which is equivalent to the usual expression for the material derivative, contains (implicitly) a vertical advection term. Thus setting $w = 0$ in the kinetic energy part of the Hamiltonian would, on the face of it, remove this term from the Eulerian equations. The key idea behind Holm & Long's approach is to note that because w only occurs in the material derivative in the HPEs, then one can remove it in a natural way from the equations by using a material surface (*e.g.* isopycnal or isentropic) as a vertical coordinate. However, in the QH equations w occurs in metric and coriolis terms in addition to the vertical advection, and it is for this reason that we prefer to work in the Eulerian framework, where we can 'keep track' of the local vertical velocity.

The plan of this paper is as follows. In §2 we begin with an introduction to Hamilton's principle in Eulerian coordinates and then use this form of the principle to derive the three-dimensional Euler equations. This exercise forms the basis of the first result we then proceed to derive: the transformation to spherical geometry and the variational formulation of the QH equations. In §3, we discuss the conservation laws for the QH equations using the symmetry properties of the Lagrangian. We demonstrate in §4, that the Legendre transformation cannot be used to go from the Lagrangian functional to the Hamiltonian, and obtain Hamilton's equations (in the otherwise usual way) for the QH model. (This fact also holds for the HPEs.) This motivates the derivation in §5, of a noncanonical Poisson bracket, from which we then obtain the Hamiltonian form of the QH equations. Finally, we give a brief summary in §6.

2 Hamilton's Principle in Eulerian Coordinates

2.1 Cartesian Geometry and the Euler Equations.

Consider the mass contained in a small volume $\mathrm{d}^3\mathbf{x}$ of fluid surrounding a fluid particle, identified initially by 3 Lagrangian labels (a,b,c) at time $t=0$, and located at position $\mathbf{x}(\mathbf{a},t)$ at time t. That is, we view the fluid motion as the time dependent map from label space $\mathbf{a}$, to position $\mathbf{x}$, with

$$\mathbf{x}(\mathbf{a},t=0)=\mathbf{a}\ .$$

The mass element $\mathrm{d}^3\mathbf{a}$ defines the density $\rho(\mathbf{x},t)$. Since this mass element was contained initially in a similar volume about the point $\mathbf{a}$, then to ensure mass conservation we must have

$$\rho(\mathbf{x},t)\mathrm{d}^3\mathbf{x}=\rho(\mathbf{a},0)\mathrm{d}^3\mathbf{a}\ .$$

Thus, given $\mathbf{x}(\mathbf{a},t)$, one may solve for $\mathbf{a}(\mathbf{x},t)$. Since the a^i $(i=1,2,3)$ are independent of time, they satisfy

$$\left.\frac{\partial a^i}{\partial t}\right|_{\mathbf{x}}+\mathbf{u}.\nabla a^i=0\ , \tag{1}$$

where $\mathbf{u}=(u,v,w)$ is the velocity and $\nabla=(\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$. (The summation convention and boldface vector notation will be used (interchangeably as appropriate) throughout this paper.)

The conservation of mass and of particle labels need to be incorporated into Hamilton's principle. These principles are incorporated by using Lagrange multipliers. Let -μ be the multiplier for the mass constraint and let $\rho\gamma^i$ be the multipliers for the label constraints. Then Hamilton's principle takes the form

$$\delta\int_{t_0}^{t_1}\mathrm{d}t\ \mathcal{L}[\mathbf{u},\rho]=\delta\int_{t_0}^{t_1}\mathrm{d}t\int\mathrm{d}^3\mathbf{x}\ (\text{kinetic energy}\ -\ \text{potential energy})=0\ ,$$

which is written in terms of the dependent variables as

$$\delta\int_{t_0}^{t_1}\mathrm{d}t\int\mathrm{d}^3\mathbf{x}\left[\frac{1}{2}\rho\mathbf{u}^2-\rho(E(\rho^{-1})+\Phi(\mathbf{x})+U(\mathbf{x}))-\mu\left(\frac{\partial\rho}{\partial t}+\nabla.(\rho\mathbf{u})\right)+\rho\gamma_i\left(\frac{\partial a^i}{\partial t}+\mathbf{u}.\nabla a^i\right)\right]=0\ . \tag{2}$$

The dependent variables $\mathbf{u}$, ρ and E (internal energy) are varied at the fixed locations $\mathbf{x}$ (U is the external potential and Φ is the gravitational potential). We do not incorporate explicitly the conservation of entropy because we shall always consider adiabatic motion.

For planetary motion the fluid velocities are measured relative to a rotating frame and therefore it is necessary to transform the action functional to the non-inertial frame. The transformation of the velocities is

$$\mathbf{u}\mapsto\mathbf{u}+\boldsymbol{\Omega}\wedge\mathbf{x}\ , \tag{3}$$

where $\boldsymbol{\Omega}$ is the rotation vector. The transformation is applied to the kinetic energy term in the Lagrangian, but not to the advecting velocities in the constraints. This is because the conservation laws are valid in any frame and are defined in terms of the velocity relative to the local fixed coordinates. Applying the transformation (3), the Lagrangian becomes

$$\mathcal{L}[\mathbf{u},\rho] = \int \mathrm{d}^3\mathbf{x} \left[\frac{1}{2}\rho\mathbf{u}^2 + \rho\mathbf{u}.(\boldsymbol{\Omega}\wedge\mathbf{x}) - \rho(E(\rho^{-1}) + \Phi'(\mathbf{x}) + U(\mathbf{x})) \right.$$

$$\left. -\mu\left(\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{u})\right) + \rho\gamma_i\left(\frac{\partial a^i}{\partial t} + \mathbf{u}.\nabla a^i\right)\right] , \tag{4}$$

where $\Phi'(\mathbf{x}) = \Phi(\mathbf{x}) - (\boldsymbol{\Omega}\wedge\mathbf{x})^2/2$, is the true gravitational potential minus the centripetal contribution from the rotation. The use of Φ', instead of Φ, is a standard approximation (Salmon 1983).

The equations of motion for adiabatic, frictionless flow, are obtained by setting $U = 0$ in (4) and taking independent arbitrary variations δa^i, $\delta\mathbf{u}$ and $\delta\rho$ which vanish at the boundary; then Hamilton's principle yields

$$\delta a^i : \quad \frac{\partial\gamma^i}{\partial t} + \mathbf{u}.\nabla\gamma^i = 0 , \tag{5}$$

$$\delta\mathbf{u} : \quad \mathbf{u} + \boldsymbol{\Omega}\wedge\mathbf{x} + \nabla\mu + \gamma.\nabla\mathbf{a} = 0 , \tag{6}$$

$$\delta\rho : \quad \frac{1}{2}\mathbf{u}^2 + \mathbf{u}.(\boldsymbol{\Omega}\wedge\mathbf{x}) - \frac{\partial\,(\rho E)}{\partial\rho} - U - \Phi' + \frac{\partial\mu}{\partial t} + \mathbf{u}.\nabla\mu = 0 . \tag{7}$$

For a discussion of these equations and of Lin's work (Lin, 1963), see Salmon (1988a) and references therein.

If the gradient of (7) is subtracted form the time derivative of (6) and then if (6) is used again to eliminate μ, we obtain an equation in $\mathbf{u}$, γ^i, a^i, ρ, E and Φ'. The particle labels and their Lagrange multipliers can then be eliminated from the expression by using (5) and the gradient of (6). Finally, use of the thermodynamic relation

$$\nabla\left(\frac{\partial(\rho E)}{\partial\rho}\right) = \frac{1}{\rho}\nabla p , \tag{8}$$

yields the three-dimensional Euler equations

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\wedge\mathbf{x} + \frac{1}{\rho}\nabla p + \nabla\Phi' = 0 . \tag{9}$$

A detailed derivation of the three-dimensional Euler equations for irrotational flow, using an Eulerian form of Hamilton's principle, can be found in Mittag, Stephen & Yourgrau (1979).

2.2 Spherical Polar Coordinates and the Quasi-Hydrostatic Equations

As mentioned in the Introduction, an important feature of the QH equations is that they retain all the metric terms from the spherical geometry and therefore we shall require the Lagrangian to be written in terms of spherical polar coordinates. We write down the result of the transformation of the Lagrangian and then consider the variations of the dependent variables.

Before we derive the Euler equations once more, it is useful to distinguish the term in $w^2/2$ in the kinetic energy part of the Lagrangian from the linear terms in w that arise in the mass and particle label constraints. Therefore, denote the former by W, so that the Lagrangian (4) written in the spherical polar coordinates becomes

$$\begin{aligned}\mathcal{L}[\mathbf{u},\rho] = \int \mathrm{d}r\mathrm{d}\phi\mathrm{d}\lambda\, r^2\cos\phi \Big[& \frac{1}{2}\rho(u^2+v^2+W^2) + \rho\Omega ur\cos\phi - \rho(E+U+\Phi') \\ & -\mu\left(\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2\rho w)}{\partial r} + \frac{1}{r\cos\phi}\frac{\partial(\rho v\cos\phi)}{\partial\phi} + \frac{1}{r\cos\phi}\frac{\partial(\rho u)}{\partial\lambda}\right) \\ & +\rho\gamma_i\left(\frac{\partial a^i}{\partial t} + w\frac{\partial a^i}{\partial r} + \frac{v}{r}\frac{\partial a^i}{\partial\phi} + \frac{u}{r\cos\phi}\frac{\partial a^i}{\partial\lambda}\right)\Big] \; . \end{aligned} \tag{10}$$

$\mathbf{u}$ are the three components of the velocity relative to the spherical polar coordinate system (*i.e.* relative to the earth), and the labels a^i and the constraints γ^i are now assumed to be specified relative to the new coordinates. The geopotential $\Phi' = gr$. Independent, arbitrary variations of the action are taken, and application of Hamilton's principle then leads to

$$\delta a^i: \quad \frac{\partial\gamma^i}{\partial t} + w\frac{\partial\gamma^i}{\partial r} + \frac{v}{r}\frac{\partial\gamma^i}{\partial\phi} + \frac{u}{r\cos\phi}\frac{\partial\gamma^i}{\partial\lambda} = 0\,, \tag{11}$$

$$\delta\mathbf{u}: \begin{cases} u + r\Omega\cos\phi + \dfrac{1}{r\cos\phi}\dfrac{\partial\mu}{\partial\lambda} + \dfrac{\gamma_i}{r\cos\phi}\dfrac{\partial a^i}{\partial\lambda} &= 0\,, \\ v + \dfrac{1}{r}\dfrac{\partial\mu}{\partial\phi} + \dfrac{\gamma_i}{r}\dfrac{\partial a^i}{\partial\phi} &= 0, \\ W + \dfrac{\partial\mu}{\partial r} + \gamma_i\dfrac{\partial a^i}{\partial r} &= 0, \end{cases} \tag{12}$$

and

$$\begin{aligned}\delta\rho: \quad & \frac{1}{2}(u^2+v^2+W^2) + ur\Omega\cos\phi - \frac{\partial(\rho E)}{\partial\rho} - U - \Phi' \\ & + \frac{\partial\mu}{\partial t} + w\frac{\partial\mu}{\partial r} + \frac{v}{r}\frac{\partial\mu}{\partial\phi} + \frac{u}{r\cos\phi}\frac{\partial\mu}{\partial\lambda} = 0\,. \end{aligned} \tag{13}$$

We sketch the derivation of the QH equations using (11)-(13). The Lagrange multipliers μ and γ^i need to be eliminated along with the particle labels a^i. This is achieved if we first take the gradient of (13) and subtract the result from the local time derivative of (12). The equations (12) are used again to eliminate the terms in μ. Using the thermodynamic relationship (8), we obtain

$$\begin{aligned}&\frac{\partial u}{\partial t} + \frac{1}{r\cos\phi}\frac{\partial}{\partial t}\left(\gamma_i\frac{\partial a^i}{\partial\lambda}\right) \\ &\quad + \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}\left(\frac{1}{2}\left(u^2+v^2-W^2\right) + Ww\right) + \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} + \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}\left(U + \Phi' + \gamma_i\mathbf{u}.\nabla a^i\right) = 0\,, \end{aligned} \tag{14}$$

$$\begin{aligned}&\frac{\partial v}{\partial t} + \frac{1}{r}\frac{\partial}{\partial t}\left(\gamma_i\frac{\partial a^i}{\partial\phi}\right) \\ &\quad + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{1}{2}\left(u^2+v^2-W^2\right) + Ww\right) + \frac{1}{\rho r}\frac{\partial p}{\partial\phi} + \frac{1}{r}\frac{\partial}{\partial\phi}\left(U + \Phi' + \gamma_i\mathbf{u}.\nabla a^i\right) = 0\,, \end{aligned} \tag{15}$$

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial t}\left(\gamma_i \frac{\partial a^i}{\partial r}\right)$$
$$+ \frac{\partial}{\partial r}\left(\frac{1}{2}\left(u^2 + v^2 - W^2\right) + Ww\right) + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left(U + \Phi' + \gamma_i \mathbf{u}.\nabla a^i\right) = 0 , \tag{16}$$

where

$$\mathbf{u}.\nabla \equiv w\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \phi} + \frac{u}{r\cos\phi}\frac{\partial}{\partial \lambda} .$$

We now expand these expressions using (11). Instead of writing out the full equations we merely write down the terms in a^i; they are

$$\frac{\partial}{\partial t}\left(\gamma_i \frac{\partial a^i}{\partial \lambda}\right) + \frac{\partial}{\partial \lambda}(\gamma_i \mathbf{u}.\nabla a^i)$$
$$= w\left(\frac{\partial \gamma_i}{\partial \lambda}\frac{\partial a^i}{\partial r} - \frac{\partial \gamma_i}{\partial r}\frac{\partial a^i}{\partial \lambda}\right) + \frac{v}{r}\left(\frac{\partial \gamma_i}{\partial \lambda}\frac{\partial a^i}{\partial \phi} - \frac{\partial \gamma_i}{\partial \phi}\frac{\partial a^i}{\partial \lambda}\right) , \tag{17}$$

$$\frac{\partial}{\partial t}\left(\gamma_i \frac{\partial a^i}{\partial \phi}\right) + \frac{\partial}{\partial \phi}(\gamma_i \mathbf{u}.\nabla a^i)$$
$$= w\left(\frac{\partial \gamma_i}{\partial \phi}\frac{\partial a^i}{\partial r} - \frac{\partial \gamma_i}{\partial r}\frac{\partial a^i}{\partial \phi}\right) + \frac{u}{r\cos\phi}\left(\frac{\partial \gamma_i}{\partial \phi}\frac{\partial a^i}{\partial \lambda} - \frac{\partial \gamma_i}{\partial \lambda}\frac{\partial a^i}{\partial \phi}\right) , \tag{18}$$

$$\frac{\partial}{\partial t}\left(\gamma_i \frac{\partial a^i}{\partial r}\right) + \frac{\partial}{\partial r}(\gamma_i \mathbf{u}.\nabla a^i)$$
$$= \frac{v}{r}\left(\frac{\partial \gamma_i}{\partial r}\frac{\partial a^i}{\partial \phi} - \frac{\partial \gamma_i}{\partial \phi}\frac{\partial a^i}{\partial r}\right) + \frac{u}{r\cos\phi}\left(\frac{\partial \gamma_i}{\partial r}\frac{\partial a^i}{\partial \lambda} - \frac{\partial \gamma_i}{\partial \lambda}\frac{\partial a^i}{\partial r}\right) . \tag{19}$$

If we now take the gradient of (12) and substitute for the terms on the right hand sides of (17)-(19), we obtain

$$\frac{1}{r\cos\phi}\left(\frac{\partial}{\partial t}\left(\gamma_i \frac{\partial a^i}{\partial \lambda}\right) + \frac{\partial}{\partial \lambda}(\gamma_i \mathbf{u}.\nabla a^i)\right)$$
$$= \frac{uw}{r} + w\frac{\partial u}{\partial r} + 2\Omega w\cos\phi - \frac{w}{r\cos\phi}\frac{\partial W}{\partial \lambda} + \frac{v}{r}\frac{\partial u}{\partial \phi} - \frac{uv}{r}\tan\phi - 2\Omega v\sin\phi - \frac{v}{r\cos\phi}\frac{\partial v}{\partial \lambda} , \tag{20}$$

$$\frac{1}{r}\left(\frac{\partial}{\partial t}\left(\gamma_i \frac{\partial a^i}{\partial \phi}\right) + \frac{\partial}{\partial \phi}(\gamma_i \mathbf{u}.\nabla a^i)\right)$$
$$= \frac{vw}{r} + w\frac{\partial v}{\partial r} - \frac{w}{r}\frac{\partial W}{\partial \phi} + \frac{u}{r\cos\phi}\frac{\partial v}{\partial \lambda} - \frac{u}{r}\frac{\partial u}{\partial \phi} + \frac{u^2}{r}\tan\phi + 2\Omega u\sin\phi , \tag{21}$$

$$\frac{\partial}{\partial t}\left(\gamma_i \frac{\partial a^i}{\partial r}\right) + \frac{\partial}{\partial r}(\gamma_i \mathbf{u}.\nabla a^i)$$
$$= \frac{v}{r}\frac{\partial W}{\partial \phi} - v\frac{\partial v}{\partial r} + \frac{u}{r\cos\phi}\frac{\partial W}{\partial \lambda} - u\frac{\partial u}{\partial r} - \frac{u^2 + v^2}{r} - 2\Omega u\cos\phi . \tag{22}$$

We now bring these results together by substituting (20)-(22) into (14)-(16), and obtain the non-hydrostatic momentum equations in the form:

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{W}{r\cos\phi}\frac{\partial(w - W)}{\partial \lambda}$$
$$- \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) + \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial \lambda} = \frac{-1}{r\cos\phi}\frac{\partial U}{\partial \lambda} , \tag{23}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \frac{W}{r}\frac{\partial(w-W)}{\partial\phi} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial\phi} = \frac{-1}{r}\frac{\partial U}{\partial\phi}\,, \tag{24}$$

$$\frac{\mathrm{D}W}{\mathrm{D}t} + W\frac{\partial(w-W)}{\partial r} - \left(2\Omega + \frac{u}{r\cos\phi}\right)u\cos\phi - \frac{v^2}{r} + g + \frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{\partial U}{\partial r}\,. \tag{25}$$

The terms on the right hand side represent any external forces, and

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial r}\,.$$

If we set $w = W$, then (23)-(25) are the Navier-Stokes equations in height coordinates. If we set $W = 0$ (and retain w), which would be equivalent to starting out with the Lagrangian

$$\begin{aligned}\mathcal{L}[\mathbf{u},\rho] = \int \mathrm{d}r\mathrm{d}\phi\mathrm{d}\lambda r^2\cos\phi \Bigg[&\frac{1}{2}\rho(u^2+v^2) + \rho\Omega ur\cos\phi - \rho(E+U+\Phi') \\ &-\mu\left(\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2\rho w)}{\partial r} + \frac{1}{r\cos\phi}\frac{\partial(\rho v\cos\phi)}{\partial\phi} + \frac{1}{r\cos\phi}\frac{\partial(\rho u)}{\partial\lambda}\right) \\ &+\rho\gamma_i\left(\frac{\partial a^i}{\partial t} + w\frac{\partial a^i}{\partial r} + \frac{v}{r}\frac{\partial a^i}{\partial\phi} + \frac{u}{r\cos\phi}\frac{\partial a^i}{\partial\lambda}\right)\Bigg]\,,\end{aligned} \tag{26}$$

then we obtain the QH equations in height coordinates:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) + \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} = \frac{-1}{r\cos\phi}\frac{\partial U}{\partial\lambda}\,, \tag{27}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial\phi} = \frac{-1}{r}\frac{\partial U}{\partial\phi}\,, \tag{28}$$

$$-\left(2\Omega + \frac{u}{r\cos\phi}\right)u\cos\phi - \frac{v^2}{r} + g + \frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{\partial U}{\partial r}\,. \tag{29}$$

This is the first of the results that we set out to achieve.

3 Conservation Laws

The application of Noether's theorem to (26) will yield the necessary information on the conservative properties of this system.

Simply by inspection of the Lagrangian (26), we see that, provided the external potential function U does not depend explicitly on time, the Lagrangian is autonomous. This symmetry corresponds (via Noether's theorem) to the conservation of energy. Also by inspection, it is observed that the Lagrangian does not depend explicitly on λ. This symmetry relates to the axial angular momentum principle (not a Lagrangian conservation law)

$$\rho\,\frac{\mathrm{D}}{\mathrm{D}t}\Big((u+\Omega r\cos\phi)r\cos\phi\Big) = -\frac{\partial p}{\partial\lambda}\,,$$

which we shall refer to later in our Hamiltonian formulation of the QH equations.

In this Eulerian variational principle we have explicitly constrained the system to conserve the Lagrangian particle labels. Let θ be any quantity that is a Lagrangian conserved quantity. Then evidently, from (11),

$$\frac{\mathrm{D}}{\mathrm{D}t}\{(\nabla\wedge\gamma).\nabla\theta\} = 0\,. \tag{30}$$

Note that, from (12), we have

$$\gamma^i = -\left(u_j + \varepsilon_{jkl}\Omega^k x^l + \nabla_j \mu\right) \frac{\partial x^j}{\partial a_i} , \tag{31}$$

where $u^i \leftrightarrow (u, v, 0)$, and ε_{jkl} is the completely skew-symmetric alternating symbol (we have used covariant notation for the sake of brevity). When (31) is substituted into (30), we obtain, after a little manipulation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left\{\frac{(\nabla \wedge \mathbf{u} + 2\mathbf{\Omega}).\nabla\theta}{\rho}\right\} = 0 . \tag{32}$$

If θ is identified with potential temperature, then this is the equation for the conservation of potential vorticity.

4 The Degenerate Legendre Transformation

It is common practice in analytical mechanics to transform the second-order Euler-Lagrange equations into a set of first-order equations of the Hamiltonian type. That is, one defines a Hamiltonian $\mathcal{H}[\mathbf{p}, \mathbf{x}]$, which is a functional of some new coordinates $\mathbf{p}$, called momenta, and the position coordinates $\mathbf{x}$. The Hamiltonian and the momenta are defined by the Legendre Transformation FL

$$FL : \ \mathcal{H}[\mathbf{p}, \mathbf{x}] \equiv \dot{\mathbf{x}}.\frac{\delta\mathcal{L}}{\delta\dot{\mathbf{x}}} - \mathcal{L} , \tag{33}$$

where

$$\mathbf{p} \equiv \frac{\delta\mathcal{L}}{\delta\dot{\mathbf{x}}} . \tag{34}$$

The functional derivative $\delta/\delta u$ is defined by

$$\delta\mathcal{F}[u] \equiv \mathcal{F}[u + \delta u] - \mathcal{F}[u] = \left\langle \frac{\delta\mathcal{F}}{\delta u} \middle| \delta u \right\rangle + \mathcal{O}(\delta u^2) ,$$

with respect to the inner product

$$\langle \mathcal{F} | \mathcal{G} \rangle \equiv \int_\Sigma \mathcal{F}\mathcal{G} \, \mathrm{d}\sigma , \quad \sigma \in \Sigma \subset \mathbb{R}^3$$

(where henceforth we only consider functions with compact support on Σ to avoid any difficulties with boundary conditions). The equations of motion can be written in terms of the $2n$ generalized coordinates $z^i \leftrightarrow (\mathbf{p}, \mathbf{x})$, where $i = 1 \ldots 2n$, as

$$\dot{z}^i = J^{ij} \frac{\delta\mathcal{H}}{\delta z^j} . \tag{35}$$

The matrix J^{ij} is known as the symplectic matrix, and if it takes the form

$$J^{ij} = \begin{pmatrix} 0^n & 1^n \\ -1^n & 0^n \end{pmatrix} , \tag{36}$$

then the system of equations is *canonical.* The symplectic matrix determines the Poisson bracket operator $\{\ ,\ \}$ via

$$\{\mathcal{F}, \, \mathcal{G}\} = \int_\Sigma \mathrm{d}\sigma \, \frac{\delta\mathcal{F}}{\delta z^i} J^{ij} \frac{\delta\mathcal{G}}{\delta z^j} . \tag{37}$$

The bracket should be skew-symmetric, bilinear and satisfy the Jacobi identity

$$\{\mathcal{E}, \{\mathcal{F}, \mathcal{G}\}\} + \{\mathcal{G}, \{\mathcal{E}, \mathcal{F}\}\} + \{\mathcal{F}, \{\mathcal{G}, \mathcal{E}\}\} = 0$$

(see Morrison 1982, or Salmon 1988a, for further details).

The significance of this procedure for the QH equations is that it fails to work. To see this we simply remind ourselves that if a coordinate transformation of the type (34) is to possess an inverse, then the Jacobian

$$\frac{\partial(\mathbf{p})}{\partial(\dot{\mathbf{x}})} = \det \left| \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \right| \neq 0 \,. \tag{38}$$

(Here, L stands for the integrand of the functional $\mathcal{L}$.) However, because the Lagrangian (26) is not quadratic in w, (38) is equal to zero.

The breakdown of the Legendre Transformation in this manner indicates the necessity to invoke the Dirac method of constraints in order to (attempt to) find canonical equations. It will transpire that equations with the form (35) can be found for the QH model, but the symplectic matrix is not canonical.

We note, somewhat as an aside, that the Legendre transformation, although degenerate in the sense outlined above, is not degenerate in the sense that it still defines a relationship between points and planes or poles and polars respectively (see Sewell, 1987). Thus, in terms of functions rather than functionals, the linearity of L (from (26)) in w, means that $H(w)$ under FL (33), is a single point. This is a geometric view of the degeneracy.

5 A Hamiltonian Formulation of the Quasi-Hydrostatic Equations

We now give a formulation of (27)-(29) (with $U = 0$), together with the thermodynamic and continuity equations

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0, \;\; \frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0 \,,$$

as a Hamiltonian system.

As we mentioned in the Introduction, our method stems from the technique used by Holm & Long for the HPEs. They considered a flow to be inviscid and incompressible, with hydrostatic balance assumed in the Boussinesq approximation. The most limiting assumptions of their model were that *a*) the fluid does not cross material (isopycnal) surfaces and *b*) the mass density remains a monotonic function of the vertical coordinate (piecewise monotonicity would lead to a multi-region model). It is the assumption of monotonicity that allows a transformation to isopycnal coordinates. We choose to use isentropic rather than isopycnal coordinates (bearing in mind IPV thinking), and therefore assume monotonicity of the θ surfaces. However, if we simply transform the vertical coordinate and then follow Holm & Long we are not able to derive a Hamiltonian version of the QH equations. An additional transformation of coordinates is required. It turns out that the use of the zonal and meridional angular momentum coordinates, in place of the u and v wind fields respectively, is the key to a noncanonical formulation.

In isentropic coordinates the θ surfaces are an independent vertical coordinate, while the height of a θ surface becomes a dependent variable. The coordinate transformation $(r, \phi, \lambda) \mapsto (\theta, \phi, \lambda)$ together with the conservation of potential temperature under adiabatic conditions leads to the following form for the material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi}\,, \tag{39}$$

where all the derivatives on the right-hand side are evaluated on surfaces of constant θ. The coordinate transformation has therefore removed the explicit vertical advection term from this derivative.

Transformation of (27) and (28) to isentropic coordinates, with the use of (39), gives

$$\begin{aligned}&\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega v\sin\phi - \frac{uv}{r}\tan\phi + 2\Omega\cos\phi\left(\frac{\partial r}{\partial t} + \frac{v}{r}\frac{\partial r}{\partial\phi}\right)\\ &\qquad + \frac{u}{r}\left(\frac{\partial r}{\partial t} + \frac{v}{r}\frac{\partial r}{\partial\phi}\right) + \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} - \frac{1}{r\cos\phi}\left(\frac{v^2}{r} + g\right)\frac{\partial r}{\partial\lambda} = 0\,,\end{aligned} \tag{40}$$

$$\begin{aligned}&\frac{\mathrm{D}v}{\mathrm{D}t} + 2\Omega u\sin\phi + \frac{u^2}{r}\tan\phi\\ &\qquad + \frac{v}{r}\left(\frac{\partial r}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial r}{\partial\lambda}\right) + \frac{1}{\rho r}\frac{\partial p}{\partial\phi} - \frac{1}{r}\left(2\Omega u\cos\phi + \frac{u^2}{r} - g\right)\frac{\partial r}{\partial\phi} = 0\,.\end{aligned} \tag{41}$$

The continuity equation becomes

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial r}{\partial\theta}\right) + \nabla.\left(\rho\mathbf{u}\frac{\partial r}{\partial\theta}\right) = 0\,, \tag{42}$$

where $\mathbf{u} = (u, v, 0)$, and

$$\nabla.\mathbf{A} \leftrightarrow \frac{1}{r^2\cos\phi}\left(\frac{\partial(rA_\lambda)}{\partial\lambda}, \frac{\partial(r\cos\phi A_\phi)}{\partial\phi}\right)\,.$$

All derivatives are evaluated on surfaces of constant θ.

The quasi-hydrostatic equation (29) has been used in (40) and (41) to replace the vertical derivative of the pressure. Thus (40), (41) and (42) are the QH equations in isentropic coordinates. The potential vorticity (PV), q, in isentropic coordinates, is given by the expression

$$q = \left(\frac{\partial r}{\partial\theta}\right)^{-1}\left[2\Omega\sin\phi - \frac{2\Omega\cos\phi}{r}\frac{\partial r}{\partial\phi} + \frac{1}{r^2\cos\phi}\left(\frac{\partial(rv)}{\partial\lambda} - \frac{\partial(r\cos\phi u)}{\partial\phi}\right)\right]\frac{1}{\rho}\,. \tag{43}$$

Before we proceed with the derivation of the Poisson bracket for the QH equations it is useful to recall the Holm & Long result for the HPEs and relate their bracket to the general bracket for an ideal fluid (Morrison & Greene 1980). Using cartesian (x, y) coordinates in the horizontal and potential temperature in the vertical, the HPEs become

$$\frac{\partial u}{\partial t} + \mathbf{u}.\nabla u - 2\Omega v \sin\phi + \frac{1}{\rho}\frac{\partial p}{\partial x} + g\frac{\partial z}{\partial x} = 0\ , \tag{44}$$

$$\frac{\partial v}{\partial t} + \mathbf{u}.\nabla v + 2\Omega u \sin\phi + \frac{1}{\rho}\frac{\partial p}{\partial y} + g\frac{\partial z}{\partial y} = 0\ , \tag{45}$$

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial \theta}\right) + \nabla.\left(\rho\mathbf{u}\frac{\partial z}{\partial \theta}\right) = 0\ , \tag{46}$$

where, again, $\mathbf{u} \leftrightarrow (u, v, 0)$. Now, the analogue of the Holm & Long result using isentropic coordinates, is that (44)-(46) have Hamiltonian form (35), where $z^i = (u, v, \rho\partial z/\partial\theta)$ and the bracket is given by

$$\{\mathcal{H},\, \mathcal{F}\} = -\int \mathrm{d}x\mathrm{d}y\mathrm{d}\theta\frac{\partial z}{\partial \theta}\left[q\left(\frac{\delta\mathcal{H}}{\delta u}\frac{\delta\mathcal{F}}{\delta v} - \frac{\delta\mathcal{H}}{\delta v}\frac{\delta\mathcal{F}}{\delta u}\right) + \frac{\delta\mathcal{F}}{\delta\mathbf{u}}.\nabla\left(\frac{\delta\mathcal{H}}{\delta\left(\rho\frac{\partial z}{\partial\theta}\right)}\right) + \frac{\delta\mathcal{F}}{\delta\left(\rho\frac{\partial z}{\partial\theta}\right)}\nabla.\left(\frac{\delta\mathcal{H}}{\delta\mathbf{u}}\right)\right]\ . \tag{47}$$

The Hamiltonian is

$$\mathcal{H}\left[\mathbf{u}, \left(\rho\frac{\partial z}{\partial\theta}\right)\right] = \int \mathrm{d}x\mathrm{d}y\mathrm{d}\theta\frac{\partial z}{\partial\theta}\rho\left[\frac{1}{2}\left(u^2+v^2\right) + C_vT + gz\right]\ , \tag{48}$$

and the PV

$$q = \left(\frac{\partial z}{\partial\theta}\right)^{-1}\left(2\Omega\sin\phi + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{1}{\rho}\ . \tag{49}$$

Note that $T \equiv \partial E/\partial S$, where S is the entropy, and C_v is the specific heat at constant volume. In (48) we do not explicitly denote the functional dependence on the internal energy because we shall not (here or elsewhere) take variations of the energy with respect to E.

The bracket (47) can be obtained from the bracket for an ideal fluid (here $\mathbf{u} \leftrightarrow (u, v, w)$)

$$\{\mathcal{F},\, \mathcal{G}\} = -\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\left[\frac{\nabla\wedge\mathbf{u}}{\rho}.\left(\frac{\delta\mathcal{F}}{\delta\mathbf{u}}\wedge\frac{\delta\mathcal{G}}{\delta\mathbf{u}}\right) + \frac{\delta\mathcal{F}}{\delta\mathbf{u}}.\nabla\left(\frac{\delta\mathcal{G}}{\delta\rho}\right) + \frac{\delta\mathcal{F}}{\delta\rho}\nabla.\left(\frac{\delta\mathcal{G}}{\delta\mathbf{u}}\right)\right]\ , \tag{50}$$

by including rotation and transforming to isentropic coordinates (see (24) in Holm & Long (1989) and Morrison (1982)). The thermodynamic equation may be included in the Hamiltonian system by augmenting the bracket with a term involving the entropy. However, because the conservation of potential temperature is implicit in our system, we choose to omit the additional term from the bracket (see Morrison (1982) for details).

We are now in a position to give a Poisson bracket formulation of the QH equations. An appropriate bracket may be obtained from (47) by making the following changes. First, transform to (λ, ϕ, θ) coordinates. The PV q, given by (49) is replaced by (43). We retain the Hamiltonian (48), but now write it in spherical polar coordinates,

$$\mathcal{H}\left[u, v, \rho\frac{\partial r}{\partial\theta}\right] = \int \mathrm{d}\lambda\,\mathrm{d}\phi\,\mathrm{d}\theta\; r^2\cos\phi\frac{\partial r}{\partial\theta}\rho\left[\frac{1}{2}\left(u^2+v^2\right) + C_vT + gr\right]\ . \tag{51}$$

Henceforth it will be convenient to define $\mathrm{d}\hat{\sigma} \equiv \mathrm{d}\lambda \mathrm{d}\phi \mathrm{d}\theta\ r^2 \cos\phi$. Note that integration should be carried out over the θ surfaces first in all the operations carried out, or implicit in our statements, hereafter. (Once the transformation to isentropic coordinates is carried out, the radial function $r(\theta)$, no longer commutes with the derivatives with respect to the angular variables. So, for instance, when integrating by parts, the integration with respect to θ is written as an integration with respect to r, and then the result of integrating over the angular variables must be expressed once again with $r(\theta)$. See also Holm & Long 1989.)

Next, introduce the expressions for the zonal and (local) meridional angular momenta ($m_\lambda,\ m_\phi$),

$$m_\lambda \equiv ru\cos\phi + \Omega r^2 \cos^2\phi\ , \tag{52}$$

$$m_\phi \equiv rv\ , \tag{53}$$

then the Hamiltonian can be expressed in terms of these coordinates as

$$\mathcal{H}\left[m_\lambda, m_\phi, \rho\frac{\partial r}{\partial \theta}\right] = \int \mathrm{d}\hat{\sigma}\,\frac{\partial r}{\partial\theta}\rho\left[\frac{1}{2}\left\{\left(\frac{m_\lambda - \Omega r^2\cos^2\phi}{r\cos\phi}\right)^2 + \left(\frac{m_\phi}{r}\right)^2\right\} + C_v T + gr\right]\ . \tag{54}$$

The functional derivatives of $\mathcal{H}$ are:

$$\frac{\delta\mathcal{H}}{\delta m_\lambda} = \rho\frac{\partial r}{\partial\theta}\frac{u}{r\cos\phi}\ ,\quad \frac{\delta\mathcal{H}}{\delta m_\phi} = \rho\frac{\partial r}{\partial\theta}\frac{v}{r}\ , \tag{55}$$

$$\frac{\delta\mathcal{H}}{\delta\left(\rho\frac{\partial r}{\partial\theta}\right)} = \frac{1}{2}\left(u^2+v^2\right) + C_p T + gr\ . \tag{56}$$

Next, transform the independent variables $(u, v, \rho\partial r/\partial\theta) \mapsto (m_\lambda, m_\phi, \rho\partial r/\partial\theta)$. Under this invertible transformation the functional derivatives become

$$\frac{\delta}{\delta u} = \frac{\delta m_\lambda}{\delta u}\frac{\delta}{\delta m_\lambda} = r\cos\phi\frac{\delta}{\delta m_\lambda}\ , \tag{57}$$

$$\frac{\delta}{\delta v} = \frac{\delta m_\phi}{\delta v}\frac{\delta}{\delta m_\phi} = r\frac{\delta}{\delta m_\phi}\ , \tag{58}$$

and using (57) and (58) in the spherical polar version of (47) leads to

$$\begin{aligned}\{\mathcal{H},\ \mathcal{F}\} = -\int_\Sigma \mathrm{d}\hat{\sigma}\frac{\partial r}{\partial\theta}&\left[qr^2\cos\phi\left(\frac{\delta\mathcal{H}}{\delta m_\lambda}\frac{\delta\mathcal{F}}{\delta m_\phi} - \frac{\delta\mathcal{H}}{\delta m_\phi}\frac{\delta\mathcal{F}}{\delta m_\lambda}\right) + \frac{\delta\mathcal{F}}{\delta m_\lambda}\frac{\partial}{\partial\lambda}\left(\frac{\delta\mathcal{H}}{\delta\left(\rho\frac{\partial r}{\partial\theta}\right)}\right)\right.\\ &\left.+\frac{\delta\mathcal{F}}{\delta m_\phi}\frac{\partial}{\partial\phi}\left(\frac{\delta\mathcal{H}}{\delta\left(\rho\frac{\partial r}{\partial\theta}\right)}\right) + \frac{\delta\mathcal{F}}{\delta\left(\rho\frac{\partial r}{\partial\theta}\right)}\frac{1}{r^2\cos\phi}\left(\frac{\partial}{\partial\lambda}\left(r^2\cos\phi\frac{\delta\mathcal{H}}{\delta m_\lambda}\right) + \frac{\partial}{\partial\phi}\left(r^2\cos\phi\frac{\delta\mathcal{H}}{\delta m_\phi}\right)\right)\right]\ .\end{aligned} \tag{59}$$

The equations of motion (40)-(42) are obtained from

$$\frac{\partial\mathcal{F}}{\partial t} = \{\mathcal{H},\ \mathcal{F}\}\ ,$$

by setting $\mathcal{F}$ equal to m_λ, m_ϕ and $\rho\partial r/\partial\theta$ respectively. This is the second result we set out to achieve.

The bracket (59) is clearly skew-symmetric (*e.g.* integrate by parts on the second and third terms) and bilinear. To see that it satisfies the Jacobi identity we make a further, invertible, coordinate transformation. After this transformation has been performed, we shall find that the resulting expression for the bracket is linear in the Eulerian variables, and there are no terms, such as ρ (*c.f.* (43)), in the denominator. We shall then refer to a result due to Morrison (1982), that shows that with such a form for an operator, the Jacobi property may be established by algebraic methods. (The significance of this procedure becomes clear when understood in terms of the underlying Lie algebra structure of the bracket. The form of the bracket that we shall obtain after the coordinate transformation can be identified with the Lie-Poisson bracket for the semi-direct product of Lie groups with functions. See Marsden, Ratiu & Weinstein (1984), for further technical details.)

Before introducing the coordinate transformation, we note that the PV (43), can be written succinctly in terms of the angular momentum variables m_λ and m_ϕ:

$$q = \frac{1}{r^2 \cos\phi}\left(\frac{\partial m_\phi}{\partial \lambda} - \frac{\partial m_\lambda}{\partial \phi}\right)\frac{1}{\rho}\left(\frac{\partial r}{\partial \theta}\right)^{-1} . \tag{60}$$

Now define the new variables $\mathbf{M} \equiv (M_\lambda,\ M_\phi)$

$$M_\lambda \equiv \rho\frac{\partial r}{\partial \theta} m_\lambda\ ,\quad M_\phi \equiv \rho\frac{\partial r}{\partial \theta} m_\phi\ , \tag{61}$$

whence, the functional derivatives (in (59)) transform as follows:

$$\frac{\delta}{\delta \mathbf{m}} = \rho\frac{\partial r}{\partial \theta}\frac{\delta}{\delta \mathbf{M}}\ ,\quad \left.\frac{\delta}{\delta\left(\rho\frac{\partial r}{\partial \theta}\right)}\right|_{\mathbf{m}} = \left.\frac{\delta}{\delta\left(\rho\frac{\partial r}{\partial \theta}\right)}\right|_{\mathbf{M}} + \mathbf{m}.\frac{\delta}{\delta \mathbf{M}}\ . \tag{62}$$

If we now re-write (59) as

$$\begin{aligned}\{\mathcal{H},\ \mathcal{F}\} = -\int_\Sigma \mathrm{d}\hat{\sigma}\frac{\partial r}{\partial \theta}\Bigg[& qr^2\cos\phi\left(\frac{\delta \mathcal{H}}{\delta m_\lambda}\frac{\delta \mathcal{F}}{\delta m_\phi} - \frac{\delta \mathcal{H}}{\delta m_\phi}\frac{\delta \mathcal{F}}{\delta m_\lambda}\right) + \frac{\delta \mathcal{F}}{\delta m_\lambda}\frac{\partial}{\partial \lambda}\left(\frac{\delta \mathcal{H}}{\delta\left(\rho\frac{\partial r}{\partial \theta}\right)}\right) \\ & + \frac{\delta \mathcal{F}}{\delta m_\phi}\frac{\partial}{\partial \phi}\left(\frac{\delta \mathcal{H}}{\delta\left(\rho\frac{\partial r}{\partial \theta}\right)}\right) - \frac{\delta \mathcal{H}}{\delta m_\lambda}\frac{\partial}{\partial \lambda}\left(\frac{\delta \mathcal{F}}{\delta\left(\rho\frac{\partial r}{\partial \theta}\right)}\right) - \frac{\delta \mathcal{H}}{\delta m_\phi}\frac{\partial}{\partial \phi}\left(\frac{\delta \mathcal{F}}{\delta\left(\rho\frac{\partial r}{\partial \theta}\right)}\right)\Bigg]\ ,\end{aligned} \tag{63}$$

where q takes the form given in (60), and then transform to the $\mathbf{M}$ coordinates, using (62), we obtain the following Lie-Poisson bracket:

$$\begin{aligned}\{\mathcal{H},\ \mathcal{F}\} = -\int_\Sigma \mathrm{d}\hat{\sigma}\frac{\partial r}{\partial \theta}\Bigg[& \rho\frac{\partial r}{\partial \theta}\left(\frac{\delta \mathcal{F}}{\delta \mathbf{M}}.\nabla\left(\frac{\delta \mathcal{H}}{\delta\left(\rho\frac{\partial r}{\partial \theta}\right)}\right)\right. \\ & \left. - \frac{\delta \mathcal{H}}{\delta \mathbf{M}}.\nabla\left(\frac{\delta \mathcal{F}}{\delta\left(\rho\frac{\partial r}{\partial \theta}\right)}\right)\right) + M^i\frac{\delta \mathcal{F}}{\delta \mathbf{M}}.\nabla\left(\frac{\delta \mathcal{H}}{\delta M^i}\right) - M^i\frac{\delta \mathcal{H}}{\delta \mathbf{M}}.\nabla\left(\frac{\delta \mathcal{F}}{\delta M^i}\right)\Bigg]\ .\end{aligned} \tag{64}$$

Here, the index i, is summed over the values 1 and 2, and the derivative operator $\nabla = (\partial/\partial\lambda,\ \partial/\partial\phi)$.

The proof of the Jacobi identity for the bracket (64) follows a standard procedure (Morrison 1982). Consider the constituent terms in the expression. The first two terms are linear in $\rho\partial r/\partial\theta$.

Together with the third and fourth terms, which are linear in $\mathbf{M}$, they can be written in the form (*c.f.* (37))

$$J^{ij} = \sum_{k,r=1}^{3} \left[\psi_r^{ijk} z_k \nabla_r + \psi_r^{jik} \nabla_r z_k \right] , \tag{65}$$

where $\psi_r^{ijk} \in \mathbb{R}$, $\forall\ i,\ j,\ k,\ r$ (which take the values 1, 2 and 3), and $z_i = (\rho\partial r/\partial\theta,\ M_\lambda,\ M_\phi)$. Then one proves that an operator of the form (65) satisfies the Jacobi identity if

$$\sum_{k=1}^{3} \left(\psi_r^{lkm} \psi_t^{ijk} - \psi_r^{ikm} \psi_t^{ljk} \right) = 0 ,$$

and

$$\sum_{k=1}^{3} \left(\psi_r^{lkm} \psi_t^{ijk} + \psi_r^{kim} \psi_t^{ljk} - \psi_t^{lkm} \psi_r^{jik} - \psi_t^{kjm} \psi_r^{lik} \right) = 0 ,$$

are satisfied $\forall\ r,\ t,\ l,\ m,\ i$ and j. These can be checked explicitly for the bracket (64), and thus the Jacobi condition is satisfied. In fact, the proof outlined here, is entirely analogous to the proof of the Jacobi identity for the bracket (50) for an irrotational, ideal fluid. (This is because (64) is isomorphic to an expression for (50) when the latter is written in the form (65).)

6 Summary

We have obtained the QH equations from Hamilton's Principle, and from a noncanonical Poisson bracket written in terms of the quasi-hydrostatic potential vorticity and the zonal and meridional angular momentum coordinates. The Hamiltonian governing the evolution of the QH model is precisely the same as for the HPEs. This Hamiltonian system is a natural embodiment of the conservative properties of the QH equations as derived by WB.

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