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VARIATIONAL ASPECTS OF SEMI-GEOSTROPHIC THEORY

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ABSTRACT

Starting with a phase-space duality principle for finite element solutions of the semi-geostrophic equations a Hamiltonian representation of these equations is derived . It is shown that a natural extension of these equations from the f-plane to a variable f and non-Euclidean horizontal domain, such as the spherical surface, leads to the finite element form of Salmon's (1983, 1985) equations, which preserve an analogue to the law of conservartion of potential vorticy in their continuous form.

The duality principle may be exploited by way of Legendre transformations to give reciprocity relations of some dynamical significance in both finite element and continuous solutions.

1. INTRODUCTION

This note explores certain properties of the semi-geostrophic (SG) equations of Hoskins and Bretherton (1972) in their finite element form which can be deduced from the underlying variational principles . The use of a finite element approach to study the solutions of these equations in the context of frontogenesis began with Cullen (1983) and the theory was developed further by Cullen and Purser (1984), henceforth abbreviated to CP. More recently Chynoweth (1987) has developed numerical code to study both static and evolutionary aspects of the finite element solutions, mainly in two dimensions, and the code has been extended by Shutts et al. (1988) for application to axisymmetric flows in which angular momentum is the conserved quantity instead of geostrophic momentum. For conventional f-plane SG theory it was recognised by CP that balanced (i.e., symmetrically stable) solutions are associated with a convex function of space. This solution may also be obtained at any instant by minimising the total (kinetic plus potential) energy with respect to fluid parcel rearrangements that conserve mass, potential temperature and geostrophic/angular momentum.

In independent investigations of reduced forms of almost geostrophic flows derived from Hamiltonian theory Salmon (1983, 1985) discovered a Hamiltonian formulation of SG theory enabling it to be generalised in a way that retains the conservation laws of energy, mass and potential vorticity to domains over which the Coriolis parameter and basic state density vary widely. In developing the variational method as it relates to the finite element formulation of SG dynamics we clarify the geometrical interpretation of Salmon's system in the case where the fluid is partitioned into finite element cells within each of which the two "dual space" (Purser and Cullen, 1987) coordinates that generalise geostrophic momentum are constant, as is the potential temperature which forms the third coordinate dual to the altitude z. We also discuss how the finite element form of Salmon's system generalises naturally to a curved horizontal domain such as the surface of a sphere. It is argued here that, from the geometrical modelling perspective, Salmon's generalisation is the most natural framework for extension of the adaptive finite element codes envisaged by Purser (1988) to more general domains. It follows from the

Hamiltonian expression of the finite element dynamics that a Liouville theorem pertains to these systems, that is, there is a dynamical phase space within which the density of a hypothetical ensemble of states is constrained to be conserved.

In addition to the Hamiltonian aspects of the dynamics it is shown as a bi-product of the variational structure that the sensitivity of centroid motion at one cell to a mass source and sink couplet at two other cells is connected by way of a reciprocity relation to the sensitivity of the geopotential difference at the two latter points to thermal and momentum changes at the first cell. This reciprocity is naturally evident in the continuous solutions also where it is associated with the self-adjoint differential operator that figures in the SG tendency equation (Schubert, 1985).

2. BASIC SEMI-GEOSTROPHIC DYNAMICS

The reader is referred to CP for an outline of the finite element structure of SG solutions and the geometric interpretation of solutions in terms of convex functions. The notation adopted here is that of Purser and Cullen (1987), which we henceforth abbreviate to PC. Chynoweth (1987) provides a discussion of geometrical algorithms that have been implemented and Purser (1988) discusses a potentially fast method of adapting geometrical solutions to small perturbations. We shall begin with a discussion of the formal geometrical assumptions and consequent properties for the SG Boussinesq dynamics on the f-plane.

Define,

$$\underset{\sim}{x} \equiv (x, y, z), \qquad (2.1)$$

to be the spatial coordinate. The SG solution may be expressed in terms of the geopotential, $\phi(x)$ or, as in CP, by the modified potential,

$$P = \frac{1}{f^2}\phi(\underset{\sim}{x}) + \frac{1}{2}(x^2+y^2), \qquad (2.2)$$

in which case the geostrophic momentum coordinates (X, Y), together with the scaled potential temperature (Z) form the "dual" coordinates:

$$\underset{\sim}{X} = (X, Y, Z)$$

$$= \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z} \right) = \left(x + \frac{v_g}{f}, \; y - \frac{u_g}{f}, \; \frac{g\theta}{f^2 \theta_0} \right) , \qquad (2.3)$$

where the geostrophic wind vector $\underset{\sim}{u}_g$ has components

$$\underset{\sim}{u}_g = (u_g, v_g, 0) = \left(-\frac{1}{f}\frac{\partial \phi}{\partial y}, \frac{1}{f}\frac{\partial \phi}{\partial x}, 0 \right) . \qquad (2.4)$$

In the continuous SG equations (Hoskins, 1975; Hoskins and Draghici, 1977) the dual coordinates evolve according to

$$\frac{D\underset{\sim}{X}}{Dt} = \underset{\sim}{u}_g , \qquad (2.5)$$

which, together with the continuity equation,

$$\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} = 0 \qquad (2.6)$$

where $\underset{\sim}{u}$ is the ordinary velocity:

$$\underset{\sim}{u} = \frac{Dx}{Dt} , \qquad (2.7)$$

imply the Lagrangian conservation of a SG potential vorticity q, where q is defined:

$$q = \det\left[\frac{\partial \underset{\sim}{X}}{\partial \underset{\sim}{x}} \right] \equiv \det\left[\frac{\partial^2 P}{\partial x_i \partial x_j} \right] . \qquad (2.8)$$

It can be shown that these equations are also consistent in their treatment of energetics provided the kinetic energy density is obtained using only the geostrophic wind components:

$$KE = \frac{1}{2} \underset{\sim}{u}_g \cdot \underset{\sim}{u}_g . \qquad (2.9)$$

The potential energy density can be written in Boussinesq form:

$$PE = -f^2 Z_z \,. \tag{2.10}$$

As shown by Hoskins (1975), the energy conversions expressed by

$$\frac{D}{Dt}(KE + PE) = -\underset{\sim}{\nabla}(\underset{\sim}{u}\,\phi) \;, \tag{2.11}$$

imply a global conservation of energy for such inviscid, adiabatic flow within a fixed bounded domain.

3. FINITE ELEMENT EQUATIONS AND HAMILTONIAN DYNAMICS ON THE f-PLANE

In the f-plane finite element form of these equations it is necessary to adopt the "centroid convention" in place of (2.5) in order to preserve energetic consistency. Essentially this consists of prescribing the change of $\underset{\sim}{X}$ for each element as a whole in terms of the geostrophic wind $\underset{\sim}{u}_g$ at that element's centroid $\overline{\underset{\sim}{x}}$, the latter defined as

$$\overline{\underset{\sim}{x}} = \frac{\int x \, d\mu}{\int d\mu} \equiv \frac{\underset{\sim}{M}^x}{M} \;, \tag{3.1}$$

where $d\mu$ = dx. dy. dz. Note that $\overline{\underset{\sim}{u}_g} = \underset{\sim}{u}_g(\overline{\underset{\sim}{x}})$ by the linearity of $\underset{\sim}{u}_g$ across an element of constant $\underset{\sim}{X}$. Each finite element α is associated with a hyperplane in the extended $(P, \underset{\sim}{x})$-space defined by the linear equation:

$$P_\alpha(\underset{\sim}{x}) = s_\alpha + \underset{\sim}{X}_\alpha \cdot \underset{\sim}{x} \;, \tag{3.2}$$

where s_α are the intercepts of the hyperplane at the x-origin and, as discussed by PC, are the negatives of the corresponding dual potential, i.e., $s_\alpha = -R_\alpha$. Hence the moments M, $\underset{\sim}{M}^x$, can be expressed as partial derivatives of the (convex) function

$$K(\{R_\alpha, \underset{\sim}{X}_\alpha\}) \equiv K(\underset{\sim}{x}) = \int_D P(\underset{\sim}{x}) \, d\mu \;, \tag{3.3}$$

with respect to the 4N individual components

$$\{R_\alpha, X_\alpha, Y_\alpha, Z_\alpha\}$$

for a "configuration" $\underset{\sim}{\chi}$ involving N finite elements within the domain D:

$$\left(M_\alpha, M_\alpha^x, M_\alpha^y, M_\alpha^z\right) = \left(-\frac{\partial K}{\partial R_\alpha}, \frac{\partial K}{\partial X_\alpha}, \frac{\partial K}{\partial Y_\alpha}, \frac{\partial K}{\partial Z_\alpha}\right) . \tag{3.4}$$

Since in adiabatic solutions the mass of each element is conserved the momentum equations may be written for finite elements in the form:

$$\left.\begin{aligned} M_\alpha \frac{dX_\alpha}{dt} &= f\left(M_\alpha^y - M_\alpha Y_\alpha\right) , \\ M_\alpha \frac{dY_\alpha}{dt} &= f\left(-M_\alpha^x + M_\alpha X_\alpha\right) . \end{aligned}\right\} \tag{3.5}$$

In order to get a scheme more closely resembling canonical Hamiltonian form we apply a Legendre transformation to obtain from K a new potential, L:

$$L\left(\{M_\alpha, \underset{\sim}{X}_\alpha\}\right) = K\left(\{R_\alpha, X_\alpha\}\right) + \sum_{\alpha=1}^{N} M_\alpha R_\alpha = \int_D \underset{\sim}{X} \cdot \underset{\sim}{x} \, d\mu , \tag{3.6}$$

from which it is evident that

$$\begin{aligned} dL &= \sum_{\alpha=1}^{N} \left(-M_\alpha dR_\alpha + \underset{\sim}{M}_\alpha^x \cdot d\underset{\sim}{X}_\alpha + M_\alpha dR_\alpha + R_\alpha dM_\alpha\right), \\ &= \sum_{\alpha=1}^{N} \left(\underset{\sim}{M}_\alpha^x \cdot d\underset{\sim}{X}_\alpha + R_\alpha dM_\alpha\right) \end{aligned} \tag{3.7}$$

Thus

$$\left.\begin{aligned} R_\alpha &= \frac{\partial L}{\partial M_\alpha} \\ \underset{\sim}{M}_\alpha^x &= \left(\frac{\partial L}{\partial X_\alpha}, \frac{\partial L}{\partial Y_\alpha}, \frac{\partial L}{\partial Z_\alpha}\right)^T \end{aligned}\right\} \tag{3.8}$$

so that, in adiabatic mass-conserving motion, the horizontal momentum equations may be written:

$$\begin{aligned} M_\alpha \frac{dX_\alpha}{dt} &= f\left(\frac{\partial L}{\partial Y_\alpha} - M_\alpha Y_\alpha\right) , \\ M_\alpha \frac{dY_\alpha}{dt} &= f\left(-\frac{\partial L}{\partial X_\alpha} + M_\alpha X_\alpha\right) , \end{aligned} \tag{3.9}$$

which now requires the trivial transformation

$$\left.\begin{aligned} X'_\alpha &= f^{\frac{1}{2}} M_\alpha^{\frac{1}{2}} X_\alpha \quad , \\ Y'_\alpha &= f^{\frac{1}{2}} M_\alpha^{\frac{1}{2}} Y_\alpha \quad , \end{aligned}\right\} \qquad (3.10)$$

to obtain the canonical Hamiltonian equations:

$$\left.\begin{aligned} \frac{dX'_\alpha}{dt} &= -\frac{\partial H}{\partial Y'_\alpha} \quad , \\ \frac{dY'_\alpha}{dt} &= \frac{\partial H}{\partial X'_\alpha} \quad . \end{aligned}\right\} \qquad (3.11)$$

Hamiltonian H is defined:

$$H = f^2\left[-L + \frac{1}{2}\sum_{\alpha=1}^{N} M_\alpha \left(X_\alpha^2 + Y_\alpha^2\right)\right] + \frac{f^2}{2}\int x^2 + y^2 \, d\mu \qquad (3.12)$$

or, in physical terms

$$H = \int KE + PE \; d\mu , \qquad (3.13)$$

as in Salmon's "L_∞-dynamics" (Salmon, 1985). Adapting his action integral for finite element solutions, we note that (3.11) are obtained by minimimising,

$$\int_{t_1}^{t_2} \mathcal{L} \; dt \qquad (3.14)$$

subject to constant M_α and Z_α, where,

$$\mathcal{L} = \sum_{\alpha=1}^{N} M_\alpha \, \underset{\sim}{A}(\underset{\sim}{X}_\alpha) \cdot \frac{d\underset{\sim}{X}_\alpha}{dt} - \int_D H \, d\mu \qquad (3.15)$$

and $\underset{\sim}{A}$ is any constant vector field in $\underset{\sim}{X}$-space satisfying

$$\left.\begin{aligned} \frac{\partial A_y}{\partial X} - \frac{\partial A_x}{\partial Y} &= f \quad , \\ A_z &= 0 \quad . \end{aligned}\right\} \qquad (3.16)$$

Liouville's theorem follows immediately from (3.13):

$$\sum_{\alpha=1}^{N} \frac{\partial}{\partial X'_\alpha} \frac{dX'_\alpha}{dt} + \frac{\partial}{\partial Y'_\alpha} \frac{dY'_\alpha}{dt} , \qquad (3.17)$$

as does conservation of energy:

$$\frac{dH}{dt} = -\sum_{\alpha=1}^{N} \frac{dX'_\alpha}{dt}\frac{\partial H}{\partial X'_\alpha} + \frac{dY'_\alpha}{dt}\frac{\partial H}{\partial Y'_\alpha} = 0 . \tag{3.18}$$

Although we have demonstrated conservation of energy directly from the Hamiltonian (i.e., by a global method) it is useful to see how energy consistency comes about for an individual element, since the pattern of cancellations that occurs in the case of an f-plane suggests the forms that most naturally generalise the finite element model in more general domains. We are at liberty to choose our coordinate origin in $\underset{\sim}{X}$-space to be the point that makes $\underset{\sim}{X}_\alpha = 0$ at time $t = 0$ for the element α of interest. Let the element occupy the region D_α and have boundary ∂D_α. Defining the total kinetic energy for the element to be E_k, then

$$E_k = \int_{D_\alpha} \frac{f^2}{2}\left[(X-x)^2 + (Y-y)^2\right] d\mu \equiv \int_{D_\alpha} \frac{f^2}{2}(x^2+y^2)\, d\mu . \tag{3.19}$$

With this choice of origin the potential energy,

$$E_p = \int_{D_\alpha} -f^2 Zz\, d\mu = 0 , \tag{3.20}$$

and remains constant for all time. Now consider the rate of change of E_k:

$$\frac{dE_k}{dt} = \int_{D_\alpha} f^2\left[\frac{dX}{dt}.(-x) + \frac{dY}{dt}.(-y)\right] d\mu + \int_{\partial D_\alpha} \frac{f^2}{2}(x^2+y^2)\, \underset{\sim}{u}\cdot d\underset{\sim}{\sigma}_\alpha , \tag{3.21}$$

where $d\underset{\sim}{\sigma}_\alpha$ is the vector outward normal infinitesimal surface element on ∂D_α. (Note that although internal velocities of a finite element of constant $\underset{\sim}{X}$ are not defined the component of movement of the element boundary in a direction normal to that boundary is a well defined quantity). As previously noted, $dE_p/dt = 0$ since $dZ/dt = 0$ but an additional contribution to the energy budget is the "pressure work" term. The rate at which element α does work on its surroundings is

$$\frac{dE_w}{dt} = \int_{\partial D_\alpha} \phi\, \underset{\sim}{u}\cdot d\underset{\sim}{\sigma}_\alpha = \int_{\partial D_\alpha} \left[\Phi_\alpha - \frac{f^2}{2}(x^2+y^2)\right] \underset{\sim}{u}\cdot d\underset{\sim}{\sigma}_\alpha . \tag{3.22}$$

where the constant Φ_α is the value of ϕ attained at $\underset{\sim}{x} = 0$. Integrating continuity equation (2.6) and applying Gauss' theorem:

$$\int_{\partial D_\alpha} \underset{\sim}{u} \cdot d\underset{\sim}{\sigma_\alpha} = 0 \quad . \tag{3.23}$$

Hence

$$\frac{dE_k}{dt} + \frac{dE_w}{dt} = \int_{D_\alpha} f^2 \left[\frac{dX}{dt} . (-x) + \frac{dY}{dt}(-y) \right] d\mu \quad . \tag{3.24}$$

But

$$\frac{dX}{dt} = \overline{u}_g = f \int_{D_\alpha} y \, d\mu \quad ,$$

$$\frac{dY}{dt} = \overline{v}_g = -f \int_{D_\alpha} x \, d\mu \quad , \tag{3.25}$$

so that

$$\frac{dE_k}{dt} + \frac{dE_w}{dt} = 0 \quad . \tag{3.26}$$

Thus, we have shown for an individual element that the rate of change of kinetic energy is compensated by the integrated pressure-work when the centroid convention is obeyed. In the next section we generalise the relations above to the case of variable Coriolis parameter and curved domains.

4 FINITE ELEMENT SEMI-GEOSTROPHIC SOLUTIONS IN CURVED DOMAINS WITH VARIABLE CORIOLIS PARAMETER.

The solution surface $\phi(\underset{\sim}{x})$ for f-plane SG theory can be thought of geometrically as the envelope of a set of intersecting similar hypersurfaces, each of the form

$$\phi_\alpha(\underset{\sim}{x}) = \Phi_\alpha - \frac{f^2}{2}\left[(X_\alpha - x)^2 + (Y_\alpha - y)^2 \right] + f^2 Z_\alpha z , \tag{4.1}$$

with each Φ_α a constant. Considering "sections" at any constant z, the solution forms the supremum of a set of paraboloids in (x, y, ϕ)-space of similar shape and with axes parallel the the ϕ-axis. It is this structure and the linearity in the z-direction that allows a simple transformation of the solutions to convex functions, as shown in CP. Since the curvature of paraboloidal sections depends on the Coriolis parameter f it would appear that to generalise this construction to a domain where f varies requires either that a form of surface more general than quadratic must be chosen or that each quadratic surface derives its curvature from a favoured value of the Coriolis parameter, such as that pertaining to the position where ϕ_α attains its maximum on a horizontal section i.e., $f_\alpha = f(\underset{\sim}{X}_\alpha)$. The latter alternative is clearly the simpler although it no longer leads to a solution that we can naturally identify with a convex function. Simplifying the scaled potential temperature to

$$Z = \frac{g\theta}{\theta_0} \quad ,$$

we modify the form of the surfaces (4.1) to

$$\phi_\alpha(\underset{\sim}{x}) = \Phi_\alpha - \frac{f^2(\underset{\sim}{X})}{2} \, |X_\alpha - \underset{\sim}{x}|_h^2 + Z_\alpha z \quad , \tag{4.2}$$

where subscript h denotes the horizontal part of a vector. Note that this formula is also valid for domains where horizontal surfaces are intrinsically curved (e.g., spherical) by generalising the horizontal vector $(\underset{\sim}{X}-\underset{\sim}{x})_h$ to lie on, and be parallel to, the geodesic arc from $\underset{\sim}{x}_h$ to $\underset{\sim}{X}_h$ with a magnitude equal to this horizontal distance. Henceforth in this section it will be implicitly assumed that the horizontal components of $\underset{\sim}{x}$ and $\underset{\sim}{X}$ are postion vectors rather than Cartesian coordinates to allow for the possibility of a curved domain. Likewise, we shall modify the definition of $d\mu$ so that it still refers to a measure of infinitesimal physical volume, not coordinate volume.

To guarantee energetic consistency for a moving finite element we note that defining the kinetic energy density,

$$KE = \frac{1}{2}\underset{\sim}{U}_g^2 = \frac{1}{2}\frac{1}{f^2(\underset{\sim}{X})}\left|\underset{\sim}{\nabla}\phi\right|_h^2 , \tag{4.3}$$

will ensure that, since,

$$\underset{\sim}{\nabla}\phi_h = f^2(\underset{\sim}{X}).(\underset{\sim}{X} - \underset{\sim}{x})_h , \tag{4.4}$$

the boundary integral terms generalising those of the change of kinetic energy (3.16) and pressure-work (3.18) will continue to cancel, both in the case of varying f and in a curved domain. It remains to determine the form of the evolution of $\underset{\sim}{X}$ that causes the body-integral generalising that of (3.16) to vanish. Because f now varies with $\underset{\sim}{X}$ this integral is a little more complicated. Writing,

$$\underset{\sim}{\nabla}_X \equiv \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right),$$

then

$$\frac{dE_k}{dt} = \frac{d\underset{\sim}{X}_\alpha}{dt} \cdot \int_{D_\alpha} f^2(\underset{\sim}{X}_\alpha).(\underset{\sim}{X}_\alpha - \underset{\sim}{x})_h + \left(\underset{\sim}{\nabla}_X f^2\right)\Big|_{X = X_\alpha} . \frac{\left|\underset{\sim}{X}_\alpha - \underset{\sim}{x}\right|_h^2}{2}\, d\mu - \frac{dE_w}{dt} , \tag{4.5}$$

where the first vector in the integrand of (4.5) is parallel-transported along the geodesic from $\underset{\sim}{x}$ to dual location $\underset{\sim}{X}$. Clearly the most natural generalisation of the evolution equation for $\underset{\sim}{X}$ that will cause the integral of (4.5) to vanish is,

$$M_\alpha \frac{d\underset{\sim}{X}_\alpha}{dt} = \frac{-\underset{\sim}{k}\times}{f(\underset{\sim}{X})}\int_{D_\alpha} f^2(\underset{\sim}{X}_\alpha).(\underset{\sim}{X}_\alpha - \underset{\sim}{x})_h + \left(\underset{\sim}{\nabla}_X f^2\right)\frac{\left|\underset{\sim}{X}_\alpha - \underset{\sim}{x}\right|_h^2}{2}\, d\mu , \tag{4.6}$$

$$= \int_{D_\alpha} \underset{\sim}{U}_g - \frac{(U_g^2 + V_g^2)}{f^2(\underset{\sim}{X}_\alpha)} . \underset{\sim}{k}\times\underset{\sim}{\nabla}_X f \, d\mu . \tag{4.7}$$

Differentiating (4.2) with respect to a locally Cartesian coordinate component X_i and using

$$\frac{\partial\phi}{\partial X} \equiv \sum_i \frac{\partial\phi}{\partial x_i}\frac{\partial x_i}{\partial X} = (X - x)\frac{\partial x}{\partial X} + (Y - y)\frac{\partial y}{\partial X} + Z\frac{\partial z}{\partial X} , \tag{4.8}$$

we find in the continuous limit of differentiable functions Φ that

$$\frac{\partial \Phi}{\partial X} = f^2(\underset{\sim}{X})(X - x) + \frac{|\underset{\sim}{X} - \underset{\sim}{x}_h|^2}{2} \frac{\partial f^2}{\partial X} \ , \tag{4.9}$$

with a similar expression for the partial derivative with respect to Y. Thus, in the continuous solutions,

$$\frac{d\underset{\sim}{X}}{dt} = \underset{\sim}{u}_g - \frac{(u_g^2 + v_g^2)}{f^2(\underset{\sim}{X})} \underset{\sim}{k} \times \underset{\sim}{\nabla}_X f \ ,$$

$$= \frac{1}{f(\underset{\sim}{X})} \hat{k} \times \underset{\sim}{\nabla}_X \Phi \ , \tag{4.10}$$

which is clearly identical to the L_∞-dynamics of Salmon (1985). It is readily verified that the form of dual-space velocity field,

$$\frac{d\hat{X}}{dt} = \frac{1}{f(\hat{X})} \hat{k} \times \underset{\sim}{\nabla}_X \Psi \tag{4.11}$$

for any scalar Ψ, together with continuity (2.6), guarantees the Lagrangian conservation of a potential vorticity q of the form

$$q = f(\underset{\sim}{X}) \frac{d\nu}{d\mu} \tag{4.12}$$

where $d\nu$ is a differential volume element of $\underset{\sim}{X}$-space (i.e., $d\nu/d\mu$ is a generalised Jacobian).

We have shown in this section that the finite element construction of SG theory can be extended naturally to domains with curved horizontals and with a spatial variations of Coriolis parameter f. The generalisation preserves the laws of conservation of mass and energy, but the geometric construction of a convex function surface $P(\underset{\sim}{x})$ that was possible in the f-plane case (CP) is no longer possible. Instead, finite element solutions ϕ are constructed from the envelope of intersecting hypersurfaces in $(\underset{\sim}{x}, \phi)$-space that have sections in $(\underset{\sim}{x}_H, \phi)$-space in the form of paraboloids. The finite element system discussed here may be thought of as the natural form taken by Salmon's (1985) model when using discrete data.

5. RECIPROCITY RELATIONS IN f-PLANE SEMI-GEOSTROPHIC THEORY.

Since in this section we shall be dealing exclusively with f-plane dynamics we shall implicitly assume units of time t that make f = 1. As shown in section 3 for the f-plane solutions, the existence of a function L in (4N-1)-dimensional space of configurations defined by 3N components $\underset{\sim}{X}_\alpha$ and N-1 independent components M_α allows one to express the vector of moments M_α of an element α by the vector of derivatives,

$$\left(\underset{\sim}{M}^{\ast}_\alpha\right)^T = \left(\frac{\partial L}{\partial X_\alpha}, \frac{\partial L}{\partial Y_\alpha}, \frac{\partial L}{\partial Z_\alpha}\right) \equiv \frac{\partial L}{\partial X_\alpha}, \tag{5.1}$$

and potential R_α by the derivative

$$R_\alpha = \frac{\partial L}{\partial M_\alpha} \qquad \alpha \neq \gamma, \tag{5.2}$$

where we assume that one element γ has fixed R and changes its mass in the sense required to conserve the total,

$$\sum_{\alpha=1}^{N} M_\alpha = \int_D d\mu \qquad = \text{constant}.$$

By taking a further partial derivative with respect to independent components $\underset{\sim}{X}^{\ast}_\beta$ or M_β we can identify certain reciprocity relations that must hold:

$$\left(\frac{\partial \underset{\sim}{M}^{\ast}_\alpha}{\partial \underset{\sim}{X}_\beta}\right)^T = \frac{\partial \underset{\sim}{M}^{\ast}_\beta}{\partial \underset{\sim}{X}_\alpha}, \tag{5.3a}$$

$$\left(\frac{\partial \underset{\sim}{M}^{\ast}_\alpha}{\partial M_\beta}\right)^T = \frac{\partial R_\beta}{\partial \underset{\sim}{X}_\alpha}, \qquad \beta \neq \gamma \tag{5.4a}$$

$$\frac{\partial R_\alpha}{\partial M_\beta} = \frac{\partial R_\beta}{\partial M_\alpha}, \qquad \alpha \neq \gamma, \quad \beta \neq \gamma. \tag{5.5}$$

To give one particular example of the physical meaning of these relations, suppose we change potential temperature Z_α of element α by heating it an infinitesimal amount δZ_α. Then the concomitant change in the geopotential difference between fixed points $\underset{\sim}{x}_\beta$ and $\underset{\sim}{x}_\gamma$ in elements β and γ is proportional to δZ_α:

$$\delta P_\beta - \delta P_\gamma = -(\delta R_\beta - \delta R_\gamma) = -\frac{\partial R_\beta}{\partial Z_\alpha}\delta Z_\alpha , \tag{5.6}$$

but the same coefficient of proportionality (apart from a change in sign) holds for the sensitivity of z-moment M^z_α of element α to the injection of mass into element β:

$$\frac{\partial R_\beta}{\partial Z_\alpha} = \frac{\partial M^z_\alpha}{\partial M_\beta}$$

Note that the vector moments $\underset{\sim}{M}^x_\alpha$ can each be expressed as a product:

$$\underset{\sim}{M}^x_\alpha \equiv M_\alpha \overline{\underset{\sim}{x}}_\alpha \tag{5.7}$$

where $\overline{\underset{\sim}{x}}_\alpha$ represents the centroid position of α. Thus (5.3a), (5.34a) are equivalently expressed:

$$\left(\frac{1}{M_\beta}\frac{\partial \overline{\underset{\sim}{x}}_\alpha}{\partial \underset{\sim}{X}_\beta}\right)^T = \frac{1}{M_\alpha}\frac{\partial \overline{\underset{\sim}{x}}_\beta}{\partial \underset{\sim}{X}_\alpha} , \tag{5.3b}$$

$$\frac{\partial \overline{\underset{\sim}{x}}_\alpha}{\partial M_\beta} = \frac{1}{M_\alpha}\frac{\partial R_\beta}{\partial X_\alpha} , \tag{5.4b}$$

in which form they more directly generalise to the continuous reciprocity relations discussed below.

The example of a mass source(β) and sink (γ) couplet might represent a simple model of penetrative convection (Shutts, 1987). Then ascent or subsidence at α induced by a unit rate of transfer of mass in such a process is in identical proportion to what the sensitivity of geopotential difference $\phi_\beta - \phi_\gamma$ would be under the influence of a unit rate of heating of the parcel at α. The other reciprocity relations implied by (5.3)-(5.5) involve various combinations of geostrophic momentum and horizontal moments as well as potential temperature, vertical position, mass transfers and geopotential differences.

Clearly we should expect to see continuous solution analogues of the above relationships in the limit of vanishing small elements. Let P be a given solution in D associated with a data distribution $\rho(\underset{\sim}{X})$:

$$\rho(\underset{\sim}{X}(\underset{\sim}{x})) = q^{-1} \equiv \left[\det Q_{ij} \right]^{-1} , \tag{5.8}$$

where

$$Q_{ij} = \frac{\partial^2 P}{\partial x_i \partial x_j} \equiv \frac{\partial X_i}{\partial x_j} . \tag{5.9}$$

Suppose that, gauged according to some convenient parameter τ, a variational adjustment of this solution is carried out by a combination of $\underset{\sim}{X}$-space rearrangements, expressed by a vector field,

$$\frac{d\underset{\sim}{X}}{d\tau} = \underset{\sim}{V} ,$$

and by mass sources and compensating sinks, expressed by the scalar field $S(\underset{\sim}{x})$ which is equated to the divergence of the resulting movement $\underset{\sim}{u}(\underset{\sim}{x})$ in physical space:

$$S(\underset{\sim}{x}) = \underset{\sim}{\nabla} \cdot \underset{\sim}{u} , \tag{5.10}$$

$$\underset{\sim}{u} = \frac{d\underset{\sim}{x}}{d\tau} . \tag{5.11}$$

Note that τ is not time, and $\underset{\sim}{u}$ is therefore strictly not a velocity, although its effect resembles what we should expect from a velocity field and the rate of change of P is to the same degree similar to a true "tendency" and will be referred to as such:

$$P'(\underset{\sim}{x}) = \left. \frac{dP}{d\tau} \right|_{\underset{\sim}{x}} . \tag{5.12}$$

The construction of the resulting "tendency equation" for P' follows closely the standard SG derivations (Schubert, 1985; PC). Since

$$\underset{\sim}{\nabla} P' = \left.\frac{d\underset{\sim}{X}}{d\tau}\right|_{\underset{\sim}{x}} = \underset{\sim}{V} - \underset{\approx}{Q} \cdot \underset{\sim}{u} \quad , \tag{5.13}$$

i.e.,

$$\underset{\sim}{u} = \underset{\approx}{Q}^{-1} \cdot \underset{\sim}{V} - \underset{\approx}{Q}^{-1} \cdot \underset{\sim}{\nabla} P' \quad , \tag{5.14}$$

it follows by taking the divergence,

$$\underset{\sim}{\nabla} \cdot \underset{\approx}{Q}^{-1} \cdot \underset{\sim}{\nabla} P' = \underset{\sim}{\nabla} \cdot \underset{\approx}{Q}^{-1} \cdot \underset{\sim}{V} - S \quad , \tag{5.15}$$

with conditions on the boundary, where the normal vector is $\underset{\sim}{n}$, given by

$$\underset{\sim}{n} \cdot \underset{\approx}{Q}^{-1} \cdot \underset{\sim}{\nabla} P' = 0 \quad . \tag{5.16}$$

Let P_1' be a solution induced by a data-displacement field $\underset{\sim}{V}$ only. Let P_2' be the solution obtained by a unit mass source at $\underset{\sim}{x}_\beta$ and unit sink at $\underset{\sim}{x}_\gamma$:

$$S_2(\underset{\sim}{x}) = \delta(\underset{\sim}{x} - \underset{\sim}{x}_\beta) - \delta(\underset{\sim}{x} - \underset{\sim}{x}_\gamma) \quad .$$

The rate of change of geopotential difference $\phi_1'(\underset{\sim}{x}_\beta) - \phi_1'(\underset{\sim}{x}_\gamma)$ of the first solution can now be related via a form of Green's theorem the solution P_2' as follows:

$$\Delta\phi_1' = \phi_1'(\underset{\sim}{x}_\beta) - \phi_1'(\underset{\sim}{x}_\gamma) = \int_D \left[\delta(\underset{\sim}{x} - \underset{\sim}{x}_\beta) - \delta(\underset{\sim}{x} - \underset{\sim}{x}_\gamma)\right] P_1' \, d\mu \quad ,$$

$$= -\int_D \left(\underset{\sim}{\nabla} \cdot \underset{\approx}{Q}^{-1} \cdot \underset{\sim}{\nabla} P_2'\right) P_1' \, d\mu \quad . \tag{5.17}$$

which, by integrating by parts twice becomes:

$$\Delta\phi_1' = -\int_D P_2' \left(\underset{\sim}{\nabla} \cdot \underset{\approx}{Q}^{-1} \cdot \underset{\sim}{\nabla} P_1'\right) d\mu \quad ,$$

$$= -\int_D P_2' \left(\underset{\sim}{\nabla} \cdot \underset{\approx}{Q}^{-1} \cdot \underset{\sim}{V}\right) d\mu \quad , \tag{5.18}$$

integrating by parts again:

$$\Delta\phi_1' = \int_D \underset{\sim}{V} \cdot \underset{\approx}{Q}^{-1} \cdot \underset{\sim}{\nabla} P_2' \, d\mu$$

$$= -\int_D \underset{\sim}{V} \cdot \underset{\sim}{u}_2 \, d\mu \qquad (5.19)$$

where $\underset{\sim}{u}_2 = -\underset{\approx}{Q}^{-1} \cdot \underset{\sim}{\nabla} P_2$ is the $\underset{\sim}{x}$-space rate of displacement associated with the source-sink solution P_2'. In particular, when

$$\underset{\sim}{V}(\underset{\sim}{x}) = \underset{\sim}{e}\, \delta(\underset{\sim}{x} - \underset{\sim}{x}_\alpha) \qquad (5.20)$$

then

$$\Delta\phi_1' = \phi_1'(\underset{\sim}{x}_\beta) - \phi_1'(\underset{\sim}{x}_\gamma) = -\underset{\sim}{u}_2(\underset{\sim}{x}_\alpha) \cdot \underset{\sim}{e} \quad , \qquad (5.21)$$

which is essentially the continuous analogue of reciprocity relation (5.4b). Corresponding continuous analogues of (5.3b) and (5.5) may be constructed in a similar manner.

We note from (5.19) that if the rate of perturbation of the data forcing tendency P_1' is a field $\underset{\sim}{V}$ bounded in magnitude uniformly by a "norm" W, then the maximum rate of change of geopotential difference $\Delta\phi_1' = [\phi_1'(\underset{\sim}{x}_\beta) - \phi_1'(\underset{\sim}{x}_\gamma)]$ provided by such a field $\underset{\sim}{V}$ occurs when $|\underset{\sim}{V}|$ achieves its maximum, W, and is parallel (or anti-parallel) to the displacement field $\underset{\sim}{u}_2$ everywhere ($\underset{\sim}{u}_2$ being associated with solution P_2' as before). Then

$$\Delta\phi_1' = \pm W \int_D |\underset{\sim}{u}_2| \, d\mu \quad . \qquad (5.22)$$

Furthermore, since $\underset{\sim}{u}_2$ integrates to a net "flow" of unity over every fixed surface separating $\underset{\sim}{x}_\beta$ from $\underset{\sim}{x}_\gamma$, the integral of (5.22) is just the flux-averaged <u>length</u> of the streamlines of $\underset{\sim}{u}_2$. In the case of two dimensional solutions in a simply connected bounded domain with perimeter $|\partial D|$ this mean flux length $\overline{l}$ is conjectured to satisfy the inequality,

$$\overline{l} = \int_D |\underset{\sim}{u}_2| \, d\mu \leq \frac{|\partial D|}{2} \quad . \qquad (5.23)$$

A proof of this conjecture has not yet been found. However, if it is true, it would effectively place a definite upper bound (namely the semiperimeter of the domain boundary) on the rate by which the geopotential difference between any fixed pair of points in the domain can change when the material rate of change of $\underset{\sim}{X}$ is everywhere bounded by unity.

6. CONCLUSION

An examination of the geometrical properties of the f-plane semi-geostrophic theory reveals a new duality principle coupling the "vector" of all components $(R_\alpha, X_\alpha, Y_\alpha, Z_\alpha)$ to the "vector" of all moments $(M_\alpha, M^x, M^y, M^z)$ of the set of finite elements. The duality structure is shown to be intimately associated with a Hamiltonian dynamical structure when the evolutionary equations follow the so-called "centroid convention". It is argued that the most natural way of extending the finite element model to domains with curved horizontals and varying Coriolis parameter leads to a dicrete form of Salmon's (1985) L_s-dynamics which he constructed explicitly to preserve analogues of the conservation laws of mass, energy and potential vorticity. The existence of a Liouville theorem places an important constraint on the dynamical behaviour of the finite element system, in particular, it precludes the possibility of chaotic attractors in the model phase-space.

The suggested generalisation of the finite element model to domains of varying Coriolis parameter no longer preserves the correspondence between solutions ϕ and the convex function P introduced by CP. However, the solution ϕ <u>is</u> still represented as an envelope of intersecting paraboloidal hypersurfaces, but now with curvatures in each (x, y)-plane characteristic of the Coriolis parameter at the dual space location with which each surface is uniquely associated. Thus, if element volumes M and dual space coordinates X are given for all elements, an iterative solution procedure along the lines of the one suggsted in CP and implemented by Chynoweth (1987) should still provide a unique solution for all but the most extreme combinations of data. Furthermore, typical finite element solutions in three dimensions will continue to display the characteristic pattern of four elements meeting at each interior vertex except momentarily at topological transitions of the vertex connectivity, as discussed by Purser (1988). Hence, locally at a vertex the pattern of joining edges can still be identified by a "tangent dual space" pattern represented by a particular tetrahedron . Likewise, the local transition of connectivity is represented by a transition in the tangent dual space pattern of adjoining tetrahedra. It would therefore appear that the essential characteristics assumed by the "panel beater" algorithm of Purser (1988) remain and may be exploited, even in a non-Euclidean domain, to effect the solution of finite element models efficiently.

In addition to its relation to a Hamiltonian, it is also demonstrated that, on the f-plane at least, the duality structure implies a family of reciprocity relations among various modes of solution sensitivity. It is conjectured that the sensitivity of geopotential to changes in the underlying data is bounded, at least in two dimensions, by an amount depending only on the linear dimensions of the domain and not on the basic solution being perturbed.

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