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METEOROLOGICAL OFFICE

FORECASTING TECHNIQUES
BRANCH MEMORANDUM No.9

THREE-PARAMETER ATMOSPHERIC MODEL USED FOR NUMERICAL WEATHER PREDICTION

by

G. A. BULL

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Part I

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Short Summary of the Main Features of the Atmospheric Model

Basic conditions

The atmosphere is taken to be a baroclinic fluid bounded below by the 1000 mb pressure surface and above by the 200 mb surface. It is separated into two layers by the 600 mb surface. In each of these two layers the thermal wind is taken to be constant in direction and to have a speed varying linearly with pressure through the layer. Pressure is used as vertical coordinate and the hydrostatic approximation is used in deriving the equations of motion thereby eliminating sound waves from the solution. The vertical velocity, which is in fact the rate of variation with time of the pressure acting on a moving particle of the air, is assumed to be zero at 200 mb and to be related at 1000 mb to the horizontal velocity at that level by conditions dependent on surface friction and topography. The vertical velocity is represented by a separate parabolic function of pressure in each of the two layers. These parabolic functions are in two parts. One of these is the same for both layers and has a coefficient proportional to the vertical velocity at 1000 mb. The second part has different coefficients in the two layers. The two functions are of course continuous at 600 mb, the upper boundary of the lower layer and bottom boundary of the upper layer.

Area of Analysis

The computations are made over an area of the earth which is a rectangle when projected stereographically on the tangent plane at the North Pole. The centre is near 74°N 13°W and the corners of the area are in the Bay of Bengal, in Cameroon, in Columbia, and near Hawaii. This area on the stereographic map specified is divided into squares by 47 lines parallel to one side and 41 parallel to the other to give 1927 points of intersection termed grid points. This array of points defines the horizontal coordinate system and the elements concerned are specified by their grid point values. The distance between the grid points is 326.68 km at the North Pole decreasing southwards to 179.25 km at the most southerly point in latitude 5° 35′N.

Fundamental dynamics; vorticity equations

The basic dynamical equations are those for the time rate of change of the vertical component of vorticity and the equation of continuity. The latter is used to eliminate the horizontal divergence from the former to give an equation specifying the "total" derivative of the vertical component of vorticity in terms of the vertical variation of vertical velocity. This last equation is integrated with respect to pressure over the whole depth from 200 to 1000 mb to give the so-called "equation of mean motion" in which the basic terms are the time-rates of variation of the vertical vorticities of the 600 mb wind and of the thermal winds of the two layers. The vertical vorticity is a function of horizontal derivatives of the horizontal components of velocity. The geostrophic approximation is now used to replace the thermal wind in each layer by the spatial gradients of the corresponding thicknesses. The geostrophic approximation is not used at 600 mb. At this level the flow is roughly non-divergent and the horizontal components of velocity are written as derivatives of a stream function as is appropriate for non-divergent flow. The stream function is not given by observation and is obtained from the spatial gradients of the 600 mb surface by means of the so-called balance equation. This is obtained from the equations of horizontal motion at the 600 mb level by using the stream function expressions for the horizontal velocity components and writing the condition for the total derivative of horizontal divergence to be zero. It has been found that derivation of wind on the 600 mb surface through the stream function gives superior forecasts to those obtained by using the geostrophic relation at that level. The equations are non-linear partial differential equations. They are of the elliptic type so that for solving them it is necessary to prescribe values on the boundary of the analysis area in advance. The boundary condition used is that there is no change in the heights of isobaric surfaces on the boundary during the period of the forecast.

Allowance for topography and surface friction

It has been stated earlier that the vertical velocity at the 100 mb level is related to the horizontal components there through the effects of topography and surface friction. The effect of topography is introduced by supposing the air has the vertical component to be expected if the geostrophic wind appropriate to the contours of the 1000 mb surface were forced up or down along the ground. The ground topography is approximated by taking at each grid point a height equal to the mean height over a symmetrical surrounding square of side one grid-length. Surface friction produces a horizontal component inwards to low pressure across the contours of the 1000 mb surface leading to an up-flow at the top of the friction layer over low pressure areas and down flow over high pressure areas. The vertical velocity at the top of the friction layer is proportional to the divergence of the horizontal velocity integrated through the friction layer. This integral is evaluated from equations of horizontal motion involving a constant coefficient of turbulence, a procedure which gives the Ekman spiral variation of wind with height in the friction layer. The surface wind is supposed over land to blow into low pressure at an angle of 30° to the 1000 mb contour and 0·35 times the corresponding geostrophic wind speed and over sea at 3° angle and 0·85 geostrophic speed. This completes the specification of vertical velocity in the two layers and so the specification of the equation of mean motion and its boundary conditions at 1000 and 200 mb.

Fundamental thermodynamics; rate of change of thickness

The next step is to obtain equations for the time rate of variation of thickness of the two layers given by advection and by both adiabatic and non-adiabatic changes of temperature. These equations are obtained from the first law of thermodynamics and express the time rate of change of thickness in terms of thickness gradients, velocity at 600 mb, an integral over the depth of each layer of the product of vertical velocity and the difference between the actual lapse-rate and the dry adiabatic lapse-rate, plus a term giving a measure of non-adiabatic heating. In evaluating the integral, the difference between actual and dry adiabatic lapse-rate is taken to be a constant through each layer; the difference is 0·4 × dry adiabatic lapse-rate in the lower layer and 0·25 × dry adiabatic lapse-rate in the upper layer. The only non-adiabatic heating or cooling effect taken into account is heating over the sea in areas where the sea is warmer than the air. This term is, over the colder sea areas, taken to be proportional to the quantity – an artificial 1000/600 mb thickness computed from mean monthly sea temperature minus the actual thickness of the layer – when it is positive and zero when negative. The formula giving the artificial thickness from sea temperature was originally computed for the sea area between Iceland and Scotland and has been found to give excessive values of artificial thickness and correspondingly excessive heating over the warmer seas of the area. Accordingly over the warmer seas the artificial thickness has been replaced by the climatological mean monthly thickness.

Two final equations are obtained by subtracting the vorticity equation for the top of each layer from that for the bottom and substituting as before described for the horizontal components of velocity.

Final equations for rates of change and method of solution

In this way five equations are obtained in which the five unknowns are the time rates of variation of the two layer thicknesses and the 600 mb stream function and two basic coefficients in the formulae for vertical velocities in the two layers. These equations are solved by replacing the derivatives by their finite difference approximations to give a series of linear equations, one of each type for each of the 1927 grid points. The linear equations are solved by an iterative technique. The 600 mb heights are obtained from the ancillary stream function at the required times by solving the balance equation in the "opposite sense" to that referred to under "Fundamental dynamics".

Computation of forecast values of heights and thickness

New values of heights and thicknesses are computed from the rates of change at intervals of time termed time-steps. The time step employed at present is ¾ hour but in certain situations a time step of ½ hour may be necessary. It is desirable to make the time-step as long as possible to minimize computations but if it is too long in relation to the grid length, random errors in the computations, which can be looked upon as spreading from grid point to grid point with roughly the wind velocity, build up rapidly and destroy the forecast. The time steps are the longest it is possible to use without running into computational instability. Even so in winter in the southern areas where there is a strong jet stream at 200 mb instability is liable to occur at the 200 mb level. The rates of change computed from the initial heights and thicknesses are added to the latter to give a new set of heights and thicknesses one time step ahead. From this new set of heights and thicknesses a further set of rates of change is computed to give a third set of heights and thicknesses one time step further on, i.e. two time steps from the initial time. Thereafter the process is gone through again and again using newly computed heights and thicknesses to compute new rates of change. Heights at an even number of time steps from the start are deduced from the last previous even values by using the time rate of change computed at the intermediate odd value. Values at an odd number of time steps are computed similarly from the last odd value and the rates of change computed from the intermediate even time step. This gives two "leap frogging" sets of values. This method is essential to maintain stability of the solution. The surface pressure is obtained by multiplying the 1000 mb height in decametres by 1·2 and adding 1000 mb. At intervals of six hours the height fields are smoothed by substituting for the "raw" directly computed value one which is a weighted mean of the "raw" value and corresponding "raw" values at adjacent grid points using weighting coefficients which have been determined to eliminate unrealistic small scale distortion of the patterns.

Final forecast

The elements forecast at present are:

Surface pressure, 500 mb and 200 mb heights, 1000/500 mb thickness: At 24, 30, 36 and 48 hours from initial time.

Mean Vertical Velocity in the lower layer. Zero, 12, 24 hours.

These are liable to change depending on operational requirements.

Summary

Basic conditions

Baroclinic atmosphere, hydrostatic in the vertical.

Boundaries in vertical: 1000 mb below, 200 mb above.

Layers: Two, separated by 600 mb surface.

Thermal wind: In each layer constant in direction and speed varying linearly with pressure.

Vertical Velocity: At 1000 mb has value produced by effect of topography and surface friction on surface wind (but see section on method below); zero at 200 mb; represented in each layer by a distinct parabolic function of pressure.

Area of analysis

Rectangular grid on stereographic projection on tangent plane at North Pole containing 1927 grid points. Corners Bay of Bengal, Cameroon, Colombia, near Hawaii. Grid length 327 km (203 st. miles, 176 nautical miles) at Pole decreasing southwards to 179 km in lat. 5° 35′N.

Fundamental dynamics

Basic equations: continuity, rate of change of vertical vorticity;

Equation of Mean Motion obtained by integrating vorticity equation over whole depth of from 1000 to 200 mb.

Relation between wind and height of isobaric surface:

Geostrophic approximation used at 1000 mb and for thermal winds but at 600 mb wind derived from stream function related to height by balance equation.

Allowance for topography and surface friction

Topography: Geostrophic wind at 1000 mb blows over a topography obtained by assigning to each grid point a height equal to mean true height over surrounding grid square.

Friction: Inflow to low pressure due to friction obtained by integrating equations of motion allowing for turbulence above surface wind proportional to geostrophic and blowing at fixed angle to isobar.

Vertical velocity: These two effects determine a vertical velocity at 1000 mb used in specification of vertical velocity in the two layers.

Fundamental thermodynamics; rate of change of thickness

Formulae for rate of change of thickness of each layer in terms of lapse-rate, vertical velocity, and non-adiabatic heating. Only non-adiabatic heating; air over a warmer sea.

Two final equations obtained from differences of rate of change of vorticity at top and bottom of each layer.

Final equations and method of solution

Five partial differential equations for five unknowns; time rates of change of thickness and stream function and two coefficients in vertical velocity formulae. 600 mb height obtained as required from stream function by solving balance equation "in reverse".

Boundary condition: no change on perimeter of analysis area.

Solution method: Derivatives replaced by finite difference approximations and resulting linear equations solved by iterative technique.

Computation of forecast heights and thicknesses and surface pressures

General procedure is to obtain heights and thicknesses at T + 2 time steps by adding to values at time T twice the rate of change computed for equations containing values for time step T + 1; the value at T = 1 being obtained from the initial values and the rates of change computed from initial values.

Smoothing: linear weighting formula applied to smooth at six-hourly intervals.

Surface pressure: obtained from 1·2 × height of 1000 mb surface in decametres ~~surface~~ plus 1000 mb.

Part II

Dynamics and Thermodynamics of the Model with Table of Constants and Bibliography

This Part contains a mathematical summary of the dynamics and thermodynamics of the numerical forecast system in use in December 1965. It does not deal with the initial data extraction or analysis of the observations or the programming for computer operations and only very summarily with the methods of numerical solution of the differential equations. The symbolism of the original papers has been preserved in almost all instances; notes have been included where symbols have been altered and also of symbols with more than one meaning.

I. Basic Assumptions

The atmosphere is taken to be a baroclinic fluid bounded by the 1000 mb (p_0) and 200 mb (p_1) surfaces and divided into two layers of different lapse-rate by the 600 mb (p_m) surface. Hydrostatic equilibrium is assumed in the vertical in order to eliminate sound waves from the solution. The rate of change of temperature with pressure in the vertical is constant in each layer; the difference between it and the dry adiabatic lapse-rate is 38°C per 900 mb in the lower and 46°C per 900 mb in the upper layer. The acceleration of gravity, g, is taken constant at 981 cm/sec^2 at all points. Vertical motion is zero at the upper boundary surface and is determined at the lower boundary by the effects of forced vertical motion over the topography and by surface friction. The thermal wind is taken to be constant in direction through each layer and to have a speed varying linearly with pressure through each layer. The only non-adiabatic heating effect included in the formulation is heating of air over a sea warmer than the air. No account is taken of cooling over a colder sea, of heating or cooling over land or ice, or of condensation processes.

II. Coordinates

The horizontal coordinates are orthogonal curvilinear distance coordinates on the spherical surface of the Earth. They are symbolized by r and s. The basic vertical coordinate is pressure, p. The function of pressure employed in the mathematics is α, where

$$\alpha = \frac{p_0 + p_1 - 2p}{p_0 - p_1}. \qquad \text{..... (1)}$$

α varies from -1 at p_0 to 0 at p_m and 1 at p_1.

In the later work horizontal coordinates on a stereographic projection of the Earth from the South Pole to the tangent plane at the North Pole are used; they are denoted by x and y. Vector quantities are underlined $\underline{V}$, the scalar product is denoted by . and vector product by $\wedge$.

III. Horizontal Velocity

The horizontal velocity vector is denoted by $\underline{V}$ and the horizontal components in the r and s directions by u and v respectively.

The assumption for the thermal wind is expressed by the formulae:

$$\underline{V} = \underline{V}_m + \alpha \underline{V}_0' \text{ for } -1 \leqslant \alpha \leqslant 0 \qquad \text{..... (2)}$$

$$\underline{V} = \underline{V}_m + \alpha \underline{V}_1' \text{ for } 0 \leqslant \alpha \leqslant 1 \qquad \text{..... (3)}$$

Where $\underline{V}_m$ is the vector wind at the mean level and $\underline{V}_0'$, $\underline{V}_1'$ are the thermal winds in the lower and upper layers respectively.

The $(_0')$ and $(_1')$ symbolism is employed systematically to denote differences of height as well as velocity between the top and bottom of a layer — suffix 0 denoting the layer 1000-600 mb and suffix 1 the layer 600-200 mb.

Caution. In reading the paper by Bushby and Whitelam (Ref.1) it should be noted that this symbolism is wrongly printed $\overset{1}{_0}$ $\overset{1}{_1}$ except at the top of p.379 of their paper.

IV. Vertical Component of Vorticity

From the expressions (2) and (3) it follows that the vertical component of the vorticity, ζ, in each layer is given by:

$$\zeta = \zeta_m + \alpha\zeta_0' \text{ for } -1 \leqslant \alpha \leqslant 0 \qquad \text{..... (4)}$$

$$\zeta = \zeta_m + \alpha\zeta_1' \text{ for } 0 \leqslant \alpha \leqslant 1 \qquad \text{..... (5)}$$

with an obvious notation. The quantities ζ_0' and ζ_1' are termed the thermal wind vorticities.

V. Vertical Motion

The vertical motion is expressed by the rate of variation, $\frac{dp}{dt}$, symbolized by ϖ, of the pressure acting on a particle moving with the fluid. $\frac{dp}{dt}$ is positive for downward and negative for upward motion and is expressed in the unit millibars per hour.

The symbol Π is used for $\frac{dp}{dt}$ in many papers, notably in Bushby-Whitelam (Ref.1).

ϖ is taken to be a separate quadratic function of pressure in each layer. These functions are written:

$$\varpi = -a + b\alpha + (a + b)\alpha^2 + \varpi_0 \left(\frac{1 - \alpha}{2}\right)^2 \quad \text{for } -1 \leqslant \alpha \leqslant 0 \qquad \ldots\ldots (6)$$

$$\varpi = -a + b\alpha + (a - b)\alpha^2 + \varpi_0 \left(\frac{1 - \alpha}{2}\right)^2 \quad \text{for } 0 \leqslant \alpha \leqslant 1 \qquad \ldots\ldots (7)$$

ϖ_0 is the value of ϖ at the lower boundary, isobaric surface p_0, and is related to the geostrophic velocity there by formulae set out in section VIII.

In these formulae:

$$a = 0{\cdot}125\,(p_0 - p_1)\;(\operatorname{div} \underline{V}_0' + \operatorname{div} \underline{V}_1') - 0{\cdot}25\,\varpi_0 \qquad \ldots\ldots (8)$$

$$b = 0{\cdot}125\,(p_0 - p_1)\;(\operatorname{div} \underline{V}_0' - \operatorname{div} \underline{V}_1') \qquad \ldots\ldots (9)$$

The operator div denotes in this Note the divergence of a horizontal vector. The quantities a and b are constant only in the vertical at any point. They vary with time and from point to point.

The formulae (6) and (7) make:

$$\varpi = \varpi_0 \quad \text{at } \alpha = -1 \qquad (1000 \text{ mb})$$

$$\varpi = -a + \frac{\varpi_0}{4} \text{ at } \alpha = 0 \qquad (600 \text{ mb})$$

$$\varpi = 0 \text{ at } \alpha = 1 \qquad (200 \text{ mb})$$

The horizontal and vertical velocities, taking into account the values of a and b of (8) and (9), satisfy the equation of continuity:

$$\operatorname{div} \underline{V} + \frac{\partial \varpi}{\partial p} = 0 \qquad \ldots\ldots (10)$$

The mean value of vertical velocity in the lower layer omitting the contribution of $\varpi_0\left(\frac{1 - \alpha}{2}\right)^2$ is one of the forecast elements.

Integrating (6), omitting the term in ϖ_0, from -1 to 0 this mean value is found to be given by

$$\bar{\varpi}_0 = -\left(\frac{2a}{3} + \frac{b}{6}\right) \qquad \ldots\ldots (11)$$

VI. Fundamental dynamics

The basic equation is the vorticity equation in the form:

$$\frac{\partial}{\partial t}(\zeta + f) + \underline{V}\cdot\nabla(\zeta + f) + \varpi\frac{\partial}{\partial p}(\zeta + f) = -(\zeta + f)\operatorname{div}\underline{V} \qquad \ldots\ldots (12)$$

This form of the vorticity equation is derived from the equations of motion for horizontal velocity by taking the curl and omitting the so-called twisting or tipping term;

$$\frac{\partial \varpi}{\partial r}\frac{\partial v}{\partial p} - \frac{\partial \varpi}{\partial s}\frac{\partial u}{\partial p}.$$

The reasons for omitting it are given in Graystone and Jones (Ref.11).

f is the Coriolis parameter $2\omega\sin\phi$, and $\underline{\nabla}$ is the horizontal gradient operator of components:

$$\frac{\partial}{\partial r}, \frac{\partial}{\partial s}.$$

Replacing div $\underline{V}$ by $-\frac{\partial\varpi}{\partial p}$ from the equation of continuity to obtain the vorticity equation in a form amenable to approximation from the geostrophic wind relation we obtain:

$$\frac{\partial}{\partial t}(\zeta + f) + \underline{V}.\nabla(\zeta + f) + \varpi\frac{\partial}{\partial p}(\zeta + f) = (\zeta + f)\frac{\partial\varpi}{\partial p} \qquad \text{..... (13)}$$

We can now substitute for ζ, $\underline{V}$, and ϖ in (13) from formulae (4), (5), (2), (3), (6) and (7), to obtain a vorticity equation for each layer in terms of a instead of p.

The vorticity equation for the lower layer is now integrated with respect to a from -1 to 0, the one for the upper layer integrated from 0 to 1, and the two integrals added to give the equation, termed the "equation of mean motion" which is, as used:

$$\frac{\partial}{\partial t}\left\{\zeta_m + \frac{1}{4}(\zeta_1' - \zeta_0')\right\} + \underline{V}_m.\nabla\left\{\zeta_m + \frac{1}{4}(\zeta_1' - \zeta_0') + f\right\}$$

$$+ \frac{1}{4}(\underline{V}_1' - \underline{V}_0').\nabla(\zeta_m + f) + \frac{1}{6}\left\{\underline{V}_0'.\nabla\zeta_0' + \underline{V}_1'.\nabla\zeta_1'\right\}$$

$$= \frac{\varpi_0}{P_0 - P_1}\bar{f} \qquad \text{..... (14)}$$

$\bar{f}$ is the mean value of f over the area of analysis.

The complete mathematics gives an additional term,

$$\frac{4a(\zeta_0' + \zeta_1') + b(\zeta_0' - \zeta_1'),}{3(P_0 - P_1)}$$

on the left-hand side of (14) while the complete expression on the right-hand side is,

$$\left(\frac{\varpi_0}{P_0 - P_1}\right)\left(\zeta_m + f + \frac{1}{6}(\zeta_0' + \zeta_1')\right).$$

The additional term on the left-hand side is ignored for the reason that it is biassed towards one sign as explained in Bushby-Whitelam (Ref.1, page 376). The right-hand side is approximated by $\bar{f}$ because the departure of f from $\bar{f}$ in combination with the other quantities is biassed (verbal information from Mr P. Graystone).

VII. Relation of Horizontal Velocity to Heights of Isobaric Surfaces

The geostrophic approximation is used for the thermal winds so that, ignoring terms expressing the variation of the Coriolis parameter with latitude, the vorticities of the thermal winds are, respectively;

$$\zeta_0' = \frac{g}{f}\nabla^2 h_0', \qquad \text{..... (15)}$$

$$\zeta_1' = \frac{g}{f}\nabla^2 h_1'. \qquad \text{..... (16)}$$

Here h_0' is the thickness of the lower layer and h_1' of the upper layer. ∇^2 is the Laplacian horizontal operator

$$\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial s^2}.$$

The wind at 600 mb on the other hand is expressed in terms of a stream function ψ.

In vector terminology,

$$\underline{V}_m = \underline{k} \wedge \nabla\psi, \qquad \text{..... (17)}$$

where $\underline{k}$ is the unit upward vector. In Cartesian form

$$u_m = -\frac{\partial\psi}{\partial s},$$

$$v_m = \frac{\partial\psi}{\partial r},$$

with an obvious notation.

The initial values of the heights of the 1000 mb and 200 mb surfaces are derived from observations at these levels. 600 mb heights are not reported and for the numerical forecast programme are derived from the formula:

$$h_m = 0{\cdot}8h_{500} + 0{\cdot}2h_{1000}.$$

This gives a sufficiently good approximation (Ref.1 page 378) for the purpose of computing derivatives. The same formula is used to obtain forecast 500 mb heights from the forecast 1000 mb and 600 mb heights.

The stream function is not given by observation and so a relation linking it with an observed quantity must be found. The relation employed is the "balance equation" which is a partial differential equation for the stream function as a function of the Laplacian of the 600 mb height field and the Coriolis parameter and its latitudinal variation.

The balance equation is obtained by writing, as shown below, the condition for the "total" variation of horizontal divergence on the 600 mb surface to be zero.

The equations of horizontal motion on the 600 mb surface are:

$$\frac{\partial u_m}{\partial t} + u_m\frac{\partial u_m}{\partial r} + v_m\frac{\partial u_m}{\partial s} - fv_m = -g\frac{\partial h_m}{\partial r} \qquad \text{..... (18)}$$

$$\frac{\partial v_m}{\partial t} + u_m\frac{\partial v_m}{\partial r} + v_m\frac{\partial v_m}{\partial s} + fu_m = -g\frac{\partial h_m}{\partial s} \qquad \text{..... (19)}$$

The condition for the total variation of divergence to be zero is:

$$\frac{\partial}{\partial t}\left(\frac{\partial u_m}{\partial r} + \frac{\partial v_m}{\partial s}\right) + u_m\frac{\partial}{\partial r}\left(\frac{\partial u_m}{\partial r} + \frac{\partial v_m}{\partial s}\right) + v_m\frac{\partial}{\partial s}\left(\frac{\partial u_m}{\partial r} + \frac{\partial v_m}{\partial s}\right) = 0. \qquad \text{..... (20)}$$

Differentiating (18) with respect to r and (19) with respect to s adding and substituting in (20) we find,

$$\frac{\partial}{\partial t}\left(\frac{\partial u_m}{\partial r} + \frac{\partial v_m}{\partial s}\right) + u_m\frac{\partial}{\partial r}\left(\frac{\partial u_m}{\partial r} + \frac{\partial v_m}{\partial s}\right) + v_m\frac{\partial}{\partial s}\left(\frac{\partial u_m}{\partial r} + \frac{\partial v_m}{\partial s}\right) =$$

$$-\left\{\left(\frac{\partial u_m}{\partial r}\right)^2 + \left(\frac{\partial v_m}{\partial s}\right)^2 + 2\frac{\partial u_m}{\partial s}\frac{\partial v_m}{\partial r} + f\left(\frac{\partial u_m}{\partial s} - \frac{\partial v_m}{\partial r}\right) + u_m\frac{\partial f}{\partial s} - v_m\frac{\partial f}{\partial r} + g\nabla^2 h_m\right\} \qquad \text{.. (21)}$$

The right-hand side of (21) is then zero since the left-hand side is zero from (20). Expressing the component velocities u_m, v_m in terms of the stream function, the balance equation is obtained in the form:

$$2\left(\frac{\partial^2\psi}{\partial r^2}\frac{\partial^2\psi}{\partial s^2} - \left(\frac{\partial^2\psi}{\partial r\partial s}\right)^2\right) + f\nabla^2\psi + \nabla f.\nabla\psi - g\nabla^2 h_m = 0. \qquad ..... (22)$$

Given h_m and f this has to be solved numerically for ψ.

The use of the geostrophic approximation for the thermal winds in the two layers and of the stream function for the wind at the middle level eliminates "noise" due to vertical-transverse gravity waves from the solution while retaining the synoptically important horizontal-transverse Rossby waves. Reference should be made to Thompson (Ref.3 Chapters 4, 5, 6 and 11) for a full discussion of the problem of filtering out the non-significant sound and vertical-transverse gravity waves which, if not excluded, give rise to computational instability. Sound waves were excluded by the use of the hydrostatic assumption. The same filtering of gravity waves can be obtained by using the geostrophic approximation for the wind at the middle level but it is found in practice that the stream-function method gives a better prediction of the significant changes in pressure distribution.

An important point regarding the balance equation is that for numerical solution it must be of the elliptic type. Using the theory given by Piaggio (Ref.12, pages 183/184) of this Monge-Ampère type of partial differential equation the condition for ellipticity is found to be:

$$g\nabla^2 h_m - \nabla\psi.\nabla f + f^2/_2 > 0 \qquad ..... (23)$$

This condition is not always satisfied over the whole area. The difficulty is overcome by making small modifications to the 600 mb heights. It will be seen that the condition (23) strictly involves the solution ψ. However, ψ is approximately equal to its geostrophic value, $\frac{gh_m}{f}$. Entering this approximation in (23) the condition becomes

$$g\nabla^2 h_m - \frac{g}{f}\left(\nabla h_m.\nabla f\right) + f^2/_2 > 0, \qquad ..... (24)$$

which involves only known quantities. The values of h_m are checked for satisfaction of the inequality (24) and over areas in which it does not hold the 600 mb heights are modified slightly to values which do satisfy it. This procedure is termed "ellipticization".

VIII. Allowance for Topography and Surface Friction

The effect of topography is to produce a forced vertical component of motion at the lower boundary. It is supposed that most of the air is forced to move over mountains and ridges.

If H is the height of the underlying surface above sea level and u, v, w are components of wind on the mountains etc. in an r, s, z coordinate system with z vertical, the conditions for zero flow normal to the underlying surface is:

$$u\frac{\partial H}{\partial r} + v\frac{\partial H}{\partial s} - w = 0. \qquad ..... (25)$$

This follows from the fact that the direction cosines of the normal to the surface $H(r,s) - z = 0$ are proportional to

$$\frac{\partial H}{\partial r}, \frac{\partial H}{\partial s}, -1.$$

Transforming to isobaric coordinates in which the vertical velocity is ϖ_{0t} the suffix t denoting topography, we have

$$w = -\frac{\varpi_{0t}}{g\rho}$$

so that

$$\varpi_{0t} = -g\rho\,\underline{V}_0 \cdot \nabla H \qquad \text{..... (26)}$$

where $\underline{V}_0$ is the vector horizontal wind on the mountains.

Taking $\underline{V}_0$ as approximately given by the geostrophic wind appropriate to the lowest pressure surface, p_0 of height h_0 above sea-level, we have:

$$\underline{V}_0 \cdot \nabla H = \frac{g}{f}\left(-\frac{\partial h_0}{\partial s}\frac{\partial H}{\partial r} + \frac{\partial h_0}{\partial r}\frac{\partial H}{\partial s}\right),$$

whence using the relation (26) and introducing the Jacobian notation

$$J(p,q) = \frac{\partial p}{\partial r}\frac{\partial q}{\partial s} - \frac{\partial p}{\partial s}\frac{\partial q}{\partial r} \quad ,$$

we derive:

$$\varpi_{0_t} = \frac{g^2\rho}{f}\, J(H,\ h_0) \qquad \text{..... (27)}$$

In the model used the topography H is mapped by assigning to each gridpoint a height equal to the mean value of the true heights taken over a square of one gridlength side centred at the gridpoint. The density ρ is taken to be constant in the model.

Turning now to the effects of friction the basic equations of motion in the friction layer are, in the r, s, z system of coordinates:

$$f(v - v_g) = -\frac{A}{\rho}\frac{\partial^2 u}{\partial z^2} \qquad \text{..... (28)}$$

$$f(u - u_g) = \frac{A}{\rho}\frac{\partial^2 v}{\partial z^2} \qquad \text{..... (29)}$$

in which u_g and v_g are horizontal geostrophic wind components and A is an "Austausch" coefficient supposed constant through the friction layer.

Suitable solutions of these equations are:

$$u - u_g = e^{-qz}(K \cos qz + B \sin qz) \qquad \text{..... (30)}$$

$$v - v_g = e^{-qz}(B \cos qz - K \sin qz) \qquad \text{..... (31)}$$

where $q^2 = \frac{\rho f}{2A}$, and B, K are quantities independent of z but variable in the horizontal.

At the level $z = o$,

$$u_0 = u_g + K,$$

$$v_0 = v_g + B.$$

It is now supposed that the surface wind speed is equal to c times the geostrophic wind speed and that it is backed through an angle θ from the geostrophic direction.

If the geostrophic wind makes an angle χ with Or then

$$K = u_o - u_g = cV_g \cos(\theta + \chi) - u_g$$

$$B = v_o - v_g = cV_g \sin(\theta + \chi) - V_g$$

where V_g is the geostrophic speed.

Hence, since $V_g \cos \chi = u_g$ etc.,

$$K = cu_g \cos \theta - cv_g \sin \theta - u_g = (c \cos \theta - 1)\, u_g - c\, v_g \sin \theta \quad \text{.... (32)}$$

$$B = c\, u_g \sin \theta + (c \cos \theta - 1)\, v_g \quad \text{..... (33)}$$

It is supposed there is a level of height Z which can be termed the top of the friction layer and at which the wind sufficiently attains its geostrophic value.

The vertical velocity ϖ_{of} in pressure coordinate terms produced by friction at the top of the friction layer is given, integrating the equation of continuity, as

$$\varpi_{of} = \int_o^Z g\rho \operatorname{div} \underline{V}\, dz\,.$$

Now consider

$$\int_o^Z (u - u_g)\, dz \text{ and } \int_o^Z (v - v_g)\, dz.$$

$$\int_{o}^Z (u - u_g)\, dz = \left| \frac{K}{2q}(1 - e^{-qz} \cos qz + e^{-qz} \sin qz) + \frac{B}{2q}(1 - e^{-qz} \cos qz - e^{-qz} \sin qz) \right|_o^Z$$

$$= \frac{B + K}{2q} - \frac{e^{-qZ}}{2q}(K \cos q Z + B \sin q Z + B \cos q Z - K \sin q Z).$$

The term in e^{-qZ} is supposed to be negligible, leaving,

$$\int_o^Z (u - u_g)\, dz = \frac{B + K}{2q} \quad \text{..... (34)}$$

Similarly

$$\int_o^Z (v - v_g)\, dz = \frac{B - K}{2q} \quad \text{..... (35)}$$

Hence, taking the divergence of the sum of these two integrals and using the fact that the divergence of the geostrophic wind is zero we have,

$$\varpi_{of} = g\rho \int_o^Z \operatorname{div} \underline{V}\, dz = \frac{g\rho}{2q} \left(\frac{\partial}{\partial r}(B + K) + \frac{\partial}{\partial s}(B - K) \right) \quad \text{..... (36)}$$

From (32) and (33)

$$B + K = u_g(c \cos \theta - 1 + c \sin \theta) + v_g(c \cos \theta - 1 - c \sin \theta)$$

$$= u_g(F_2 - 1) + v_g F_1. \quad \text{..... (37)}$$

Similarly,
$$B - K = -u_g F_1 + v_g(F_2 - 1) \quad \text{..... (38)}$$

where
$$F_1 = c(\cos\theta - \sin\theta) - 1$$

$$F_2 = c(\cos\theta + \sin\theta).$$

Hence,

$$\varpi_{0f} = \frac{g\rho}{2q}\left\{\frac{\partial u_g}{\partial r}(F_2 - 1) + u_g\frac{\partial F_2}{\partial r} + v_g\frac{\partial F_1}{\partial r} - \frac{\partial u_g}{\partial s}F_1 - u_g\frac{\partial F_1}{\partial s} + v_g\frac{\partial F_2}{\partial s}\right.$$

$$\left. + \frac{\partial v_g}{\partial s}(F_2 - 1)\right\} = g\sqrt{\frac{A\rho}{2f}}\left\{F_1\zeta_g - \underline{k}.\underline{V}_g \wedge \nabla F_1 + \underline{V}_g . \nabla F_2\right\} \quad \text{..... (39)}$$

ζ_g is the vertical component of vorticity of the geostrophic wind which, as in the derivation of the forced ascent produced by topography, is taken to be the geostrophic wind associated with the 1000 mb height field. The quantities F_1 and F_2 are functions of the horizontal coordinates. In the present forecasting model c is taken to be 0·35 over land and 0·85 over the sea while the angle of inflow into low heights, θ, is taken to be 30° over land and 3° over sea. The term $F_1\zeta_g$ has a value all over the chart but the terms involving gradients of F_1 and F_2 have non-zero values only at coast lines. It is possible that these values may be changed as a result of operational experience.

It is now supposed that ϖ_{0f} which in the derivation was vertical velocity at the top of the friction layer, is with sufficient accuracy, the vertical velocity produced by friction at the 1000 mb level. Reference should be made to Graystone (Ref. 2, page 256) for a justification of these approximations. The suffixes t and f to ϖ_0 are not used in Graystone's paper.

The final vertical velocity, ϖ_0, of Section $\underline{V}$, at the 1000 mb level is the sum of the vertical velocities produced by topography and friction. ϖ_0 is, in this way, expressed in terms of the gradients of the 1000 mb height field and of constants depending on position over land, sea or coastline.

Values of the constants concerned are given in the Appendix.

IX. Thermodynamics: Equations for Rates of Change of Thickness of the Layers

The thickness, h, of the layer between pressures p_0 and p is given in terms of pressure and temperature T by the formula, based on the hydrostatic assumption:

$$h = \frac{R}{g}\int_p^{p_0} T\frac{dp}{p}$$

so
$$\frac{\partial h}{\partial t} = \frac{R}{g}\int_p^{p_0}\frac{\partial T}{\partial t}\cdot\frac{dp}{p}$$

The first law of thermodynamics gives for the rate of change of temperature of a volume of unit mass which receives a non-adiabatic heat supply dQ and undergoes pressure change dp,

$$\frac{dT}{dt} = \frac{1}{c_p}\frac{dQ}{dt} + \frac{\gamma}{g\rho}\frac{dp}{dt}.$$

γ is the numerical value of the dry-adiabatic lapse rate.

Now $$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \underline{V} . \nabla T + \varpi \frac{\partial T}{\partial p}.$$

Hence for the rates of change of the thickness we have, using expressions (2) and (3) for the horizontal velocities in the two layers:

$$\frac{\partial h_0'}{\partial t} = \frac{R}{g}\int_{p_m}^{p_0}\left(\frac{1}{c_p}\frac{dQ}{dt} + \frac{\gamma\varpi}{g\rho} - \left(\underline{V}_m . \nabla T + a\underline{V}_0' . \nabla T\right) - \varpi\frac{\partial T}{\partial p}\right)dp$$

$$\frac{\partial h_1'}{\partial t} = \frac{R}{g}\int_{p_1}^{p_m}\left(\frac{1}{c_p}\frac{dQ}{dt} + \frac{\gamma\varpi}{g\rho} - \left(\underline{V}_m . \nabla T + a\underline{V}_1' . \nabla T\right) - \varpi\frac{\partial T}{\partial p}\right)dp$$

Transforming to a from p these integrals become

$$\frac{\partial h_0'}{\partial t} = \frac{R}{g}\frac{(p_0 - p_1)}{2}\int_{-1}^{0}\left(\frac{1}{c_p}\frac{dQ}{dt} - \underline{V}_m . \nabla T - a\underline{V}_0' . \nabla T + \varpi\left(\frac{\gamma}{g\rho} - \frac{\partial T}{\partial p}\right)\right)\frac{da}{p} \quad \text{..... (40)}$$

$$\frac{\partial h_1'}{\partial t} = \frac{R}{g}\frac{(p_0 - p_1)}{2}\int_{0}^{1}\left(\frac{1}{c_p}\frac{dQ}{dt} - \underline{V}_m . \nabla T - a\underline{V}_1' . \nabla T + \varpi\left(\frac{\gamma}{g\rho} - \frac{\partial T}{\partial p}\right)\right)\frac{da}{p} \quad \text{..... (41)}$$

Now collect terms in (40) and (41).

The non-adiabatic heating terms are symbolized by Q_0 in the lower layer and Q_1 in the upper layer, i.e.:

$$Q_0 = \frac{R}{g}\frac{(p_0 - p_1)}{2c_p}\int_{-1}^{0}\frac{dQ}{dt}\frac{da}{p} \quad \text{..... (42)}$$

$$Q_1 = \frac{R}{g}\frac{(p_0 - p_1)}{2c_p}\int_{0}^{1}\frac{dQ}{dt}\frac{da}{p} \quad \text{..... (43)}$$

The term $-\frac{R}{g}\frac{(p_0 - p_1)}{2}\int \underline{V}_m . \nabla T \frac{da}{p}$ is equal to $-\underline{V}_m . \nabla h_0'$ in the integral for the lower layer and to $-\underline{V}_m . \nabla h_1'$ in the upper layer, since V_m is constant through each layer and the thickness of each layer is

$$\frac{R}{g}\frac{(p_0 - p_1)}{2}\int T\frac{da}{p}.$$

The contributions proportional to $\int_{-1}^{0}(a\underline{V}_0' . \nabla T)\frac{da}{p}$ in the lower layer and $\int_{0}^{1}(a\underline{V}_1' . \nabla T)\frac{da}{p}$ in the upper layer are zero because the thermal wind in each layer is constant in direction.

To prove this last statement we note first that the equations of geostrophic motion in r, s, p coordinates are:

$$u = -\frac{g}{f}\left(\frac{\partial z}{\partial s}\right)_p, \quad v = \frac{g}{f}\left(\frac{\partial z}{\partial r}\right)_p,$$

where u, v are the components of geostrophic wind at height z.

Now to obtain the thermal wind of components u_T, v_T in say the lower layer, we consider the height z as measured upward from the lower pressure surface, p_0.

Then,

$$u_T = -\frac{R}{f}\int_p^{p_0}\left(\frac{\partial T}{\partial s}\right)_\rho \frac{dp}{p},$$

$$v_T = \frac{R}{f}\int_p^{p_0}\left(\frac{\partial T}{\partial r}\right)_\rho \frac{dp}{p}.$$

Hence

$$\frac{\partial u_T}{\partial p} = \frac{R}{fp}\left(\frac{\partial T}{\partial s}\right)_p$$

$$\frac{\partial v_T}{\partial p} = -\frac{R}{fp}\left(\frac{\partial T}{\partial r}\right)_p$$

which is $\frac{\alpha}{1+\alpha}$ times

so $\alpha \underline{V}_0' . \nabla T$, the scalar product of thermal wind and temperature gradient in the lower layer, is equal to

$$-\frac{\alpha}{1+\alpha} f\frac{p}{R}\left(-u_T\frac{\partial v_T}{\partial p} + v_T\frac{\partial u_T}{\partial p}\right)$$

i.e. to

$$-\frac{\alpha}{1+\alpha} f\frac{p}{R}u_T^2\frac{\partial}{\partial p}\left(\frac{v_T}{u_T}\right),$$

which is zero because of the constancy of direction of thermal wind through the layer. The same method applies also in the upper layer.

The quantity $\dfrac{\partial T}{\partial p} - \dfrac{\gamma}{g\rho}$ which is proportional to the difference between the dry-adiabatic lapse-rate and the actual lapse-rate is taken to be constant in each layer and denoted by Γ_0 in the lower layer and Γ_1 in the upper.

In the following text these quantities Γ are termed the "lapse-rate" in the layer following the terminology usual in the literature.

The equations for the rate of change of thickness have now taken the forms:

$$\frac{\partial h_0'}{\partial t} = -\underline{V}_m . \nabla h_0' - \frac{(p_0 - p_1)R\Gamma_0}{2g}\int_{-1}^{0}\varpi\frac{d\alpha}{p} + Q_0 \qquad \text{..... (44)}$$

$$\frac{\partial h_1'}{\partial t} = -V_m . \nabla h_1' - \frac{(p_0 - p_1)R\Gamma_1}{2g}\int_{0}^{1}\varpi\frac{d\alpha}{p} + Q_1 \qquad \text{..... (45)}$$

Q_0 is computed by the method described in Section XII and Q_1 is taken to be zero.

Now consider the integrals in (44) and (45).

Substituting for ϖ from (6) and (7) these become, respectively,

$$-\frac{(p_0 - p_1)R\Gamma_0}{g}\int_{-1}^{0}\frac{-a + b\alpha + (a+b)\alpha^2 + \varpi_0\left(\frac{1-\alpha}{2}\right)^2}{p_0 + p_1 - \alpha(p_0 - p_1)}\,d\alpha,$$

and,

$$-\frac{(p_0 - p_1)R\Gamma_1}{g}\int_{0}^{1}\frac{-a + b\alpha + (a-b)\alpha^2 + \varpi_0\left(\frac{1-\alpha}{2}\right)^2}{p_0 + p_1 - \alpha(p_0 - p_1)}\,d\alpha$$

which reduce on integration to:

$$Aa + Bb + E'\varpi_0 \text{ and}$$

$$Ca + Db + F'\varpi_0 \text{ respectively where}$$

$$A = \frac{R\Gamma_0}{2g}\left[\frac{8p_0p_1}{(p_0 - p_1)^2}\log_e\left(\frac{p_0 + p_1}{2p_0}\right) + \frac{(p_0 + 3p_1)}{(p_0 - p_1)}\right]$$

(This quantity conventionally denoted by A has, of course, no connection with the "Austausch" coefficient of Section VII.)

$$B = \frac{R\Gamma_0}{2g}\left[\frac{4p_0(p_0 + p_1)}{(p_0 - p_1)^2}\log_e\left(\frac{p_0 + p_1}{2p_0}\right) + \frac{(3p_0 + p_1)}{(p_0 - p_1)}\right]$$

$$E' = \frac{R\Gamma_0}{g}\left[\frac{7p_1 - 3p_0}{8(p_0 - p_1)} + \frac{p_1^{\,2}}{(p_0 - p_1)^2}\log_e\left(\frac{p_0 + p_1}{2p_0}\right)\right]$$

$$C = \frac{R\Gamma_1}{2g}\left[\frac{8p_0p_1}{(p_0 - p_1)^2}\log_e\left(\frac{2p_1}{p_0 + p_1}\right) + \frac{(3p_0 + p_1)}{(p_0 - p_1)}\right]$$

$$D = \frac{R\Gamma_1}{2g}\left[-\frac{4p_1(p_0 + p_1)}{(p_0 - p_1)^2}\log_e\left(\frac{2p_1}{p_0 + p_1}\right) - \frac{(p_0 + 3p_1)}{(p_0 - p_1)}\right]$$

$$F' = \frac{R\Gamma_1}{g}\left[\frac{5p_1 - p_0}{8(p_0 - p_1)} - \frac{p_1^{\,2}}{(p_0 - p_1)^2}\log_e\left(\frac{p_0 + p_1}{2p_0}\right)\right]$$

The numerical values of these quantities are given in the Appendix. The quantities denoted here by E' and F' are written in some working papers E_{FTB} and F_{FTB}.

The final equations for the rates of change of thickness are thus:

$$\frac{\partial h_0'}{\partial t} = -\underline{V}_m \cdot \nabla h_0' + Q_0 + Aa + Bb + E'\varpi_0 \qquad \text{..... (46)}$$

$$\frac{\partial h_1'}{\partial t} = -\underline{V}_m \cdot \nabla h_1' + Ca + Db + F'\varpi_0 \qquad \text{..... (47)}$$

X. Development Equations

Two further equations, termed "development equations", are obtained by subtracting the vorticity equation for the base of each layer from that for the top.

Applying this technique we find, for the lower layer:

$$\frac{\partial \zeta_0'}{\partial t} + \underline{V}_m \cdot \nabla \zeta_0' + \underline{V}_0' \cdot \nabla(\zeta_m - \zeta_0' + f) = -\frac{2}{p_0 - p_1}\left\{2(a + b)(\zeta_m + f) - (a + b)\zeta_0'\right\} - \frac{\varpi_0}{p_0 - p_1}\left\{\zeta_m + f - \frac{\zeta_0'}{2}\right\} \qquad \text{..... (48)}$$

and for the upper layer,

$$\frac{\partial \zeta_1'}{\partial t} + \underline{V}_m \cdot \nabla \zeta_1' + V_1' \cdot \nabla(\zeta_m + \zeta_1' + f) = -\frac{2}{p_0 - p_1}\left\{2(a-b)(\zeta_m + f)\right.$$

$$\left. + (a-b)\zeta_1'\right\} - \frac{\varpi_0}{p_0 - p_1}\left\{\zeta_m + f + \frac{\zeta_1'}{2}\right\} \qquad \text{..... (49)}$$

The equation of mean motion (14), the equations for the rates of change of the layer thickness (46) and (47) and the development equations express the basic dynamics and thermodynamics of the model. The horizontal velocities and the vorticities which occur in them are expressible in terms of the two thicknesses and the 600 mb height through the geostrophic equation for the thermal wind and the balance equation. The two other unknowns are the coefficients a and b of the vertical velocity formulae. Thus there are five equations for five unknowns. The following sections describe the transformations of these fundamental equations to equations for the elements required in the forecast.

XI. Introduction of Middle-Level Stream Function and of the Geostrophic Approximation for Thermal Winds

Entering the stream function representation of middle-level wind in equation (46) the term $-\,V_m \cdot \nabla h_0'$ becomes

$$+\frac{\partial \psi}{\partial s}\frac{\partial h_0'}{\partial r} - \frac{\partial \psi}{\partial r}\frac{\partial h_0'}{\partial s}$$

that is, $-J(\psi, h_0')$. The corresponding term in (47) becomes $-J(\psi, h_1')$.

Now transform to the stereographic projection from the South Pole on the tangent plane at the North Pole in which the horizontal coordinates are x, y. This involves replacing $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial s}$ by $\beta\frac{\partial}{\partial x}$ and $\beta\frac{\partial}{\partial y}$ respectively where β is the map magnification factor $\frac{2}{1 + \sin\phi}$ (ϕ latitude) from Earth to plane. β is, of course, a function of position on the plane. $J(\psi, h)$ thus becomes $\beta^2 J(\psi, h)$ with the partial differentiations in the new J taken with respect to x and y.

The equations for the rate of change of thickness can now be written in a form suitable for future elimination as follows:

$$\frac{\partial h_0'}{\partial t} = -H + A\left(a + \frac{\varpi_0}{4}\right) + Bb, \text{ where } H = \beta^2 J(\psi, h_0') - Q_0 - \left(E - \frac{A}{4}\right)\varpi_0$$

$$\frac{\partial h_1'}{\partial t} = -I + C\left(a + \frac{\varpi_0}{4}\right) + Db, \text{ where } I = \beta^2 J(\psi, h_1') - \left(F - \frac{C}{4}\right)\varpi_0 .$$

Solving for $a + \frac{\varpi_0}{4}$ and b we have

$$a + \frac{\varpi_0}{4} = \frac{D}{AD - BC}\left(H + \frac{\partial h_0'}{\partial t}\right) - \frac{B}{AD - BC}\left(I + \frac{\partial h_1'}{\partial t}\right) \qquad \text{..... (50)}$$

$$b = \frac{-C}{AD - BC}\left(H + \frac{\partial h_0'}{\partial t}\right) + \frac{A}{AD - BC}\left(I + \frac{\partial h_1'}{\partial t}\right) \qquad \text{..... (51)}$$

It may be noted at this point that in the British Meteorological Office contribution to W.M.O. Technical Note 67 (Appendix 3,c, pages 37/38) the equations (50) and (51) appear as equations (4) and (5).

The next step is the replacement of $\frac{\partial \zeta_0'}{\partial t}$ and $\frac{\partial \zeta_1'}{\partial t}$ in the equation of mean motion (14) by substitution from the development equations (48) and (49) which gives the equation,

$$\frac{\partial \zeta_m}{\partial t} + \underline{V}_m \cdot \nabla(\zeta_m + f) - \frac{1}{12}\underline{V}_0' \cdot \nabla \zeta_0' - \frac{1}{12}\underline{V}_1' \cdot \nabla \zeta_1' = - \frac{1}{2(p_0 - p_1)} \Big\{ 4b(\zeta_m + f)$$

$$- a(\zeta_0' + \zeta_1') + b(\zeta_0' - \zeta_1') \Big\} + \frac{\varpi_0}{2(p_0 - p_1)} \left\{ \frac{\zeta_0' + \zeta_1'}{4} + 2\bar{f} \right\}.$$

Now write the velocity at the mean level in terms of the stream function and the thermal winds in terms of thickness gradients.

ζ_m becomes $\nabla^2\psi$, ζ_0' becomes $\frac{g}{f}\nabla^2 h_0'$ and ζ_1', $\frac{g}{f}\nabla^2 h_1'$. Transforming to the stereographic projection multiplies these quantities by β^2.

The term $\underline{V}_m \cdot \nabla(\zeta_m + f)$ becomes

$$- \beta \frac{\partial \psi}{\partial y} \cdot \beta \frac{\partial}{\partial x}(\beta^2 \nabla^2 \psi + f) + \beta \frac{\partial \psi}{\partial x} \cdot \beta \frac{\partial}{\partial y}(\beta^2 \nabla^2 \psi + f),$$

$$\text{that is, } \beta^2 J(\psi, \beta^2 \nabla^2 \psi + f).$$

The term $\underline{V}_0' \cdot \nabla \zeta_0'$ becomes

$$- \frac{\beta g}{f} \frac{\partial h_0'}{\partial y} \frac{\beta \partial}{\partial x}\left(\beta^2 \frac{g}{f} \nabla^2 h_0'\right) + \frac{\beta g}{f} \frac{\partial h_0'}{\partial x} \frac{\beta \partial}{\partial y}\left(\frac{\beta^2 g}{f} \nabla^2 h_0'\right),$$

$$\text{that is, } \frac{\beta^2 g}{f} J\left(h_0', \frac{\beta^2 g}{f} \nabla^2 h_0'\right).$$

Performing these operations throughout and clearing a common factor β^2 we finally derive the equation:

$$\nabla^2\left(\frac{\partial \psi}{\partial t}\right) + J(\psi, \beta^2 \nabla^2 \psi + f) + \frac{g}{12f} J\left(\frac{\beta^2 g}{f} \nabla^2 h_0', h_0'\right)$$

$$+ \frac{g}{12f} J\left(\frac{\beta^2 g}{f} \nabla^2 h_1', h_1'\right) = \frac{-1}{2(p_0 - p_1)} \Big\{ - 4b\left(\nabla^2 \psi + \frac{f}{b^2}\right)$$

$$+ \frac{ag}{f}(\nabla^2 h_0' + \nabla^2 h_1') - \frac{bg}{f}(\nabla^2 h_1' - \nabla^2 h_0') \Big\}$$

$$+ \frac{\varpi_0}{2(p_0 - p_1)} \left\{ \frac{g}{8f}(\nabla^2 h_0' + \nabla^2 h_1') + \frac{\bar{f}}{\beta^2} \right\} \qquad \text{..... (52)}$$

This is a Poisson equation for the unknown $\frac{\partial \psi}{\partial t}$.

Equation (52) corresponds to equation (1) in the previously mentioned British Meteorological Office contribution to W.M.O. Technical Note 67 except that $\frac{\partial \zeta_0'}{\partial t}$ and $\frac{\partial \zeta_1'}{\partial t}$ have not been eliminated by using the development equations but remain in the form

$$\frac{g}{f}\,\nabla^2\,\frac{\partial h_0'}{\partial t} \text{ and } \frac{g}{f}\,\nabla^2\,\frac{\partial h_1'}{\partial t}\,.$$

The same procedure is now applied to the development equations (48) and (49).

The term $\underline{V}_m \cdot \nabla \zeta_0'$ in (48) becomes

$$-\,\beta\,\frac{\partial \psi}{\partial y}\cdot \beta\,\frac{\partial}{\partial x}\left(g\,\frac{\beta^2}{f}\nabla^2 h_0'\right) + \beta\,\frac{\partial \psi}{\partial x}\cdot\beta\,\frac{\partial}{\partial y}\left(\frac{g\beta^2}{f}\nabla^2 h_0'\right),$$

that is $\beta^2\, J\left(\psi,\ \frac{g\beta^2}{f}\nabla^2 h_0'\right)$. The term $\underline{V}_0' \cdot \nabla(\zeta_m - \zeta_0' + f)$ becomes

$$-\,\frac{\beta g}{f}\,\frac{\partial h_0'}{\partial y}\,\frac{\beta\partial}{\partial x}\left(\beta^2\,\nabla^2\psi + f - \frac{g\beta^2}{f}\nabla^2 h_0'\right)$$

$$+\,\frac{\beta g}{f}\,\frac{\partial h_0'}{\partial x}\,\frac{\beta\partial}{\partial y}\left(\beta^2\,\nabla^2\psi + f - \frac{g\beta^2}{f}\nabla^2 h_0'\right)$$

which is $\frac{\beta^2 g}{f}\, J\left(h_0',\ \beta^2\,\nabla^2\psi + f - \frac{g\beta^2}{f}\nabla^2 h_0'\right)$.

The corresponding terms in (48) transform similarly into Jacobians.

Collecting terms and clearing a common fact or β^2 the development equations become:

$$\nabla^2\left(\frac{\partial h_0'}{\partial t}\right) + \frac{f}{g}\,J\left(\psi,\ \frac{\beta^2 g}{f}\,\nabla^2 h_0'\right) + J\left(h_0',\ \beta^2\,\nabla^2\psi + f - \frac{\beta^2 g}{f}\,\nabla^2 h_0'\right)$$

$$= -\,\frac{2}{(p_0 - p_1)}\left(a + \frac{\varpi_0}{4} + b\right)\left(2\left(\frac{f}{g}\,\nabla^2\psi + \frac{f^2}{g\beta^2}\right) - \nabla^2 h_0'\right), \quad \text{..... (53)}$$

and
$$\nabla^2\left(\frac{\partial h_1'}{\partial t}\right) + \frac{f}{g}\,J\left(\psi,\ \frac{\beta^2 g}{f}\,\nabla^2 h_1'\right) + J\left(h_1',\ \beta^2\,\nabla^2\psi + f + \frac{\beta^2 g}{f}\,\nabla\, h_1'\right)$$

$$= -\,\frac{2}{(p_0 - p_1)}\left(a + \frac{\varpi_0}{4} - b\right)\left(2\left(\frac{f}{g}\,\nabla^2\psi + \frac{f^2}{g\beta^2}\right) + \nabla^2 h_1'\right) \quad \text{..... (54)}$$

Equations (53) and (54) are equations (2) and (3) of the British Meteorological Office contribution to W.M.O. Technical Note No. 67.

We now substitute for $a + \frac{\varpi_0}{4}$ and b from equations (50) and (51) to obtain the final simultaneous partial differential equations of Helmholtz type for $\frac{\partial h_0'}{\partial t}$ and $\frac{\partial h_1'}{\partial t}$.

Using the notation of Bushby-Whitelam (Ref. 1), extended to include the topography and friction terms, these are

$$\nabla^2\left(\frac{\partial h_0'}{\partial t}\right) + E\left(\frac{\partial h_0'}{\partial t}\right) + F\left(\frac{\partial h_1'}{\partial t}\right) + G = 0 \quad \text{..... (55)}$$

$$\nabla^2\left(\frac{\partial h_1'}{\partial t}\right) + L\left(\frac{\partial h_0'}{\partial t}\right) + M\left(\frac{\partial h_1'}{\partial t}\right) + N = 0 \quad \text{..... (56)}$$

The symbols E, F, G, L, M, N are abbreviations for the expressions defined below:

$$E = \frac{2f(D - C)\left(2\left(\nabla^2\psi + \frac{f}{\beta^2}\right) - \frac{g}{f}\nabla^2 h_0'\right)}{g(p_0 - p_1)(AD - BC)},$$

$$F = \frac{2f(A - B)\left(2\left(\nabla^2\psi + \frac{f}{\beta^2}\right) - \frac{g}{f}\nabla^2 h_0'\right)}{g(p_0 - p_1)(AD - BC)},$$

$$G = \frac{2f\beta^2\left(2\left(\nabla^2\psi + \frac{f}{\beta^2}\right) - \frac{g}{f}\nabla^2 h_0'\right)\left(\frac{H(D - C)}{\beta^2} + \frac{I(A - B)}{\beta^2}\right)}{g(p_0 - p_1)(AD - BC)}$$

$$+ \frac{f}{g} J\left(\psi, \frac{\beta^2 g}{f}\nabla^2 h_0'\right) + J\left(h_0', \beta^2\nabla^2\psi + f - \frac{\beta^2 g}{f}\nabla^2 h_0'\right),$$

$$L = \frac{2f(D + C)\left(2\left(\nabla^2\psi + \frac{f}{\beta^2}\right) + \frac{g}{f}\nabla^2 h_1'\right)}{g(p_0 - p_1)(AD - BC)},$$

$$M = \frac{-2f(A + B)\left(2\left(\nabla^2\psi + \frac{f}{\beta^2}\right) + \frac{g}{f}\nabla^2 h_1'\right)}{g(p_0 - p_1)(AD - BC)},$$

$$N = \frac{2f\beta^2\left(2\left(\nabla^2\psi + \frac{f}{\beta^2}\right) + \frac{g}{f}\nabla^2 h_1'\right)\left(\frac{H(D + C)}{\beta^2} - \frac{I(A + B)}{\beta^2}\right)}{g(p_0 - p_1)(AD - BC)}$$

$$+ \frac{f}{g} J\left(\psi, \frac{\beta^2 g}{f}\nabla^2 h_1'\right) + J\left(h_1', \beta^2\nabla^2\psi + f + \frac{\beta^2 g}{f}\nabla^2 h_1'\right).$$

The balance equation (22) retains its form identically on transformation to stereographic coordinates as

$$\nabla^2\psi = \frac{g}{f}\nabla^2 h_m - \frac{2}{f}\left(\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right) - \frac{\nabla f \cdot \nabla\psi}{f} \qquad \text{..... (57)}$$

XII. Computation of Heating Term Q_0

As already stated the only non-adiabatic heating effect allowed is heating of the air over a sea warmer than the air.

The method of computation uses the thickness of the lower layer as a measure of the air temperature and the rate of heating is taken to be proportional to the difference between an artificial thickness representing the sea temperature and the thickness of the lower layer.

Over the colder seas of the area the formula employed is

$$Q_0 \text{ (metres/hour)} = 2{\cdot}068 \times 10^{-2}\,(h_s' - h_0') \qquad \text{..... (58)}$$

for $h_s' > h_0'$, where

$$h_s'(\text{metres}) = 12{\cdot}97\,(T_s + 282{\cdot}5)\,, \qquad \ldots\ldots (59)$$

in which T_s is the mean monthly sea temperature at the point concerned in degrees Fahrenheit, and Q_0 is zero for $h_s' \leqslant h_0'$. Thus it is supposed that when the sea is warmer than the air it is heating the air at a rate tending to cause the lower thickness to increase at a rate proportional to the difference between the artificial thickness and the actual thickness but if the sign of this difference is negative it is taken to be zero. The heating formulae (58) and (59) were derived by Bushby and Hinds (Ref. 13) who state it is based on the work of Craddock (Ref. 14) and on a statistical survey which they made of the relation between sea surface temperature and 1000/500 mb thickness at Ocean Weather Stations I and J.

The formula (59) is now (1966) suspected of giving excessive heating over sub-tropical seas by giving an artificial thickness which is too high as a measure of the sea temperature. It is thought the formula may be valid only where the artificial thickness it gives is less than the climatological mean thickness. Over sub-tropical seas the thickness computed from formula (59) certainly gives appreciably higher values of thickness than the climatological mean. It is intended to make a trial of using whichever is less of climatological mean thickness and the artificial thickness of formula (59).

XIII. Surface Pressure

The computing system forecasts among other elements heights of the 1000 mb surface. The pressure at mean sea level is obtained by multiplying the (signed) 1000 mb height in decametres by 1·2 and adding 1000 mb. This procedure is equivalent to supposing the mean temperature between the 1000 mb level and sea level is 10°C. The error caused by divergence of the true mean temperature from 10°C is not large since at 30°C the factor is 1·13 and at 0°C it is 1·25.

XIV. Analysis Area

The computations are made over an area of the Earth which is a rectangle on the stereographic plane described in Section II. The North Pole is the origin of rectangular Cartesian coordinates on this plane in the forecast system. The projection is conformal so that angles between directions at a point in the Earth are preserved in the projection. The positive x axis is along the meridian 55°E and the positive y axis along meridian 145°E. The number of grid-points is 47 × 41, the range in x being −21 to +25 and in y, −25 to +15.

The corners of the area are at:

x	y	Lat.	Long.	Country or Sea
+25	−25	5° 35′ 15″ N.	10°E.	Cameroon Republic
+25	+15	16° 25′ N.	85° 57′ 50″ E.	Bay of Bengal
−21	+15	22° 59′ 24″ N.	160° 32′ 15″ W.	Pacific, near Hawaiian Islands
−21	−25	10° 6′ 24″ N.	75° 1′ 42″ W.	Columbia.

The distance between grid-points (grid-length) is 326·68 km at the North Pole decreasing southwards to 179·25 km at the most southerly point in lat. 5° 35′N.

On a chart such as METFORM 2215 which has the specified stereographic projection and a representative fraction of 30 × 10⁶ at lat. 60° the grid-length is 1 centimetre.

XV. Computation of Forecast Values of 1000/500 mb Thickness, 500 and 200 mb Heights and Surface Pressure

The final equations used in forecasting the 1000/600 mb and 600/200 mb thickness and the height of the 600 mb surface are (55), (56), ~~(53)~~, ~~(54)~~ and (52).

The first stage in the procedure is the calculation of the set of initial values of 1000 mb, 600 mb and 200 mb heights and the thicknesses of the two layers. This is the analysis problem which is not considered in this note.

The stages of the solution of the equations are, in outline,

a. Compute the stream function ψ from the 600 mb height field by means of the balance equation after ellipticizing the 600 mb height field as described in Section VII.

b. Compute ϖ_0 and the derivatives of the height and thickness fields such as $\nabla^2 h_0'$. This will provide all the quantities required for computing the coefficients E, F, G, L, M, N of equations ~~(53)~~ and ~~(54)~~ for the rates of change of thickness.
55 56

c. Compute the initial values of E, F, G, L, M, N from initial values of ϖ_0, thickness fields and stream function.

d. Compute rates of change of thickness from equations (55) and (56).

e. Compute a and b from equations (50) and (51).

f. Compute the rate of change of ψ from (52); this requires a and b in addition to the initial values of ψ and the thickness fields.

g. Compute new values of thickness and ψ from initial values and rates of change.

h. When required for output compute values of 1000, 500 and 200 mb height, 1000/500 mb thickness and the mean vertical velocity (excluding the effect of ϖ_0) in the lower layer. The 600 mb heights are computed from the balance equation which, given ψ, is a Poisson equation for h_m. The 500 mb heights are obtained by using formula $h_m = 0{\cdot}8\, h_{500} + 0{\cdot}2\, h_{1000}$.

The balance equation, the equations for rates of change of thickness and the equation for rate of change of stream function are elliptic partial differential equations. In order to solve them it is necessary to prescribe in advance values on the boundary for the whole period of the forecast. The boundary condition used is that there is no change there in heights of isobaric surfaces. The partial differential equations are solved by replacing the derivatives by their finite difference approximations to give a series of linear equations, one of each type for each of the 1755 inner grid-points. These linear equations are solved by an iterative technique which is a combination of the Liebmann and alternating-direction-implicit techniques.

The equations provide rates of change which are added to the initial values to obtain new values used in turn as new initial values. The system is described in Part I of this Memorandum under the heading "Computation of forecast values of height and thickness".

XVI. Smoothing

The isopleths of heights and thickness computed in the manner described are found to contain an unrealistic small-scale waviness caused by mild computational instability. This small-scale waviness is removed by replacing the "raw" values at six-hourly intervals by smoothed values. The smoothed values are a weighted sum of "raw" values at the grid-point and at grid-points distant one and two grid-lengths away in both the x and y directions. The basic theory is explained by Wallington (Ref. 4) and Shuman (Ref. 5). They show how by forming such weighted sums of fields represented by a sum of trigometric functions of position it is possible to eliminate waves of particular wave-length.

Smoothing is applied during the course of the computations as well as to the final output in order to provide smoother sets of coefficients of the equations thereby facilitating the iteration process and minimizing computational instability. The smoothing system used leaves waves of length of more than three grid-lengths scarcely changed while reducing the amplitude of those of shorter wave-length at a rate increasing sharply with decreasing wave-length.

To set out the formulae, suppose that H_0 is the "raw" value to be smoothed and H_{+1}, H_{+2}, H_{-1}, H_{-2} are the corresponding "raw" values at grid-points distant respectively +1, +2, −1, and −2 grid-lengths away in either the x or y directions. The smoothing formulae:

$$H_0' = 3{\cdot}8798\, H_0 - 1{\cdot}77097\, (H_{+1} + H_{-1}) + 0{\cdot}331065\, (H_{+2} + H_{-2}),$$

$$H_0'' = 0{\cdot}375\, H_0' + 0{\cdot}25\, (H_{+1}' + H_{-1}') + 0{\cdot}625\, (H_{+2}' + H_{-2}'),$$

are applied successively. Each is applied first in the y direction and then in the x direction The final smoothed value is H_0'' obtained after the x direction smoothing.

The formulae clearly cannot be applied at grid-points on rows and columns adjacent to the boundary since some points of type + 2 and − 2 would be outside the area. Values on the boundary are fixed in any case. The method used for smoothing at points one grid-length inside is to substitute $2H_{+1} - H_0$ for H_{+2} if +2 gives a point outside the area and $2H_{-1} - H_0$ for H_{-2} if − 2 gives a point outside.

Appendix I

SYMBOLS

General

ϕ	latitude
f	Coriolis parameter
g	acceleration of gravity
$\underline{k}$	vertical vector
γ	dry-adiabatic lapse rate
$\frac{d}{dt}$	differentiation following the motion
r, s	orthogonal curvilinear coordinates on the Earth
x, y	horizontal coordinates on a stereographic projection

Pressure

General, p 1000 mb, p_0 600 mb, p_m 200 mb, p_1

$$\alpha = \frac{p_0 + p_1 - 2p}{p_0 - p_1}$$

, representation of pressure in the mathematics

Height and Thickness

General, h 1000 mb, h_0 600 mb, h_m

h'_0	1000/600 mb thickness
h'_1	600/200 mb thickness
h'_s	Thickness of 600/1000 mb layer representative of sea temperature
Z	Top of friction layer
H	Smoothed height, Topography.

Horizontal velocity

$\underline{V}$	Vector general
$\underline{V}_m$	at 600 mb
u_m, v_m	components
ψ	stream function at 600 mb

Horizontal velocity (contd.)

$\underline{V}_g$, Geostrophic wind of 1000 mb surface

u_g, v_g components

V_g speed

u_0, v_0 wind at ground level components

u_T, v_T Thermal wind, general components

$\underline{V}'_0$ Vector Thermal wind 1000/600 layer

$\underline{V}'_1$ Vector Thermal wind 600/200 layer

Vertical velocity

$\frac{dp}{dt}$, ϖ. ϖ at 1000 mb. ϖ_0

Value of ϖ at 1000 mb due to topography ϖ_{0t}

Value of ϖ at 1000 mb due to surface friction ϖ_{0f}

a, b coefficients in expressions for ϖ as function of a

Friction and Topography

$K = u_0 - u_g$,

$B = v_0 - v_g$

A Austausch coefficient

$q^2 = \frac{\rho f}{2A}$

C ratio of ground level wind speed to geostrophic speed

θ angle between geostrophic and ground level wind

χ angle between geostrophic wind and axis Or

Vertical component of Vorticity

ζ General

ζ_m 600 mb level

ζ'_0 ζ of 1000/600 mb thermal wind

ζ'_1 ζ of 600/200 mb thermal wind

Lapse Rate and Heating

R Gas constant

$\left(\frac{\gamma}{g\rho} - \frac{\partial T}{\partial p}\right) = \Gamma$ Difference between dry adiabatic and actual lapse rate

Γ_0 Constant Γ layer 1000/600 mb

Γ_1 Constant Γ layer 600/200 mb

Lapse Rate and Heating (contd.)

Q_0 Rate of heating of lower layer over warmer sea

T_s Mean monthly temperature of sea surface

Constant Coefficients in Basic Rate of Change of Thickness Equations

A, B, E′, C, D, F′

E′ and F′ are denoted by E_{FTB} and F_{FTB} in some working papers

Variable Coefficients in Helmholtz equations for Rates of Change of Thickness

E, F, G, L, M, N

Appendix II

Table of Numerical Values, Constants and Coefficients

The units of the basic elements of length, time and temperature used in the operational system are metres, hours, and degrees Celsius. Pressure is expressed in millibars. The dimensions of the quantities are included. The dimensions of absolute temperature are written [K] and of pressure [P]. The numerical values are those in use in the programme in December 1965.

Quantity	Symbol	Dimensions	Numerical value
Acceleration of gravity	g	$[L]\ [T^{-2}]$	Constant at 127 137 600 m/hr^2
Gas Constant	R	$[L^2]\ [T^{-2}]\ [K^{-1}]$ or $[P]\ [L^3]\ [M^{-1}]\ [K^{-1}]$	3719908800 $m^2/hr^2\ {}^\circ K$ (equivalent to $2{\cdot}8703 \times 10^6$ c.g.s. units or $2{\cdot}8703 \times 10^3$ mb/(grams/cm^3) °K)
Lapse-rate	Γ	$[K]\ [P^{-1}]$ or $[K]\ [L]\ [T^2]\ [M^{-1}]$	
Lower layer value	Γ_0		$-0{\cdot}0422222222$ °C/mb (equivalent to −38°C/ 900 mb)
Upper layer value	Γ_1		$-0{\cdot}0511111111$ °C/mb equivalent to −46°C/ 900 mb)
Multipliers	$\frac{R\Gamma}{g}$	$[L]\ [P^{-1}]$ or $[L^2]\ [T^2]\ [M^{-1}]$	
Lower layer value	$\frac{R\Gamma_0}{g}$		$-1{\cdot}23538$ m/mb
Upper layer value	$\frac{R\Gamma_1}{g}$		$-1{\cdot}49548$ m/mb

Friction Constants

Austausch Coefficient	A	$[M]\ [L^{-1}]\ [T^{-1}]$	160 grams/cm.sec.
Density	ρ	$[M]\ [L^{-3}]$	Constant at $1276{\cdot}2$ grams/m^3
Coefficient	$\frac{g\rho}{4}$	$[P]\ [L^{-1}]$	$0{\cdot}0313$ mb/m
Coefficient	$g\sqrt{\frac{A\rho}{2}}$	$[P]\ [T^{-\frac{1}{2}}]$	$18{\cdot}8076$ mb/hour$^{\frac{1}{2}}$

Friction Constants (contd.)

Coefficient, F_1 $c(\cos\theta - \sin\theta) - 1$	Pure number	−0·87189111 (land) −0·195650725 (sea)
Coefficient, F_2 $c(\cos\theta + \sin\theta)$	Pure number	0·47811 (land) 0·89332 (sea)
Coefficient $F_1 g\sqrt{\frac{A\rho}{2}}$		−16·4 (land) −3·686 (sea)
Difference, land-sea $\frac{F_1 g}{4}\sqrt{\frac{A\rho}{2}}$		−3·178
Difference, land-sea $\frac{F_2 g}{4}\sqrt{\frac{A\rho}{2}}$		−1·951

Coefficients in Equations for Rate of Change of Thickness

A	$[L]\ [P^{-1}]$	−0·4465490669 m/mb
B	"	−0·1042706009 m/mb
E'	"	0·3483 m/mb
C	"	−0·9372539997 m/mb
D	"	0·02632612258 m/mb
F'	"	−0·0477 m/mb
$-\frac{(A+B)}{(C+D)}$	Pure number	−0·8172486215
$\frac{(D-C)}{(A-B)}$	"	−3·5074226111
$\frac{A}{(AD-BC)}$	$[P]\ [L^{-1}]$	2·0742026939 mb/m
$\frac{B}{(AD-BC)}$	"	0·4843328014 mb/m
$\frac{C}{(AD-BC)}$	"	4·3535076230 mb/m
$\frac{D}{(AD-BC)}$	"	−1·2228379473 mb/m
$\frac{(A+B)}{(AD-BC)}$	"	2·5585354924 mb/m
$\frac{A-B}{(AD-BC)}$	"	1·5898698925 mb/m
$\frac{C+D}{(AD-BC)}$	"	3·1306696683 mb/m
$\frac{C-D}{(AD-BC)}$	"	5·5763455778 mb/m

The coefficients A, B, C, D, and the derived ones $-\dfrac{(A + B)}{(C + D)}$ to $\dfrac{(C - D)}{(AD - BC)}$

were computed on METEOR by an Autocode programme written by Mr D.E. Jones.

Geographical Constants and Coefficients

Grid length at North Pole a =	326·683635 km
	= 202·992 statute miles
	= 176·2842 nautical miles
Radius of Earth R	6367·3 km
Coefficient $\dfrac{a^2}{8R^2}$	$3{\cdot}2901007 \times 10^{-4}$
Coefficient $\dfrac{\beta g}{af}$	$\dfrac{1486{\cdot}5}{\sin\phi(1 + \sin\phi)}$ hr^{-1}
Coefficient $\dfrac{2R}{a}$	38·983

Formulae for latitude ϕ and longitude θ of grid points:

$$(1 + \sin\phi)^{-1} = 0{\cdot}5 + \frac{a^2(x^2 + y^2)}{8R^2}$$

$$\tan(\theta^\circ - 55^\circ) = \frac{y}{x}$$

Magnification factor Earth to Stereographic plane

$$\beta \doteq \left(\frac{2}{1 + \sin\phi}\right)$$

Grid length on stereographic chart of scale 30×10^6 at lat. 60°N (METFORM 2215): 1 cm

Height gradient ΔH per grid length for geostrophic wind to exceed one grid length time-step of q hours (useful in considering stability)

$$\Delta H > \frac{4\pi a^2 \sin\phi(1 + \sin\phi)^2}{24\, q\, g\, 4}$$

metres/grid length

On a stereographic projection of scale 30×10^6 at lat. 60°N on which grid length is 1 cm, the above inequality is equivalent to an interval between 6 dkm contours of less than

$$\frac{6q}{11 \sin\phi(1 + \sin\phi)^2} \text{ cm.}$$

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