THE BALANCE EQUATION

1. The present method of solution

The current operational technique for solving the balance equation

$$2\left(\psi_{xx}\psi_{yy}-\psi_{xy}^2\right)+f\nabla^2\psi+\nabla\psi\cdot\nabla f-g\nabla^2 h=0 \qquad \ldots\ldots (1)$$

is to write it in the form

$$\nabla^2\psi = F(\psi) \equiv -4f+\Big[(\nabla^2\psi)^2-16\left(\psi_{xx}\psi_{yy}-\psi_{xy}^2\right)$$
$$+12f^2+8g\Big\{G+\varepsilon+\frac{1}{f}\nabla h\cdot\nabla f-\frac{1}{g}\nabla\psi\cdot\nabla f\Big\}\Big]^{1/2} \qquad \ldots\ldots (2)$$

and solve it using the iterative scheme

$$\nabla^2\psi_{n+1}=(1-a)\nabla^2\psi_n+a\,F(\psi_n) \qquad \ldots\ldots (3)$$

ψ is the stream function, h is the ellipticised pressure surface contour height satisfying the approximate ellipticity condition.

$$G\equiv\nabla^2 h-\frac{1}{f}\nabla h\cdot\nabla f+f^2/2g-\varepsilon>0 \qquad \ldots\ldots (4)$$

and the positive number ε is chosen so that, on all conceivable synoptic conditions,

$$\varepsilon>\frac{1}{g}\nabla\psi\cdot\nabla f-\frac{1}{f}\nabla h\cdot\nabla f \qquad \ldots\ldots (5)$$

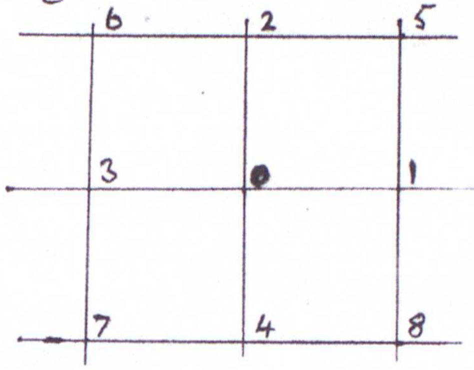
i.e. so that the true ellipticity condition

$$\nabla^2 h-\frac{1}{g}\nabla\psi\cdot\nabla f+f^2/2g>0 \qquad \ldots\ldots (6)$$

is satisfied. The parameter a in (3) is usually taken to be 0.6, but is reduced to 0.4 if the solution diverges, and alternating direction implicit (A.D.I.) methods are used to solve the Poisson equation at each iteration. The finite difference representations adopted for the second order differentials are

$$\nabla^2\psi = (\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0)/d^2 \qquad \text{........(7)}$$

$$(\psi_{xx}\psi_{yy} - \psi_{xy}^2) = \{(\psi_5 + \psi_7 - 2\psi_0)(\psi_6 + \psi_8 - 2\psi_0) - (\psi_2 - \psi_1 - \psi_3 + \psi_4)^2\}/4d^4$$



Note that, for consistency, the terms in the expression

$(\psi_{xx}\psi_{yy} - \psi_{xy}^2)$ are all evaluated on a diagonal grid with all first order differences taken over a uniform grid length of $\sqrt{2}\,d$. For complete consistency the term $\nabla^2\psi$ ought to be evaluated in the same way but such a procedure leads to the separation of the solution into two nearly independant fields. A grid length of d is therefore used for this term.

While using the above method in Met O 11 (in the initialisation procedure for the ten level model) we have come across a difficulty, namely that the expression under the square root may become negative. This is readily seen to be possible since on rearrangement $F(\psi)$ takes the form

$$F(\psi) = -4f + \Big[-3(\nabla^2\psi)^2 + 4(\psi_{xx} - \psi_{yy})^2 + 16\psi_{xy}^2 + 12f^2 + 8g\Big\{G + \mathcal{E} + \frac{1}{f}\nabla h.\nabla f - \frac{1}{g}\nabla\psi.\nabla f\Big\}\Big]^{1/2}$$

and it is clear that negative values of the radicand are bound to arise when the relative vorticity $\nabla^2\psi$ is too large. This is surprising since it is precisely in such regions of large cyclonic vorticity that the equation is most elliptic and, one would think, most easily solved.

The procedure adopted by Met O 2b when this happens during an operational forecast is to set the square root term equal to zero. However on some occasions the method does not then converge (even if it does the stream function ψ so obtained is not the solution of the balance equation (1)). Operationally the computer is programmed to stop the calculation as soon as its starts to diverge and use the last iterate as the best available estimate for ψ. It must be emphasised that this procedure is entirely incorrect, since it is clear from (2) that setting the square root equal to zero (in moderately strong cyclonic regions) is equivalent to setting the relative vorticity equal to $-4f$, a (totally unrealistic) anticyclonic value. The adoption of this practise in conjunction with the primitive equations will almost certainly be disast~~e~~rous and it probably introduces serious errors in a balanced vorticity model. What we require is a form of the equation which does not have negative numbers under the square root if the contour field is elliptic.

2. The modified scheme

A more suitable method is one which is essentially the same as that described by Bushby and Huckle (1956). Instead of (7) we adopt the finite difference approximations

$$\nabla^2\psi = (\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0)/d^2$$

$$\begin{aligned}(\psi_{xx}\psi_{yy} - \psi_{xy}^2) &= (\psi_1 + \psi_3 - 2\psi_0)(\psi_2 + \psi_4 - 2\psi_0)/d^4 \qquad \text{..........(8)} \\ &\quad - (\psi_5 - \psi_6 - \psi_8 + \psi_7)^2/16d^4\end{aligned}$$

It will be noted that the terms in $\psi_{xx}\psi_{yy} - \psi_{xy}^2$ are inconsistent since ψ_{xx} and ψ_{yy} are evaluated on a grid length d and ψ_{xy} on a grid length $2d$. The balance equation may now be written

$$\nabla^2\psi = -f + \Big[\frac{1}{d^4}(\psi_2 - \psi_1 - \psi_3 + \psi_4)^2 + \frac{1}{4d^4}(\psi_5 - \psi_6 - \psi_8 + \psi_7)^2$$

$$+ 2g \Big\{ G + \mathcal{E} + \frac{1}{f}\nabla h . \nabla f - \frac{1}{g}\nabla\psi . \nabla f \Big\} \Big]^{1/2} \qquad \text{......(9)}$$

and this may be solved using the same iterative scheme (3). Clearly negative values under the square root never occur if the approximate ellipticity condition (4) (together with (5)) is satisfied. However they may still arise if the first guess for ψ is a poor one, since (5) may be violated to such an extent that condition (6) is not satisfied. Nevertheless they are unlikely if the first guess is obtained from the linear balance equation

$$f\nabla^2\psi + \nabla\psi . \nabla f - g\nabla^2 h = 0$$

If negative radicands do occur then probably the correct operational procedure would be to re-ellipticise the contour field using the current estimate for ψ in (6).

It has been argued that the inconsistent finite difference approximation for $\psi_{xx}\psi_{yy} - \psi_{xy}^2$ in (8) introduces errors which are not compensated elsewhere however, as has already been mentioned, the current operational scheme is itself not completely consistent, and it is difficult to say which method contains the worse inconsistency

From our limited experience in the use of (9) we have found that (i) no negative numbers under the square root have arisen (even for cases when (2) does give negative radicands), (ii) the method has always converged (even when (2) has not), (iii) when both (9) and (2) converge the solutions are almost identical (winds differ by about 1 m/sec) and (9) seems to converge slightly more rapidly than (2).

3. References

Bushby F. H. and Huckle V. M. 1956 'The use of a stream function in a two-parameter model of the atmosphere'. Quart.J. R.Met.Soc., 82, pp409-418.