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A COMPARISON OF THE CONFLICTING WIND-STRESS RELATIONSHIPS FORMULATED BY LETTAU AND BY SWINBANK ON THE BASIS OF THE LEIPZIG WIND PROFILE

by

D J CARSON and F B SMITH

Boundary Layer Research
Met O 14
Meteorological Office HQ
Bracknell

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TABLE OF NOTATION

$A(z)$		scalar austausch coefficient, ρK
A_i		$i = 1, 2, 3$. Particular representations of the austausch coefficient $A(z)$
$B(z)$		Used as a scalar coefficient in the statement of Swinbank's hypothesis, $\underset{\sim}{\tau}(z) = B(z)\, \underset{\sim}{V}(z)$
f		Coriolis parameter, $f = 1.14 \times 10^{-4}\ \mathrm{sec}^{-1}$ at Leipzig
g		acceleration due to gravity, $g = 980.665\ \mathrm{cm\ sec}^{-2}$ at Leipzig
h		the height of the boundary layer defined so that, $u \longrightarrow Vg,\ v \longrightarrow 0,\ \tau_x \longrightarrow 0,\ \tau_y \longrightarrow 0$ as $z \longrightarrow h$
H		used as a depth over which the errors ε_L and ε_s are applicable.
$\underset{\sim}{k}$		unit vector along the z - axis
$K(z)$		diffusivity for momentum, $A = \rho K$
L	(i)	used as a temperature lapse rate
	(ii)	as a subscript it refers to the Leipzig Wind Profile (LWP)
p		pressure
R		gas constant for dry air
$R_1(z)$		$f \int_0^z \rho u \, dz$
$R_2(z)$		$f \int_0^z \rho v \, dz$
s		$\lvert \underset{\sim}{V} - \underset{\sim}{Vg} \rvert$
$T(z)$		temperature, $^{\circ}K$
u		wind velocity component in the direction of the geostrophic wind
v		wind velocity component perpendicular to $\underset{\sim}{k}$ and the geostrophic wind.
$\underset{\sim}{V}$		wind vector in the horizontal plane
V		$\lvert \underset{\sim}{V} \rvert$
$\underset{\sim}{Vg}$		$\frac{1}{\rho f} (\underset{\sim}{k} \times \nabla p)$, the geostrophic wind vector.
Vg		$\lvert \underset{\sim}{Vg} \rvert = \frac{1}{\rho f} \left\lvert \frac{\partial p}{\partial y} \right\rvert$

z	height above ground
z_o	surface roughness length
z_1	height where v has a local maximum value
z_2	height where u has a local maximum value
z^*	height at which $v = 0$, $u \neq 0$.
$\alpha(z)$	angle between the wind vector and the geostrophic wind
$\beta(z)$	angle between the vertical gradient of the wind, $\frac{\partial \underset{\sim}{V}}{\partial z}$, and the geostrophic wind
δ	deviation of the wind from its direction as determined by the LWP, $\delta = \alpha - \alpha_L$
Δ	$-\frac{\partial \theta}{\partial z} \tau \tan(\alpha - \theta)$
ε_L	$\varepsilon_L(0\text{-}H) = \left\{\frac{1}{H}\int_0^H (\beta - \theta)^2 \, dz\right\}^{\frac{1}{2}}$
ε_s	$\varepsilon_s(0\text{-}H) = \left\{\frac{1}{H}\int_0^H (\alpha - \theta)^2 \, dz\right\}^{\frac{1}{2}}$
η	depth below the top of the boundary layer, strictly $\eta = h - z$; this is sometimes equivalent to $(z^* - z)$
$\theta(z)$	angle between the stress vector and the geostrophic wind
ρ	air density
$\underset{\sim}{\tau}$	the shearing stress
τ	$\lvert\underset{\sim}{\tau}\rvert$
τ_x	component of shearing stress parallel to the geostrophic wind
τ_y	component of shearing stress perpendicular to $\underline{k}$ and the geostrophic wind
τ_s	$\tau_s(z) = \left\lvert\frac{\partial p}{\partial y}\right\rvert \int_z^h \sin\alpha \, dz$, Swinbank's formula for the magnitude of the shearing stress.
ψ	$\tan^{-1}\left(\frac{-v}{u - Vg}\right)$

Note: Throughout this paper it will, in general, be convenient to express angular measurements in terms of degrees, whereas in the mathematical and computational formulae it is to be understood that the angles are required in radian measure

1. Introduction

In the atmospheric boundary layer many basic questions remain unanswered to any reasonable degree of satisfaction and, in particular, the variation of the wind vector with height and its relation to the shearing stress have not been resolved. One of the main difficulties has been the lack of good observational data so essential for tackling such problems. As a result of this the careful and detailed wind measurements made by Mildner on 20 October 1931 at Leipzig have provided, until recently (see eg Clarke (1970)), the most reliable data for well-monitored, steady conditions.

The Leipzig data was obtained from a series of twenty-eight double theodolite pilot balloon ascents made during a seven-hour period (0915 - 1615 GMT), the wind field was stated to be steady and uniform and the measured mean lapse rate throughout the layer of 0.65°C per 100 m indicated that the air was slightly stable. A full account of the observational study was given by Mildner (1932).

Further analyses of these data, by Lettau (1950) and by Swinbank (1970) have led to the formulation of two conflicting hypotheses for the relationship between the wind vector and the associated shearing stress throughout the boundary layer. The hypotheses are:

(i) The stress vector is parallel to the vertical shear of the wind vector (Lettau). This relationship is the classical, semi-empirical, flux-gradient relationship which, although somewhat supported by physical concepts, has never been adequately verified in the atmosphere. Indeed Taylor (1963) has advanced theoretical reasons to indicate that the assumption is probably not valid and he suggests that in our present state of knowledge it cannot be relied upon.

(ii) The stress vector is parallel to the wind vector (Swinbank). This hypothesis is based entirely on a study of the Leipzig data and at present there is no theoretical justification to support it. Swinbank's formulation has appeared in a section dealing with the determination of the flow field near the top of the Ekman layer, in a study of the atmospheric boundary layer by Plate (1971). It is clearly stated that Swinbank's assumption remains questionable.

The situation we are faced with is that of having two conflicting hypotheses, neither of which has been adequately verified in the atmosphere, both claiming support from the same data. The present study is designed to provide an objective assessment of the two theories and to decide which, if either, provides the more consistent representation in relation to the Leipzig data. In order to provide a complete study the procedure for each hypothesis will follow the pattern:

(i) Summary of the original analysis.

(ii) Criticism of the original analysis.

(iii) Reanalysis and objective assessment.

2. Lettau's Analysis of the Leipzig Data

Let us assume that we have a horizontally uniform, atmospheric boundary layer under conditions of unaccelerated motion and in which we may ignore all vertical velocity components. Horizontal homogeneity is assumed for all variables except pressure which is assumed to have a constant horizontal gradient. Under such conditions the general equation for the horizontal mean motion in the atmospheric boundary layer reduces to

$$\frac{\partial \underset{\sim}{\tau}}{\partial z} = -f\varrho\,(\underset{\sim}{V} - \underset{\sim}{V}_g) \times \underset{\sim}{k} \quad , \tag{1}$$

where,

$$\underset{\sim}{V}_g = \frac{1}{\varrho f}\left(\underset{\sim}{k} \times \nabla p\right)$$

defines the geostrophic wind vector at any height z.

Lettau adopts the classical mixing length concept of the flux-gradient relationship,

$$\underset{\sim}{\tau} = A \frac{\partial \underset{\sim}{V}}{\partial z} \tag{2}$$

where he defines A (= ρ K) as a scalar austausch coefficient, such that

$$A(z) > 0 \text{ for all } z > 0.$$

He also assumes that the air density and the geostrophic wind are independent of height, ie there is no thermal wind component between any two levels in the boundary layer. Right-handed axes are chosen such that the x-direction is parallel to $\underline{V}g$ and so the component equations for the motion become,

$$\frac{\partial \tau_x}{\partial z} = -f\rho v , \tag{3}$$

$$\frac{\partial \tau_y}{\partial z} = f\rho (u - V_g) , \tag{4}$$

where $$V_g = -\frac{1}{\rho f} \cdot \frac{\partial p}{\partial y} = \frac{1}{\rho f} \left| \frac{\partial p}{\partial y} \right| , \tag{5}$$

with the surface boundary conditions that,

(i) the stress vector $\underset{\sim}{\tau}(z)$ is non-zero and parallel to the limiting direction of the wind vector $\underline{V}$ as $z \longrightarrow 0$ (strictly this should be as $z \longrightarrow zo$, but we shall not wish to distinguish between the surface and zo throughout this study).

(ii) $\underline{V}(0) = 0$ and $0 < \alpha(0) < \pi/2$,

where $\alpha(z) = \tan^{-1}(v/u)$

and $\alpha(0) = \lim_{z \to 0} [\tan^{-1}(v/u)]$.

If the wind hodograph has the characteristics of the Ekman spiral, then equation (2) implies the existence of two significant levels, z_1 and z_2, such that

τ_y = 0 and v has a local maximum value at $z = z_1$,

and τ_x = 0 and u has a local maximum value at $z = z_2$.

The above equations and boundary conditions yield, after integration,

$$\tau_x = \tau_x(0) - f\rho\int_0^z v\,dz = A\,\frac{\partial u}{\partial z}\,, \tag{6}$$

$$\tau_y = \tau_y(0) + f\rho\int_0^z (u - V_g)\,dz = A\,\frac{\partial v}{\partial z}\,, \tag{7}$$

and, in particular,

$$\tau_x(0) = f\rho\int_0^{z_2} v\,dz\,, \tag{8}$$

$$\tau_y(0) = -f\rho\int_0^{z_1} (u - V_g)\,dz\,. \tag{9}$$

One of the principal shortcomings of the Leipzig data is the lack of reliable measurements of the magnitude and the direction of the geostrophic wind. In order to obtain the best possible estimate of α (0) Lettau chooses four different values, spread about a value estimated from a synoptic weather map, and carries out the following procedure.

(i) The tentative value of α (0) and the Leipzig wind data yield u, v, z_1, z_2.

(ii) $\tau_x(0)$ is computed from equation (8) and $\tau_y(0)$ is obtained from the boundary condition,

$$\tau_y(0) = \tau_x(0)\tan\alpha(0)\,.$$

(iii) Equation (9) gives a geostrophic wind value

$$V_g = \frac{1}{z_1}\left\{\int_0^{z_1} u\,dz + \frac{\tau_y(0)}{f\rho}\right\}. \tag{10}$$

(iv) The stress profiles, $\tau_x(z)$, $\tau_y(z)$, are obtained from equations (6) and (7).

All the computations are carried out for successive 50 m-levels with the aid of graphical-numerical integrations. The scalar austausch theory implies

$$A = \frac{\tau_x}{u'} = \frac{\tau_y}{v'} \quad \text{for all values of } z,$$

where ' denotes differentiation with respect to z,

$$\text{ie} \quad E = \frac{\tau_y}{v'} - \frac{\tau_x}{u'} \equiv 0 \quad \text{for all values of } z \ . \tag{11}$$

Lettau appeals to the above result in order to choose his most suitable value for $\alpha(0)$. Height averages of E between 50 and 400 m are evaluated and the $\alpha(0)$ which corresponds to the minimum mean value is considered to give the best value.

This method leads Lettau to choose $\alpha(0) = 26.1^{\circ}$ for which value equation (10) gives Vg = 17.51 m sec^{-1}, which is in very close agreement with Vg deduced from Mildner's measured value of the pressure gradient. The corresponding wind components, (u_L, v_L), parallel to and perpendicular to the geostrophic wind direction form Lettau's "representative" wind profile. This profile is given from 50 - 950 m at 50 m intervals in Table 1 and is generally referred to in the literature as the Leipzig Wind Profile (LWP). The wind and corresponding stress hodographs are shown in Figures 1 and 4.

As a consequence of his method Lettau is able to evaluate A(z) from three different relationships. The expressions are quoted here for completeness sake and their full derivations are given by Lettau (1950).

$$A_1 = \frac{1}{V^2\alpha'}\int_0^z \rho f \left(V^2 - V V_g \cos\alpha\right) dz , \tag{12}$$

$$A_2 = \tau \Big/ \left|\frac{\partial \underline{V}}{\partial z}\right| , \tag{13}$$

and $$A_3 = \left(\tau(0) V_g \sin\alpha(0) - f\rho\int_0^z s^2\, dz\right) \Big/ \left(s^2 \frac{\partial}{\partial z}\psi\right) , \tag{14}$$

where $$s = \left|\underline{V} - \underline{V_g}\right|$$

and $$\tan\psi = \frac{-v}{u - V_g} .$$

The third of these expressions makes use of the fact that since $\underline{V}g' = 0$ then

$$\underset{\sim}{\tau} = A(\underline{V} - \underline{V}g)', \tag{15}$$

and so the introduction of a thermal wind would invalidate A_3. From the LWP Lettau produces profiles of A_1, A_2 and A_3 and a representative average of the austausch distribution as an arithmetic average of those three. Certain of these results are illustrated in Figure 6.

3. Criticism of Lettau's Analysis

Lettau's hypothesis and assumptions are quite straightforward and have the distinct advantage of not requiring any assumptions at the "top" of the boundary layer (at such a top we might expect $u \longrightarrow V_g$, $v \longrightarrow 0$, $\tau_x = \tau_y \longrightarrow 0$). However, the application of his methods requires certain refinements.

(a) A criticism raised against Lettau's theory by Swinbank is his assumption that the atmosphere was essentially barotropic throughout the period which resulted in the LWP. If instead we assume that the horizontal pressure gradient is constant throughout the layer, as stated by Mildner, we can use the measured, slightly stable, constant lapse rate of 0.65°C per 100 m and an estimate of the surface temperature, $T(0)$, to give a density profile through the layer. From the definition of $\underset{\sim}{V_g}$ it is noted that the geostrophic wind will now increase in magnitude throughout the layer but will remain constant in direction such that,

$$\begin{aligned} \rho \underset{\sim}{V_g} &= \rho(0)\, \underset{\sim}{V_g}(0) \\ \text{and} \quad \rho^2 \underset{\sim}{V_g'} &= -\rho(0)\, \underset{\sim}{V_g}(0)\, \rho' \,. \end{aligned} \tag{16}$$

Swinbank suggests from his profiles a geostrophic wind increase from 17.5 m sec^{-1} at the surface to 19.1 m sec^{-1} at 950 m and this implies the density stratification of Table 1, given by

$$\rho = \rho(0) \left[1 + \frac{Lz}{T(0)} \right]^{-(1 + g/RL)}, \tag{17}$$

where $L = -0.65\ ^{o}C/100\ m = -6.5 \times 10^{-5}\ ^{o}C\ cm^{-1}$

$g = 980.665\ cm\ sec^{-2}$,

$R = 2.87 \times 10^{6}\ erg\ gm^{-1}\ (deg\ K)^{-1}$,

$\rho(0) = 1.25 \times 10^{-3}\ gm\ cm^{-3}$,

and $T(0) \div 291.5\ ^{o}K$.

With these values equation (17) reduces to

$$\rho = \rho(0)\ (1 - 2.23 \times 10^{-5} z)^{4.256}, \qquad (18)$$

where z is in metres.

This thermal wind effect will be included in the reanalysis using Lettau's hypothesis, the choice of axes remains the same but the density variation with height must now be allowed for in all equations. In particular equation (15) will no longer be true and A_3 as defined in equation (14) will no longer give an estimate of $A(z)$.

(b) Lettau points out that the relation for A_1 is quite sensitive to small changes in the direction assumed for $\underset{\sim}{V}_g$ and that $\underset{\sim}{V}_g$ is not determined accurately enough from the synoptic charts. The method applied to obtain the best value of $\alpha(0)$ to fit the theory appears to be rather too crude to do it justice, in particular;

(i) Too few values of $\alpha(0)$ were considered.

(ii) E is not the best parameter for indicating the error in assuming the parallelism between $\underset{\sim}{\tau}$ and $\underset{\sim}{V}'$. This parameter is particularly difficult to evaluate in the immediate neighbourhoods of z_1 and z_2 where v' and u', respectively, tend to zero.

(iii) The averaging process was applied over 50 - 400 m which is only half the range for z, however the profiles finally adopted are given up to 950 m.

A method based on evaluating the root-mean-square angular deviation between $\underset{\sim}{\tau}$ and $\underset{\sim}{V}'$ for a given $\alpha(0)$ will be employed in the reanalysis which follows.

(c) Lettau's method of deteriming z_1 and z_2 is rather inaccurate so that the expected, almost linear, increase of z_1 and z_2 as $\alpha(0)$ is increased is not evident in his tabulated values. We can obtain z_1, z_2 as continuous functions of $\alpha(0)$ from the LWP where $\alpha_L(0) = 26.1^{\circ}$.

If at height z' in the LWP $$\frac{\partial v_L}{\partial u_L} = -\tan\delta ,$$

and at height z'' $$\frac{\partial u_L}{\partial v_L} = \tan\delta ,$$

then when the axes are rotated such that $\alpha = \alpha_L + \delta$, we have for the new coordinates (u, v),

$$\frac{\partial v}{\partial u} = 0 \quad \text{at } z = z'$$

and

$$\frac{\partial u}{\partial v} = 0 \quad \text{at } z = z'',$$

i.e. z', z'' are z_1, z_2 respectively for the profile with $\alpha(0) = \alpha_L(0)+\delta$. By measuring δ in the neighbourhood of z_{1L}, z_{2L} we can find z_1, z_2 as continuous functions of $\alpha(0)$. See Figure 2 and Table 2.

4. Re-analysis Based on Lettau's Hypothesis

The main changes introduced are the thermal wind effect suggested by Swinbank and a general refinement of the averaging technique for determining the best $\alpha(0)$ which fits the theory. The basic data are the LWP, (u_L, v_L), and the density profile $\rho(z)$ of Table 1. The reanalysis is carried out for twelve values of $\alpha(0)$ in the range $22.7^{\circ} \leq \alpha(0) \leq 29.0^{\circ}$. If we write $\alpha(0) = \alpha_L(0)+\delta$ where $\alpha_L(0) = 0.4555$ radians (26.1 degrees) then the values for $\alpha(0)$ are obtained by taking $\delta = -0.06\ (0.01)\ 0.05$ radians. Profiles corresponding to each $\alpha(0)$ are obtained from the LWP components such that

$$u = u_L \cos\delta - v_L \sin\delta , \tag{19}$$

$$v = u_L \sin\delta + v_L \cos\delta , \tag{20}$$

with similar expressions for u', v'.

We define

$$R_1(z) = f\int_0^z \rho u \, dz = R_{1L}(z)\cos\delta - R_{2L}(z)\sin\delta , \tag{21}$$

$$R_2(z) = f\int_0^z \rho v \, dz = R_{1L}(z)\sin\delta + R_{2L}(z)\cos\delta , \tag{22}$$

where $R_{1L}(z)$, $R_{2L}(z)$ are obtained from the LWP by graphical integration, the values being given in Table 1. In order to increase the number of steps available for numerical integration the LWP was interpolated to give values at 25 m intervals.

Direct measurement of the gradients gives u'_L, v'_L implying $\beta_L = \tan^{-1}\left(\frac{v'_L}{u'_L}\right)$ which was compared with $\beta_L = \tan^{-1}\left(\frac{dv_L}{du_L}\right)$ measured from the wind hodograph.

For each $\alpha(0)$ in the specified range the following procedure is followed.

(i) Knowing δ, the Leipzig data yields the profiles $u, v, u', v', V = |\underline{V}|$.

z_1, z_2 are obtained from Figure 2 and $R_{1L}(z_i)$, $R_{2L}(z_i)$, where i = 1, 2, from interpolation of the values in Table 1.

(ii)
$$\tau_x(0) = R_2(z_2) = R_{1L}(z_2)\sin\delta + R_{2L}(z_2)\cos\delta , \tag{23}$$
$$\tau_y(0) = \tau_x(0)\tan\alpha(0) , \tag{24}$$
$$\tau(0) = |\underset{\sim}{\tau}(0)| .$$

(iii) The pressure gradient is obtained from equations (5) and (9),

$$\begin{aligned} \left|\frac{\partial p}{\partial y}\right| &= \frac{1}{z_1}\left(\tau_y(0) + R_1(z_1)\right) \\ &= \frac{1}{z_1}\left(\tau_y(0) + R_{1L}(z_1)\cos\delta - R_{2L}(z_1)\sin\delta\right) , \end{aligned} \tag{25}$$

and the variation of the geostrophic wind speed with height from the definition,

$$V_g = \frac{1}{\rho f}\left|\frac{\partial p}{\partial y}\right| . \tag{26}$$

(iv) Knowing the pressure gradient we now evaluate the stress components at 25 m intervals, from

$$\tau_x(z) = \tau_x(0) - R_2(z) , \tag{27}$$

$$\tau_y(z) = \tau_y(0) - \left|\frac{\partial p}{\partial y}\right| z + R_1(z) . \tag{28}$$

(v) The austausch coefficient A(z) is estimated by means of the expression for A_2,

i.e. $$A(z) = \tau / |\underline{V}'| , \qquad (29)$$

and we note that $|\underline{V}'|$ is never zero.

(vi) The angle, β , between the wind gradient vector and the geostrophic wind direction is evaluated from

$$\beta = \tan^{-1}\left(\frac{v'}{u'}\right) , \qquad (30)$$

and the angle, θ , between the stress vector and the geostrophic wind direction is evaluated from

$$\theta = \tan^{-1}\left(\frac{\tau_y}{\tau_x}\right) , \qquad (31)$$

care being taken to record the correct quadrants in which β and θ lie.

(vii) The root-mean-square angular deviation

$$\varepsilon_L (0-H) = \left\{ \frac{1}{H} \int_0^H (\beta - \theta)^2 \, dz \right\}^{1/2} , \qquad (32)$$

is evaluated numerically, where the meaning process is carried out over a range 0 - H. In our particular study this process is carried out over two ranges for which H = 400 m and 800 m respectively. The resulting r.m.s. errors, ε_L , are tabulated for the range of $\alpha(0)$, see Table 2, and are plotted against $\alpha(0)$ in Figure 3.

5. Results from the Lettau Reanalysis

The value of $\alpha(0)$ which best suits Lettau's hypothesis that the stress vector is parallel to the vertical shear of the wind vector is defined to be that which corresponds to the minimum value of the computed ε_L . Figure 3 indicates that when averaged over the range 0 - 800 m the error, ε_L , has a well-defined minimum at $\alpha(0) = 25.0^\circ$, its value being approximately 2°. When the error is evaluated over the range 0 - 400 m the minimum value drops to about 1.4°, however the turning point is not quite so well defined and could reasonably be accepted anywhere in the range 24.5° to 27.5°.

In general, then, the angular deviation between the stress and the wind shear increases with height and the theory fits best nearer to the surface. If we wish to apply the hypothesis to the whole depth of the boundary layer we should consider ε_L (0 - 800 m) and from this we conclude that the wind profile with $\alpha(0) = 25.0^\circ$ provides the best overall fit to Lettau's hypothesis. The wind and stress components for $\alpha(0) = 25.0^\circ$ are listed in Table 1, and in Figures 4 and 5 we compare shearing stresses obtained on the basis of Lettau's hypothesis with the thermal wind effect included against those derived by Lettau for the LWP. It is interesting to note the effect that the assumed density stratification has on the stress values near the "top" of the layer. (End values for all but the LWP stress hodograph are for heights expected to be near to the top of the boundary layer. The derivation of these heights will be discussed later.)

With the thermal wind effect included we see that for $\alpha(0) = 26.1^\circ$ the stress values near the top do not tend to a small value as rapidly as suggested by Lettau's stresses for the LWP, however for $\alpha(0) = 25.0^\circ$, where ε_L (0 - 800 m) $\simeq 2^\circ$, we recover this rate of decay. We can reasonably expect the magnitude of the stress near the top to be small, certainly much less than $\tau(0)$. In the cases illustrated we find:

For the LWP, $\alpha(0) = 26.1^\circ$, $V_g' \equiv 0$, $\dfrac{\tau(950\text{ m})}{\tau(0)} \simeq 0.08$;

for, $\alpha(0) = 26.1^\circ$, $V_g' \neq 0$, $\dfrac{\tau(1058\text{ m})}{\tau(0)} \simeq 0.18$;

for, $\alpha(0) = 25.0^\circ$, $V_g' \neq 0$, $\dfrac{\tau(1010\text{ m})}{\tau(0)} \simeq 0.04$.

If we enforce the restriction that $\tau(h) / \tau(0) \lesssim 0.1$, where h denotes the top of the Ekman layer, then we see that this criterion is met for $\alpha(0) \simeq 25.0^\circ$.

Figure 6 illustrates austausch distributions obtained from the reanalysis and compares them with Lettau's representative average distribution for the LWP and his distribution $A_2(z)$ based on equation (13) (or equation (29)). For $\alpha(0) = 26.1^\circ$

the shape for the reanalysis $A(z)$ above 450 m differs markedly from Lettau's distributions although for $\alpha(0) = 25.0^\circ$ the shape is closer to the original. The new $A(z)$ profile is given in Table 1.

In view of the likely errors arising from the computations we can accept our best fit profiles as giving a good representation of the boundary layer profiles which is consistent with Lettau's hypothesis, the physics of the situation, and the observations. The profiles with $\alpha(0) = 25.0^\circ$ exhibit the following characteristics:

$$\varepsilon_L(0 - 800\text{ m}) = 2.1^\circ, \quad \varepsilon_L(0 - 400\text{ m}) = 1.5^\circ;$$

$$\tau(0) = 4.69 \text{ dyne cm}^{-2};$$

$$\left|\frac{\partial p}{\partial y}\right| = 2.33 \times 10^{-4} \text{ dyne cm}^{-3};$$

$v = 0$ at $z = 1010$ m (estimated by extrapolation of the v profile);

$$V_g(0) = 16.33 \text{ m sec}^{-1}$$

$$V_g(h) \simeq V_g(1010\text{ m}) = 18.0 \text{ m sec}^{-1};$$

$u(1010\text{ m}) = 18.5 \text{ m sec}^{-1}$ (estimated by extrapolation of the u profile).

Note that this implies that geostrophic balance is not quite achieved at $h \simeq 1010$ m, where $\tau(1010\text{ m})$ is estimated at 0.2 dyne cm^{-2}, i. e. approximately 4% of $\tau(0)$.

6. Swinbank's Analysis of the Leipzig Data

Swinbank's analysis of the Leipzig data leads him to adopt a formulation for the boundary layer which proposes that the stress vector is parallel to the wind vector rather than to the vertical shear of the wind vector (Swinbank (1970)). Support for his hypothesis is drawn from the same data used by Lettau to "verify" the conventional flux-gradient relationship.

Using the LWP data and accepting Mildner's measured horizontal pressure gradient as 2.5×10^{-4} dyne cm^{-3}, Swinbank criticises Lettau's assumption that the geostrophic wind is constant with height throughout the layer and instead accepts constancy of the horizontal pressure gradient and a density stratification in accordance with the measured lapse rate of 0.65°C per 100 m. The resulting effects on the geostrophic wind profile have already been discussed and were included in the reanalysis based on Lettau's hypothesis.

Initially, using the integrated boundary layer equations for unaccelerated flow, Swinbank's only condition is that there exists a height, the top of the boundary layer, where the shearing stress is negligibly small. From equations (3) and (4) the shearing stress components at any level below this height, h , are given in terms of the integrals of the geostrophic departure;

$$\tau_x = -f\int_h^z \rho v \, dz \, , \tag{33}$$

$$\tau_y = -f\int_h^z \rho\,(V_g - u)\, dz \, . \tag{34}$$

The height, h , has to be determined before the stress profiles can be obtained. Swinbank eventually uses his theory of parallel stress and wind vectors to deduce that for the LWP, h = 1070 m, and it would appear from his graphical results that this value has been accepted for the top of the Ekman layer in the evaluation of the stress hodograph which was carried out prior to the new formulation.

The knowledge of the boundary layer top, the wind profile and the horizontal pressure gradient allows the geostrophic departures and hence the stress components to be computed. Swinbank observes from a comparison of his computed stress components with the LWP that the directions of the wind and shearing stress agree closely, level for level, throughout the layer.

On the basis of this apparent parallelism Swinbank proceeds to formulate relationships between the pressure gradient, shearing stress vectors and wind vectors throughout the boundary layer. These relationships are then tested against the LWP data with an apparently high degree of success. The graphical representation of the results with their lack of scatter, generally uncharacteristic of observational boundary layer studies, appear to imply powerful support for Swinbank's hypothesis. This is an important point which we shall return to at a later stage.

7. Criticism of Swinbank's Analysis

At this particular stage let us only consider that part of Swinbank's analysis which leads to his observation that the LWP suggests a parallelism between the wind and the stress at all levels within the boundary layer.

(a) A major criticism is that Swinbank has based his observations on the profile which was chosen by Lettau as the best fit to the flux-gradient theory. It should be recalled that a principal difficulty with the Leipzig data is the absence of reliable information on the magnitude and the direction of the geostrophic wind which were only roughly determined from synoptic measurements. Lettau's analysis is designed to provide the $\alpha(0)$ which best suits his hypothesis and it is hard to believe that a profile chosen in this way would also be the best fit for a hypothesis which requires the stress parallel to the wind. Remember too that Lettau did not incorporate into his analysis the thermal wind effect suggested by Swinbank. It is therefore inconsistent of Swinbank to reject certain aspects of Lettau's formulation and yet accept his representative profile as a basis for a new hypothesis.

It is necessary to redo this part so as to obtain the profile which best fits Swinbank's theory. The reanalysis will follow closely that used to test Lettau's hypothesis and the error computed will be the root-mean-square angular deviation between $\underset{\sim}{\tau}$ and $\underset{\sim}{V}$ for a given $\alpha(0)$.

(b) Following on from the first criticism we note that the derived stress hodographs are very sensitive to small changes in the estimate of the pressure gradient and so the geostrophic wind taken from weather maps is too inaccurate for comparing different theories. Swinbank has accepted Mildner's measured value of 2.5×10^{-4} dyne cm^{-3} for the horizontal pressure gradient. This leads to a value of V_g at $z = h$, the top of the boundary layer, which is not consistent with the shape of the LWP, ie V_g (1070m) = 19.44m sec^{-1}. It is necessary then to deduce a value of $\left|\frac{\partial p}{\partial y}\right|$ from the theory.

(c) Swinbank has only needed to enforce a boundary condition at the top of the Ekman layer. Unfortunately, although the condition,

$$u \longrightarrow V_g\ ,\ v \longrightarrow 0\ ,\ \tau_x \longrightarrow 0\ ,\ \tau_y \longrightarrow 0 \text{ as } z \longrightarrow h \quad , \tag{35}$$

is readily acceptable, the height, h , remains to be determined and for $V_g(h)$, a prior knowledge of the horizontal pressure gradient is required.

We need not be so demanding on our upper boundary condition. Just as Lettau's hypothesis produces the two significant heights z_1, z_2, Swinbank's hypothesis yields the level z* such that,

$$v = 0\ ,\ \tau_y = 0\ ,\ \tau_x' = 0 \qquad \text{at } z = z^*. \tag{36}$$

This level coincides with h , the level Swinbank deduces as the top of the boundary layer; however, note that at z^* conditions need not be geostrophic and $\tau_x(z^*)$ need not be zero, although such conditions are not ruled out.

(d) Swinbank takes no account of the lower boundary condition which requires that the stress vector is parallel to the limiting direction of the wind vector as $z \longrightarrow zo$ ("the surface"). This boundary condition is necessary on physical grounds and indeed is consistent with Swinbank's own hypothesis.

In the reanalysis which follows the above boundary conditions at the surface and z* will be enforced, the pressure gradient will be determined from the profile and the stress components will be obtained by integration upwards from the surface.

8. Reanalysis Based on Swinbank's Hypothesis

The reanalysis is carried out for the same range of $\alpha(0) = \alpha_L(0) + \delta$ defined in the reanalysis based on Lettau's hypothesis, and all numerical integrations are carried out with the same 25m step in the vertical.

For each value of $\alpha(0)$ in the specified range the following procedure is followed:

(i) Knowing δ , the LWP yields the profiles u, v, $V = |\underline{V}|$.

(ii) The level z^* is found by extrapolating the curve for $v(z)$ and noting the value of z where $v(z) = 0$. In fact z^* was obtained in this way for the whole range of $\alpha(0)$ and the resulting values were smoothed by eye as shown in Figure 7. Whenever such a smoothing process is carried out the smoothed values are used thereafter. The wind speed $u(z^*)$ is obtained in a similar fashion by extrapolation of the wind profile and the smoothing is indicated in Figure 8.

(iii) By graphical integration we evaluate,

$$R_1(z^*) = f\int_0^{z^*} \rho u \, dz \tag{37}$$

and

$$R_2(z^*) = f\int_0^{z^*} \rho v \, dz, \tag{38}$$

the smoothed values being illustrated in Figure 7.

(iv) To find the pressure gradient we suppose that Swinbank's hypothesis can be taken to hold at $z = z^*$, ie $\underset{\sim}{\tau}(z) = B(z)\underset{\sim}{V}(z)$ at $z = z^*$.

The momentum equations and boundary conditions yield, after integration,

$$\tau_x(z^*) = \tau_x(0) - R_2(z^*) = B(z^*)u(z^*) \tag{39}$$

$$\tau_y(z^*) = \tau_y(0) + R_1(z^*) - \left|\frac{\partial p}{\partial y}\right| z^* = 0. \tag{40}$$

Now, on the basis of Swinbank's theory,

$$\tau_y'(z) = B'(z)v(z) + B(z)v'(z), \tag{41}$$

therefore,

$$\tau_y'(z^*) = B(z^*)v'(z^*)$$

$$= f\rho u(z^*) - \left|\frac{\partial p}{\partial y}\right|,$$

ie

$$B(z^*) = \left(f\rho u(z^*) - \left|\frac{\partial p}{\partial y}\right|\right) \Big/ v'(z^*). \tag{42}$$

From equations (39), (40) and (42), and the surface boundary condition,

$$\tau_y(0) = \tau_x(0)\tan\alpha(0),$$

we obtain,

$$\left|\frac{\partial p}{\partial y}\right| \left[z^*\cot\alpha(0) + \frac{u(z^*)}{v'(z^*)}\right] = \frac{f\rho u^2(z^*)}{v'(z^*)} + R_1(z^*)\cot\alpha(0) + R_2(z^*). \tag{43}$$

Knowing $v'(z^*)$ and $f\rho u(z^*)$ in addition to the parameters already obtained we can now evaluate the pressure gradient.

Equation (42) is valid provided that $v'(z^*) \neq 0$ and it turns out that for all values of $\alpha(0)$ the estimated value of $v'(z^*) = -8 \times 10^{-3}\ \mathrm{sec}^{-1}$.

The smoothing of $f\rho u(z^*)$ is given in Figure 8.

If the wind is geostrophic at z* and $v'(z^*) \neq 0$ then,

$$\tau'_y(z^*) = 0\,,$$

$$B(z^*) = 0\,,$$

and

$$\tau_x(z^*) = 0\,.$$

Since, now,

$$\left|\frac{\partial p}{\partial y}\right| = f\rho u(z^*),$$

equation (43) reduces to

$$\left|\frac{\partial p}{\partial y}\right| = \frac{1}{z^*}\Big[R_1(z^*) + R_2(z^*)\tan\alpha(0)\Big]. \qquad (44)$$

Thus equation (43) will give the pressure gradient to balance the coriolis term when conditions are geostrophic at z*, otherwise $\left|\frac{\partial p}{\partial y}\right|$ will be such that Swinbank's hypothesis is true at z*, ie $\underset{\sim}{\tau}(z^*)$ parallel to $\underset{\sim}{V}(z^*)$ The smoothed values obtained are given in Figure 8. The geostrophic wind at z* corresponding to the pressure gradient is calculated and by comparing the values of $V_g(z^*)$ with $u(z^*)$ it is found that the derived profiles suggest geostrophic balance at z* for $\alpha(0)$ between 26.7° and 27.2°.

(v) With the above parameters we can now evaluate the stress components at the surface and the residual stress component at the height z* as follows:

$$\tau_y(0) = \left|\frac{\partial p}{\partial y}\right| z^* - R_1(z^*)\,, \qquad (45)$$

$$\tau_x(0) = \tau_y(0)\cot\alpha(0)\,, \qquad (46)$$

$$\tau(0) = |\underset{\sim}{\tau}(0)|\,,$$

$$\tau_x(z^*) = \tau_x(0) - R_2(z^*)\,. \qquad (47)$$

The residual stress is of course zero if conditions are geostrophic at z*.

(vi) The stress components at 25m intervals are given by

$$\tau_x(z) = \tau_x(0) - R_2(z) \quad , \tag{48}$$

$$\tau_y(z) = \tau_y(0) - \left|\frac{\partial p}{\partial y}\right| z + R_1(z) . \tag{49}$$

(vii) The angle, α , between the wind vector and the geostrophic wind direction is evaluated from

$$\alpha = \tan^{-1}\left(\frac{v}{u}\right) , \tag{50}$$

and the stress angle θ , from

$$\theta = \tan^{-1}\left(\frac{\tau_y}{\tau_x}\right) , \tag{51}$$

the usual care being taken over the correct quadrants in which α, θ lie.

(viii) The root-mean-square angular deviation

$$\varepsilon_s(0-H) = \left\{\frac{1}{H}\int_0^H (\alpha-\theta)^2 \, dz\right\}^{1/2} , \tag{52}$$

is evaluated numerically, where the meaning process is carried out over the two ranges 0-400m and 0-800m.

The resulting r.m.s. errors, ε_s , are tabulated in Table 2 and are plotted against $\alpha(0)$ in Figure 9.

9. <u>Results from the Swinbank Reanalysis</u>

The value of $\alpha(0)$ which is best suited to Swinbank's hypothesis that at every level in the atmospheric boundary layer the stress vector is parallel to the wind vector should be taken as that which corresponds to the minimum value of ε_s observed in Figure 9. However, we note from the figure that Swinbank's hypothesis does not give a minimum root-mean-square error anywhere near the angle $\alpha(0)$ suggested by both the synoptic estimate and Lettau's hypothesis. Although over part of the range the actual values of Swinbank's r.m.s. errors are much smaller than the minimum ε_L recorded in the Lettau reanalysis, the corresponding profiles, pressure gradients and geostrophic wind speeds are not acceptable on physical grounds.

This is best illustrated by the ratio of the computed residual stress at z* to the stress magnitude at the surface. Although z* may not be the exact height where the stress vanishes we would certainly expect the stress in this neighbourhood to be very small. From Figure 8 we see that the criterion,

$$\tau(z^*)/\tau(0) < 0.1,$$

is satisfied only in a very small neighbourhood of $\alpha(0) = 27.0^\circ$, i.e. in this neighbourhood the ratio $\tau(z^*)/\tau(0)$ is extremely sensitive to changes in $\alpha(0)$. The application of this criterion is consistent with Swinbank's choice of upper boundary condition; moreover it implies an angle near the estimated synoptic geostrophic wind direction and provides a geostrophic wind and pressure gradient comparable to those estimated from and suggested by the wind profile. For $\alpha(0) = 27.0^\circ$, ε_s (0-800m) $\simeq 22^\circ$, and the corresponding wind and stress components are given in Table 3. For $\alpha(0) = 26.9^\circ$, $\tau(z^*)/\tau(0) \simeq 0.16$ and the error has dropped to ε_s(0-800m) $\simeq 12^\circ$.

It should be emphasised that the profile which corresponds to geostrophic conditions at z* is chosen as that which gives the best fit to the observed data based on an attempt to comply with Swinbank's hypothesis and upper boundary condition. The only extra condition, not enforced by Swinbank, is the requirement that the surface stress be parallel to the limiting direction of the wind. The reanalysis stress hodograph for $\alpha(0) = 27.0^{\circ}$, and Swinbank's original stress hodograph are presented in Figure 10.

The r.m.s. error for the chosen Swinbank reanalysis profiles is much larger than that obtained in the Lettau reanalysis and so we must conclude that, using the Leipzig data, Lettau's hypothesis provides the more consistent representation of the stress and wind profiles.

10. Discussion of Swinbank's Theoretical Formulation

The conclusion that the Leipzig data do not support Swinbank's hypothesis appears at first sight to be in violent contradiction to the results presented by Swinbank in his formulation of the wind-stress relationship. We shall see that all of the results produced in support of the validity of a wind-stress parallelism can in fact be obtained without invoking such a theory.

On the assumption that the wind-stress parallelism is real, Swinbank derives certain relationships for the boundary layer parameters which he tests against the LWP. We shall follow the same lines as Swinbank without making that assumption and compare the general results with those obtained by him.

The general boundary layer equations for horizontal motion are,

$$\rho \frac{du}{dt} - f\rho v = \frac{\partial}{\partial z}(\tau \cos\theta) , \tag{53}$$

$$\rho \frac{dv}{dt} + f\rho u = \left|\frac{\partial p}{\partial y}\right| + \frac{\partial}{\partial z}(\tau \sin\theta) . \tag{54}$$

Swinbank points out that,

$$\frac{\frac{du}{dt} - fv}{\frac{dv}{dt} + fu} = -\frac{v}{u} \qquad \text{if} \qquad \frac{d}{dt}(u^2+v^2) = 0 . \tag{55}$$

This is a less restrictive condition than the general assumption of unaccelerated flow, however it should be noted that Swinbank's computed shearing stress from integration of geostrophic departure assumes that the Leipzig flow is unaccelerated.

Division of (53) and (54), with condition (55), gives

$$\frac{\partial \tau}{\partial z} = -\left|\frac{\partial p}{\partial y}\right| \frac{\sin\alpha}{\cos(\alpha-\theta)} - \frac{\partial \theta}{\partial z} . \tau \tan(\alpha-\theta) . \tag{56}$$

If the flow is unaccelerated, elimination of τ from equations (53) and (54) produces the more useful form

$$\frac{\partial \tau}{\partial z} = -\left|\frac{\partial p}{\partial y}\right| \sin\theta - f\rho V \sin(\alpha-\theta) . \tag{57}$$

With the hypothesis that $\alpha(z) = \theta(z)$ throughout the layer equations (56) and (57) reduce to Swinbank's formula,

$$\frac{\partial \tau_s}{\partial z} = -\left|\frac{\partial p}{\partial y}\right| \sin\alpha . \tag{58}$$

We can express the full equation (56) in the form,

$$\frac{\partial \tau}{\partial z} = \frac{\partial \tau_s}{\partial z} \sec(\alpha-\theta) + \Delta \quad , \tag{59}$$

where $\Delta(z) = -\frac{\partial \theta}{\partial z} . \tau \tan(\alpha-\theta)$, using (56),

$$= -\left|\frac{\partial p}{\partial y}\right| \left[\sin\theta + \sin\alpha \sec(\alpha-\theta)\right] - f\rho V \sin(\alpha-\theta), \tag{60}$$

when we eliminate τ' from equation (56) by means of equation (57), and compare the relative contribution each term on the R.H.S. makes to the magnitude of τ' for any value of z. This has been carried out for the Swinbank best choice profile with $\alpha(0) = 27.0^\circ$ and the terms $(\alpha-\theta)$, τ'/τ_s', $\sec(\alpha-\theta)$, $\Delta/[\tau_s' \sec(\alpha-\theta)]$ are given in Table 3. Recall that for this profile the r.m.s. angular deviation error is of the order of 20° and so, in this case, there is no question of Swinbank's hypothesis being valid. However we note that the ratio of the first term on the R.H.S. of equation (59) to τ_s' , given by sec $(\alpha-\theta)$, is very close to unity, especially in the lower levels where the stress magnitudes are greatest, and, on average, the second term on the R.H.S. is about one order of magnitude less than $\tau_s' \sec(\alpha-\theta)$. The result is that τ'/τ_s' always lies in the range 0.9 to 1.1.

It is evident then that, in general, the full expression for τ' is dominated by the term τ_s' which Swinbank uses for τ' on the basis of his new formulation. Exceptional circumstances could no doubt arise where the relative behaviour of θ', τ, $(\alpha-\theta)$ is uncertain.

At the level z* where the wind becomes parallel to the geostrophic direction equation (57) reduces to

$$\left[\frac{\partial \tau}{\partial z}\right]_{z=z^*} = \left[\left(f\rho V - \left|\frac{\partial p}{\partial y}\right|\right)\sin\theta\right]_{z=z^*} . \qquad (61)$$

Therefore $\tau'(z^*) = 0$ if $\theta = 0$

or if conditions are geostrophic at z*.

If the latter condition is taken to hold for unaccelerated flow at z* then that and the equations of motion imply

$$\tau'(z^*) = \tau(z^*) = 0, \qquad (62)$$

where z* is now the top of the boundary layer as designated by Swinbank.

Let $\eta = z^* - z$ and expand τ close to z* as Swinbank suggests,

$$\tau(z) = \tau(z^*) - \eta\,\tau'(z^*) + \frac{\eta^2}{2!}\tau''(z^*) + O(\eta^3) \qquad (63)$$

$$= \frac{\eta^2}{2!}\tau''(z^*) + O(\eta^3) \quad , \qquad \text{by virtue of equation (62),}$$

and

$$\tau'(z) = \tau'(z^*) - \eta\,\tau''(z^*) + \frac{\eta^2}{2!}\tau'''(z^*) + O(\eta^3)$$

$$= -\eta\,\tau''(z^*) + \frac{\eta^2}{2!}\tau'''(z^*) + O(\eta^3). \qquad (64)$$

Equations (56), (63) and (64) give

$$-\eta\,\tau''(z^*) + \frac{\eta^2}{2!}\,\tau'''(z^*) + O(\eta^3) = \tau_s'(z)\sec(\alpha-\theta) - \frac{\eta^2}{2!}\,\tau''(z^*).\,\theta'(z)\tan(\alpha-\theta) + O(\eta^3). \tag{65}$$

Therefore the leading term in the series expansion for $\tau_s'(z)\sec(\alpha-\theta)$ for small η , yields

$$\tau_s'(z) = -\left|\frac{\partial p}{\partial y}\right|\sin\alpha = -\eta\,\tau''(z^*)\cos(\alpha-\theta) + O(\eta^2). \tag{66}$$

We note that for $|\alpha-\theta| \leq 30^\circ$, $\cos(\alpha-\theta)$ takes values in the range 0.86 to 1, and so, for a given occasion, equations (66) and (63) imply

$$\sin\alpha(z) \sim \eta \quad , \tag{67}$$

and

$$\tau(z) \sim \eta^2 \quad , \tag{68}$$

for small values of $\eta = z^* - z$. Again we note that these relationships have been obtained without any assumption of a wind-stress parallelism.

In order to verify that the above results are independent of the claims of the two conflicting hypotheses we can attempt to choose a best fit profile from the Leipzig data without invoking either of them. For each $\alpha(0)$ estimate the level of the top of the boundary layer by finding the height $z = h$ where $v(z) = 0$. The upper boundary condition,

$u = V_g$, $v = 0$, $\tau_x = \tau_y = 0$ at $z = h$, is adopted, and extrapolation of the $u(z)$ profile to $z = h$ provides the value $V_g(h)$ and hence the pressure gradient.

The surface stress components are evaluated by graphical integration,

$$\tau_x(0) = f\int_0^h \rho v\,dz ,$$

$$\tau_y(0) = \left|\frac{\partial p}{\partial y}\right| \cdot h - f\int_0^h \rho u\,dz ,$$

which give

$$\theta(0) = \tan^{-1}\left(\frac{\tau_y(0)}{\tau_x(0)}\right).$$

The resulting values of $\theta(0)$ are compared with the corresponding $\alpha(0)$; see Figure 11. If we now demand that the stress vector at the surface be parallel to the limiting direction of the wind vector, i.e. $\alpha(0) = \theta(0)$, then we see from Figure 11 that this condition implies a well-defined value of $\alpha(0) = 27.1^{\circ}$.

Although arrived at in a different manner this profile agrees well, as it should, with the profile we were eventually obliged to choose in our Swinbank reanalysis. The stress components are generated in the usual way by integration of the equations of motion. Figure 12 illustrates how $|\theta - \alpha|$ and $|\theta - \beta|$ behave throughout the boundary layer for these profiles and it is obvious that neither Swinbank's nor Lettau's hypothesis applies, in fact for the profiles obtained ε_s (0-800m) $\simeq 20^{\circ}$, which was the value already found for the best choice profiles which could be attributed to the Swinbank model.

Having established that the stress and the wind are not parallel in this case we can now consider each of Swinbank's claims in relation to the profiles.

(i) Figure 13 shows $\sin\alpha(z)$ versus z and we note the near linearity of the relationship, which was suggested by equation (67) for values of z close to h ($\equiv z^*$).
(c.f. Swinbank (1970), Figure 3)

(ii) The stress magnitudes are derived in two ways. Swinbank's formula, equation (58), is integrated to provide the estimate $\tau_s(z)$ and integration of the equations of motion provides $\tau(z)$. Figures 14 and 15 compare τ and τ_s throughout the layer (c.f. Swinbank (1970), Figure 5), and Figure 16 shows τ versus η^2 where η is the depth below the top of the boundary layer (c.f. Swinbank (1970), Figure 4). We note then that although there is no question of a wind-stress parallelism, Swinbank's formula for the magnitude of the stress provides a very good estimate. The relationship between the stress and η derived for small values of η in equation (68), appears to hold over a wider range of z, going astray near the surface where presumably the neglected higher order terms become more important.

(iii) It also turns out that if the full relationship for τ' is used in the redetermination of the wind components instead of τ_s' then the wind profiles obtained agree with the initial data; there is no need to invoke acceleration terms. Although the error involved in approximating τ' by τ_s' does not lead to significant errors in the shearing stress values arrived at by integration of τ_s', when used to re-derive the wind profile components it certainly does involve important errors.

The form of $v(z)$ can be obtained approximately from the observation that

$$v \simeq V_g \sin\alpha ,$$

for as long as $\underset{\sim}{V}(z)$ is near geostrophic in magnitude, therefore

$$v \sim \eta \quad , \text{ for small values of } \eta .$$

This linearity is clearly noted down to about 500m (c.f. Swinbank (1970) Figure 6(b))

To conclude, then, we note that all of the results which Swinbank claims as direct support for his new wind-stress formulation can be obtained independently of any such hypothesis and in fact hold for cases where there is no semblance of a wind-stress parallelism.

11. Conclusions

The aim of the present study is to provide an objective assessment of two conflicting wind-stress formulations throughout the atmospheric boundary layer in relation to the Leipzig data. The nature of the basic data allows us an important degree of freedom in our choice of representative wind profile because of the lack of reliable measurements of the direction of the geostrophic wind. It is unfortunate that both theories are very sensitive to the choice of the internal parameter $\alpha(0)$, the angle between the surface wind and the isobars.

Allowing for a reasonable range of possible values of $\alpha(0)$, an attempt has been made to ensure that each method is applied to the wind-profile best suited to the relevant hypothesis. The reanalysis based on Lettau's hypothesis presented no problems in this respect and the outcome was a set of wind and shearing stress profiles which complied admirably with the synoptic estimates made by Mildner and with our present understanding of the structure of the boundary layer, and which also produced a root-mean-square error for the hypothesis of only 2° over a great depth of the whole boundary layer. On the other hand a similar method of reanalysis based on Swinbank's hypothesis did not provide any internal consistency between the physics of the situation and the degree of fit to the theory. It was deemed necessary to accept certain physical restrictions at the top of the boundary layer and as a consequence we were obliged to choose a value for $\alpha(0)$ with corresponding profiles not compatible with Swinbank's theory, the fitting error being of the order of 20°.

Our conclusions are:

(i) The Leipzig observations do not support the Swinbank hypothesis that the wind vector is parallel to the stress vector throughout the boundary layer; however, Swinbank has derived expressions which, although they are not dependent on such a hypothesis, appear to provide very useful means of computing the stress magnitude throughout the layer.

(ii) Lettau's conventional, flux-gradient approach is internally consistent and is compatible with the Leipzig data to a high degree.

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Boundary Layer Research
Met O 14
Meteorological Office HQ
Bracknell

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Table 1 Vertical profiles at 50m intervals for,

(1) in general (i) the density stratification, ρ , $X10^{3}$ (gm cm^{-3}),

(ii) the Leipzig wind speed, V, (m sec^{-1}),

(2) $\alpha(0)=26.1^{o}$ (i) the LWP components, (u_L v_L), (m sec^{-1})

(ii) the functions R_{1L}, R_{2L}, (dyne cm^{-2}),

(3) $\alpha(0)=25.0^{o}$, the best choice $\alpha(0)$ for the Lettau reanalysis,

(i) the wind components, (u,v), (m sec^{-1}),

(ii) the stress components, (τ_x, τ_y), and magnitude, τ,(dyne cm^{-2}),

(iii) the austausch coefficient, A, (gm cm^{-1} sec^{-1})

(iv) the modulus of the angular deviation between the wind gradient vector and the shearing stress vector, $|\beta-\theta|$, (degree).

			The LWP $\alpha(0) = 26.1^{o}$				Lettau reanalysis $\alpha(0) = 25.0^{o}$						
z	ρ	V	u_L	v_L	R_{1L}	R_{2L}	u	v	τ_x	τ_y	τ	A	$\lvert\beta-\theta\rvert$
0	1.250	0	0	0	0	0	0	0	4.25	1.98	4.69	0	0
50	1.244	10.13	9.15	4.35	0.44	0.26	9.24	4.17	4.00	1.26	4.20	96.4	0.6
100	1.238	11.43	10.45	4.64	1.13	0.58	10.54	4.43	3.70	0.79	3.78	154	1.5
150	1.232	12.54	11.58	4.80	1.91	0.91	11.67	4.57	3.38	0.41	3.41	157	0.1
200	1.226	13.54	12.60	4.95	2.75	1.25	12.70	4.70	3.06	0.10	3.06	160	1.8
250	1.221	14.36	13.48	4.96	3.66	1.60	13.58	4.69	2.73	-0.15	2.73	161	0.4
300	1.215	15.12	14.30	4.90	4.62	1.94	14.40	4.61	2.41	-0.35	2.43	162	0.3
350	1.209	15.72	14.97	4.78	5.63	2.28	15.06	4.48	2.09	-0.50	2.15	159	0.5
400	1.203	16.28	15.62	4.60	6.68	2.60	15.71	4.29	1.79	-0.61	1.89	136	3.1
450	1.197	16.84	16.28	4.29	7.77	2.90	16.36	3.96	1.51	-0.67	1.65	122	3.5
500	1.192	17.30	16.83	4.00	8.90	3.19	16.91	3.66	1.25	-0.70	1.43	121	1.8
550	1.186	17.69	17.30	3.71	10.05	3.45	17.37	3.36	1.01	-0.71	1.23	114	1.8
600	1.180	18.02	17.70	3.37	11.23	3.69	17.76	3.02	0.79	-0.69	1.05	111	2.9
650	1.175	18.25	17.99	3.07	12.42	3.90	18.05	2.71	0.60	-0.66	0.89	107	3.9
700	1.169	18.43	18.23	2.73	13.63	4.10	18.28	2.37	0.43	-0.61	0.74	96.4	2.5
750	1.163	18.58	18.42	2.43	14.85	4.27	18.46	2.06	0.28	-0.55	0.62	80.7	0.5
800	1.158	18.71	18.60	2.06	16.07	4.41	18.64	1.69	0.16	-0.49	0.52	66.4	1.1
850	1.152	18.74	18.66	1.70	17.25	4.55	18.69	1.33	-	-	-	-	-
900	1.147	18.73	18.68	1.31	18.50	4.63	18.70	0.94	-	-	-	-	-
950	1.141	18.64	18.62	0.91	19.70	4.73	18.63	0.54	-	-	-	-	-
1010	1.134	18.50	-	-	-	-	18.50	0	-0.06	-0.18	0.19	-	-

Table 2 The following parameters are listed for values of $\alpha(0)$ (degrees) in the specified range.

(1) For the Lettau reanalysis.

(i) Significant levels z_1 and z_2, (m)

(ii) The root-mean-square angular deviations ε_L (0-400m) and ε_L (0-800m), (degrees).

(2) For the Swinbank reanalysis.

(i) The root-mean-square angular deviations ε_s (0-400m) and ε_s(0-800m), (degrees)

$\alpha(0)$	z_1	z_2	ε_L (0-400m)	ε_L (0-800m)	ε_s (0-400m)	ε_s (0-800m)
22.7	207	872	3.9	> 50 *	0.5	0.4
23.2	211	875	2.4	37.6 *	0.6	0.4
23.8	217	878	2.1	18.7 *	0.6	0.5
24.4	223	882	1.9	7.5	0.9	0.7
25.0	228	885	1.5	2.1	1.2	1.0
25.5	233	889	1.4	2.4	1.7	1.3
26.1	238	893	1.5	4.2	3.2	2.9
26.7	243	897	1.4	4.7	7.1	7.6
27.0	-	-	-	-	13.7	21.8
27.2	247	901	1.5	6.1	> 50	> 50
27.8	252	905	1.6	7.2	**	**
28.4	256	910	1.7	7.9	**	**
29.0	261	915	2.2	9.2	**	**

* Computations imply $A(z) < 0$ for a range of z below 800m

** Computations imply negative surface stress components.

Table 3 Vertical profiles for the Swinbank reanalysis case with $\alpha(0)$ = 27.0 degree.

z (m)	u (m sec^{-1})	v (m sec^{-1})	τ_x (dyne cm^{-2})	τ_y (dyne cm^{-2})	τ (dyne cm^{-2})	$\alpha-\theta$ (degree)	τ'/τ'_s	sec $(\alpha-\theta)$	$\Delta/[\tau'_s \sec(\alpha-\theta)]$
0	0	0	5.08	2.59	5.71	0			
50	9.08	4.49	4.82	1.85	5.16	5.3			
100	10.38	4.80	4.49	1.36	4.69	7.9	0.94	1.01	-0.09
150	11.50	4.98	4.14	0.96	4.25	10.4			
200	12.52	5.14	3.79	0.62	3.84	13.0	0.91	1.03	-0.12
250	13.40	5.17	3.43	0.35	3.45	15.3			
300	14.22	5.12	3.07	0.13	3.08	17.5	0.91	1.05	-0.13
350	14.89	5.01	2.72	-0.05	2.72	19.6			
400	15.55	4.84	2.38	-0.17	2.39	21.5	0.92	1.07	-0.14
450	16.21	4.54	2.06	-0.26	2.08	22.9			
500	16.77	4.26	1.76	-0.31	1.79	24.3	0.96	1.10	-0.13
550	17.24	3.98	1.48	-0.34	1.52	25.9			
600	17.65	3.64	1.22	-0.34	1.27	27.3	1.01	1.13	-0.11
650	17.94	3.35	0.99	-0.33	1.04	28.9			
700	18.19	3.01	0.78	-0.30	0.83	30.3	1.05	1.16	-0.09
750	18.38	2.72	0.59	-0.26	0.64	31.9			
800	18.57	2.35	0.42	-0.21	0.47	34.0	1.10	1.21	-0.09
1094	18.34	0	-0.06	0	0.06	18 0			

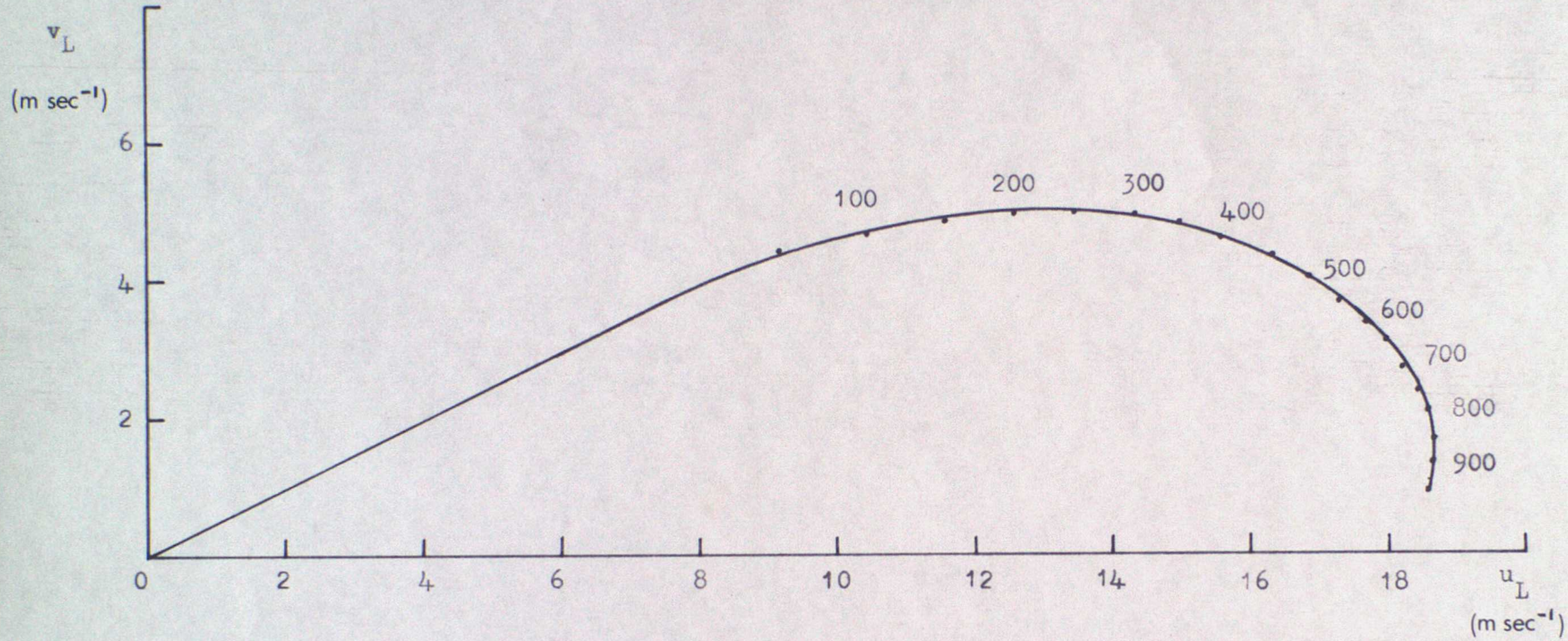

Figure 1. The Leipzig Wind Profile from 50 - 950m at 50m intervals, as presented by Lettau (1950). For this profile $\alpha(0) = \alpha_L(0) = 26{\cdot}1$ degrees.

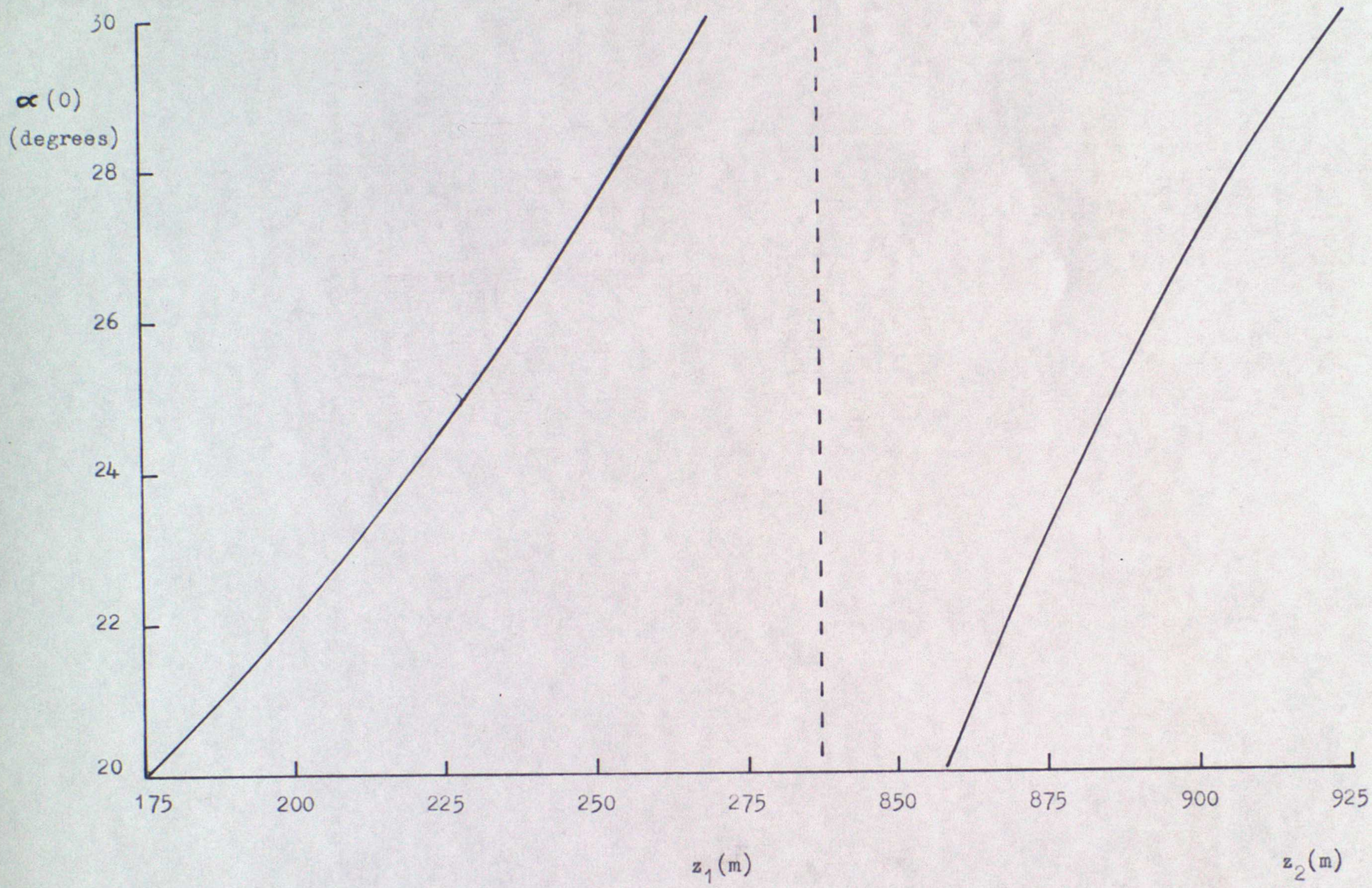

Figure 2. The curves for Lettau's significant levels z_1 and z_2 , given as functions of $\alpha(0)$.

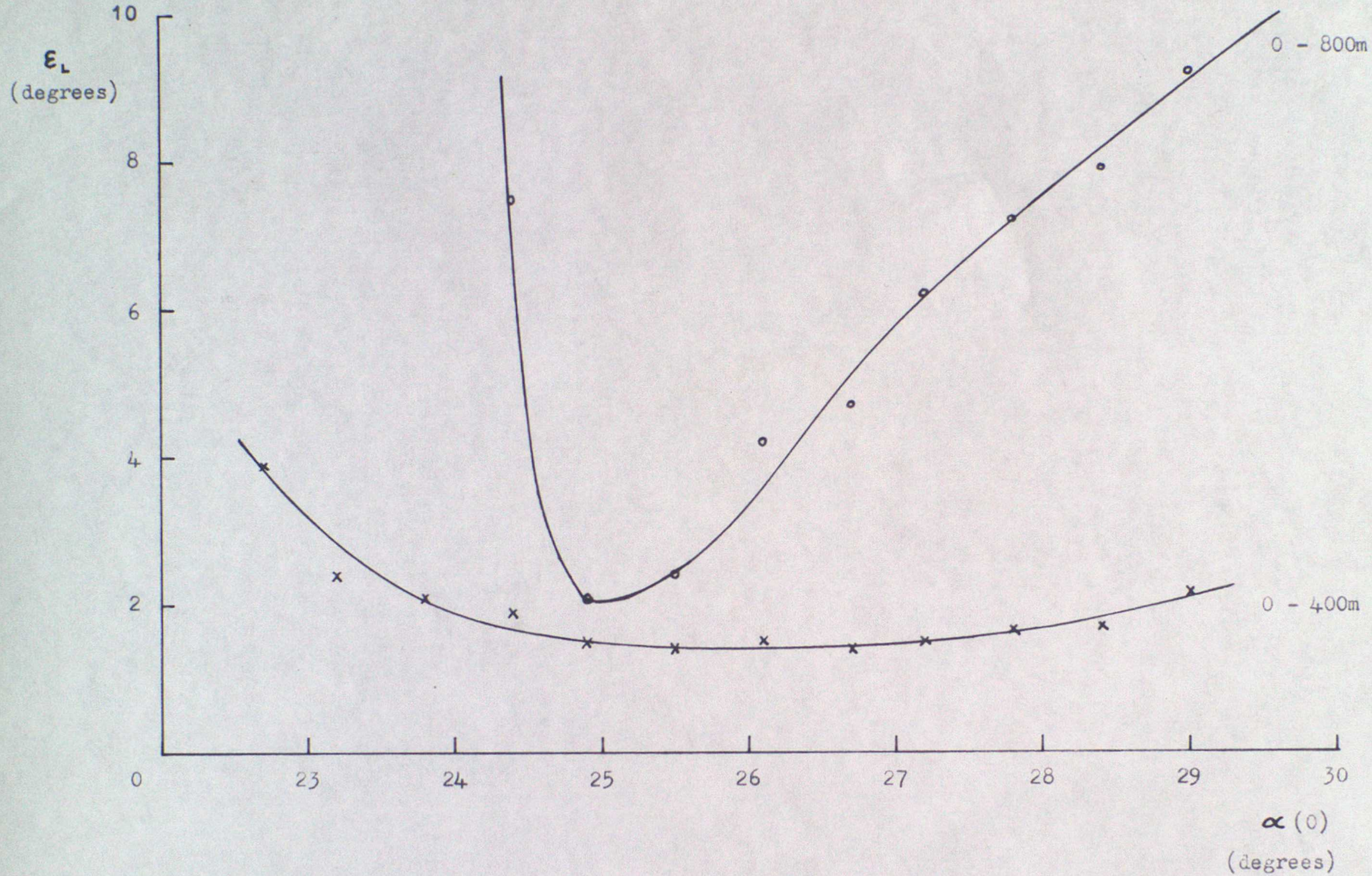

Figure 3. The root-mean-square angular deviation for Lettau's hypothesis, ε_L (degrees), measured over the vertical ranges 0 - 400m and 0 - 800m, and presented as a function of $\alpha(0)$ (degrees).

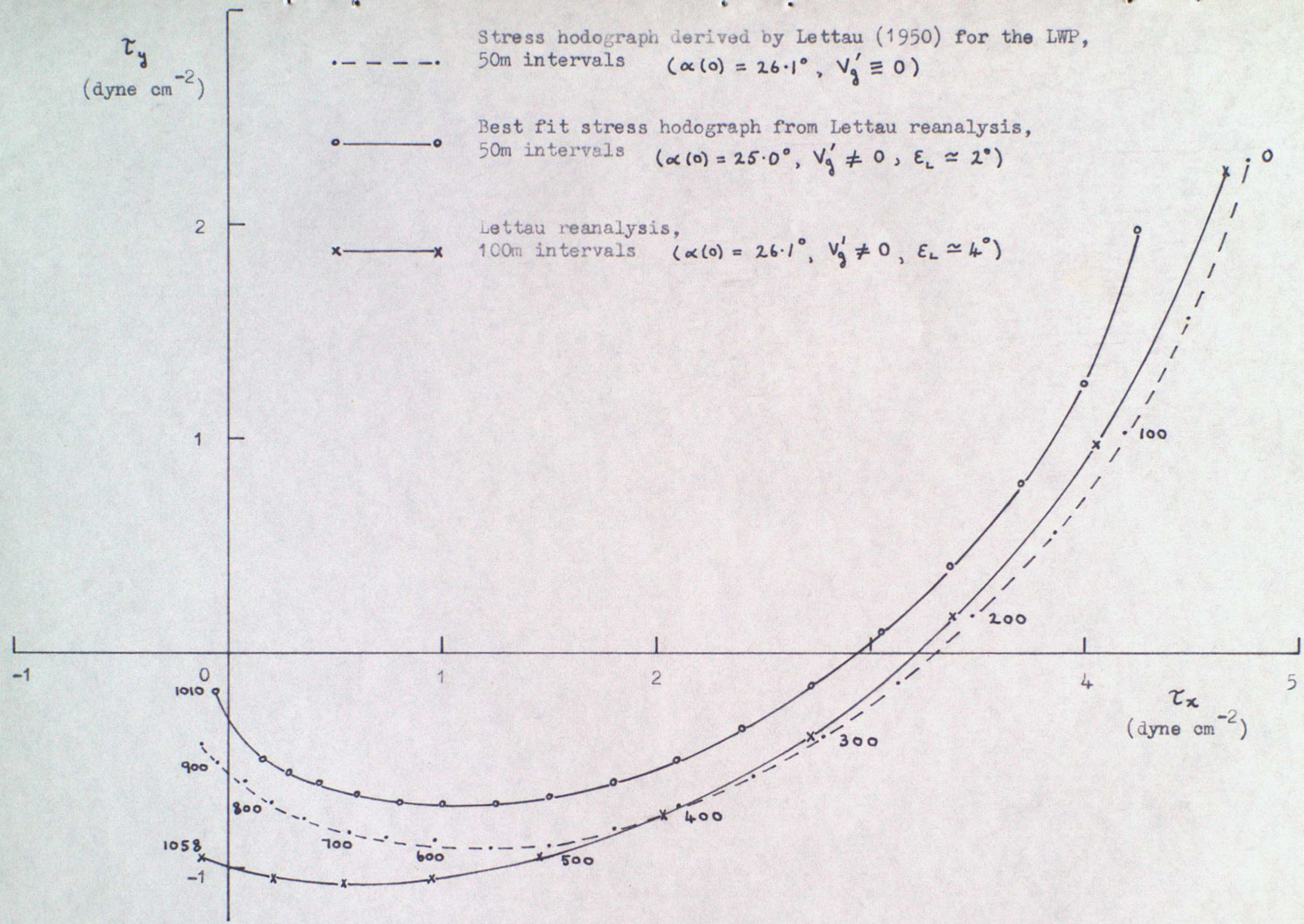

Figure 4. Shearing stress components from the surface to 800m at specified intervals, computed on the basis of Lettau's hypothesis. Values at specific levels above 800m are indicated for each hodograph.

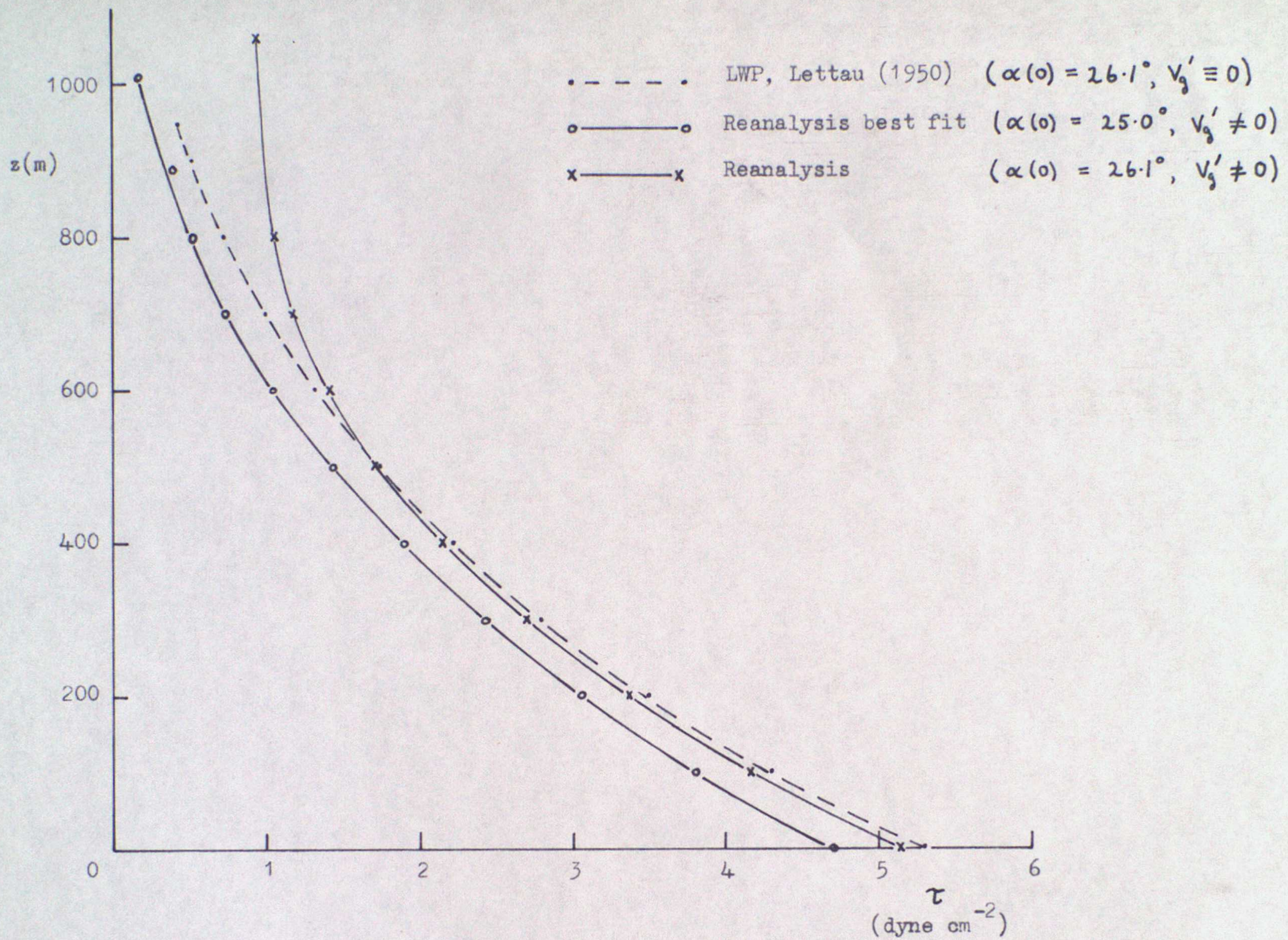

Figure 5. The variation of the magnitude of the shearing stress, τ (dyne cm^{-2}), with height, z (m), for specified cases based on Lettau's hypothesis.

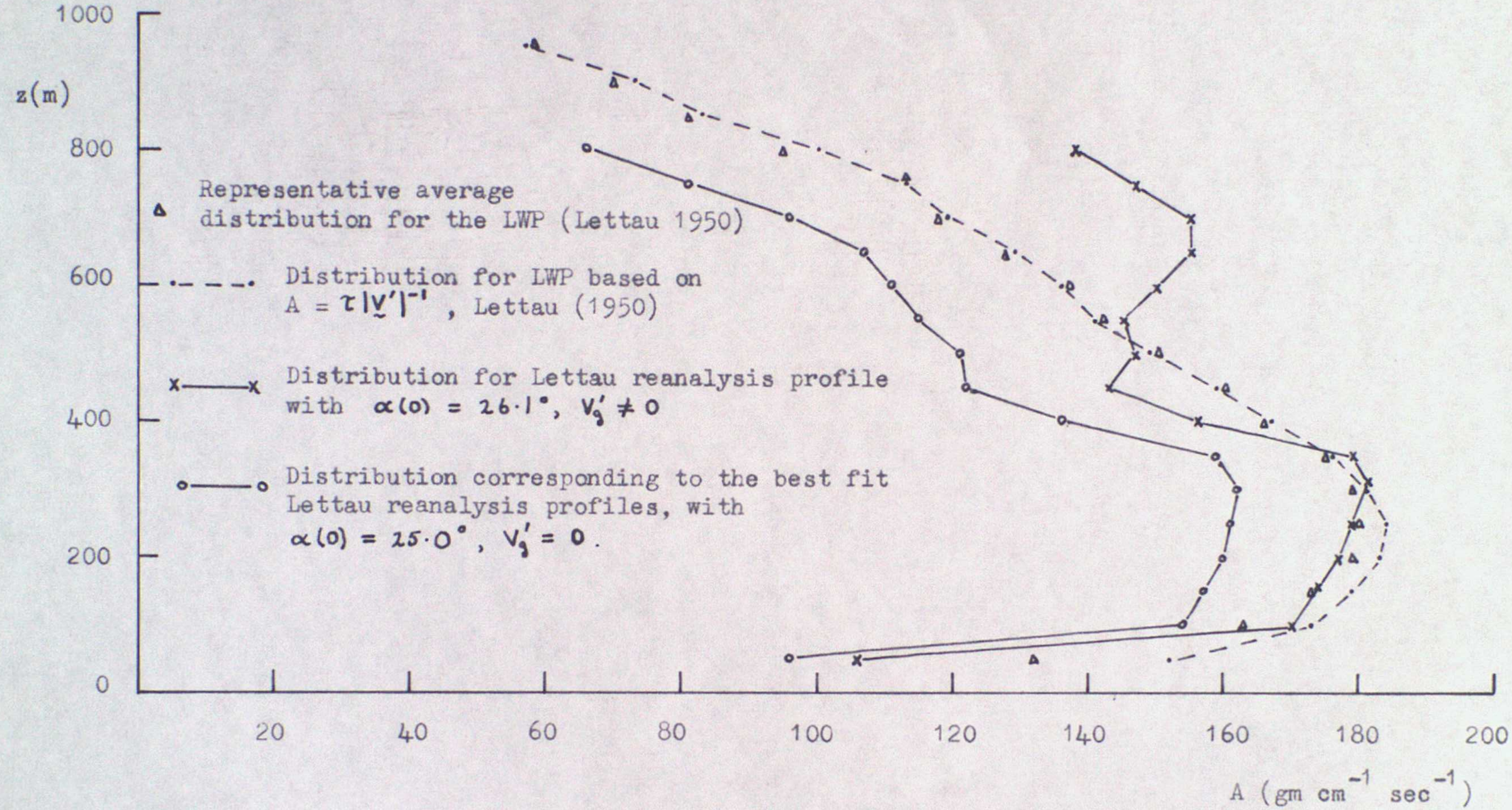

Figure 6. Vertical profiles of the austausch coefficient $A(\mathbf{z})$ $(=\rho K)$, (gm cm^{-1} sec^{-1}), under specified conditions.

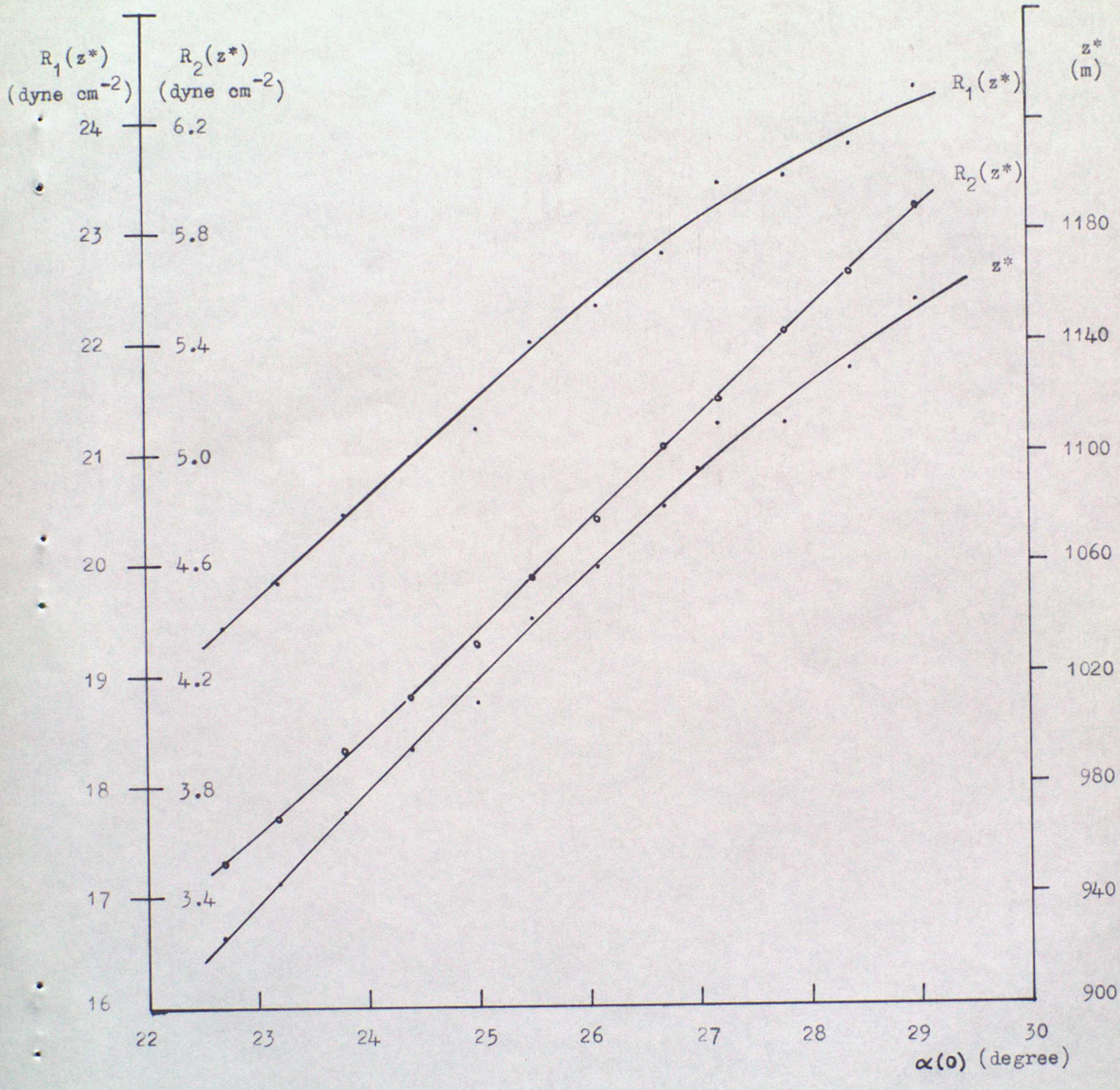

Figure 7. Swinbank reanalysis parameters z^*(m), $R_1(z^*)$ (dyne cm^{-2}), and $R_2(z^*)$ (dyne cm^{-2}) as functions of $\alpha(0)$ (degree).

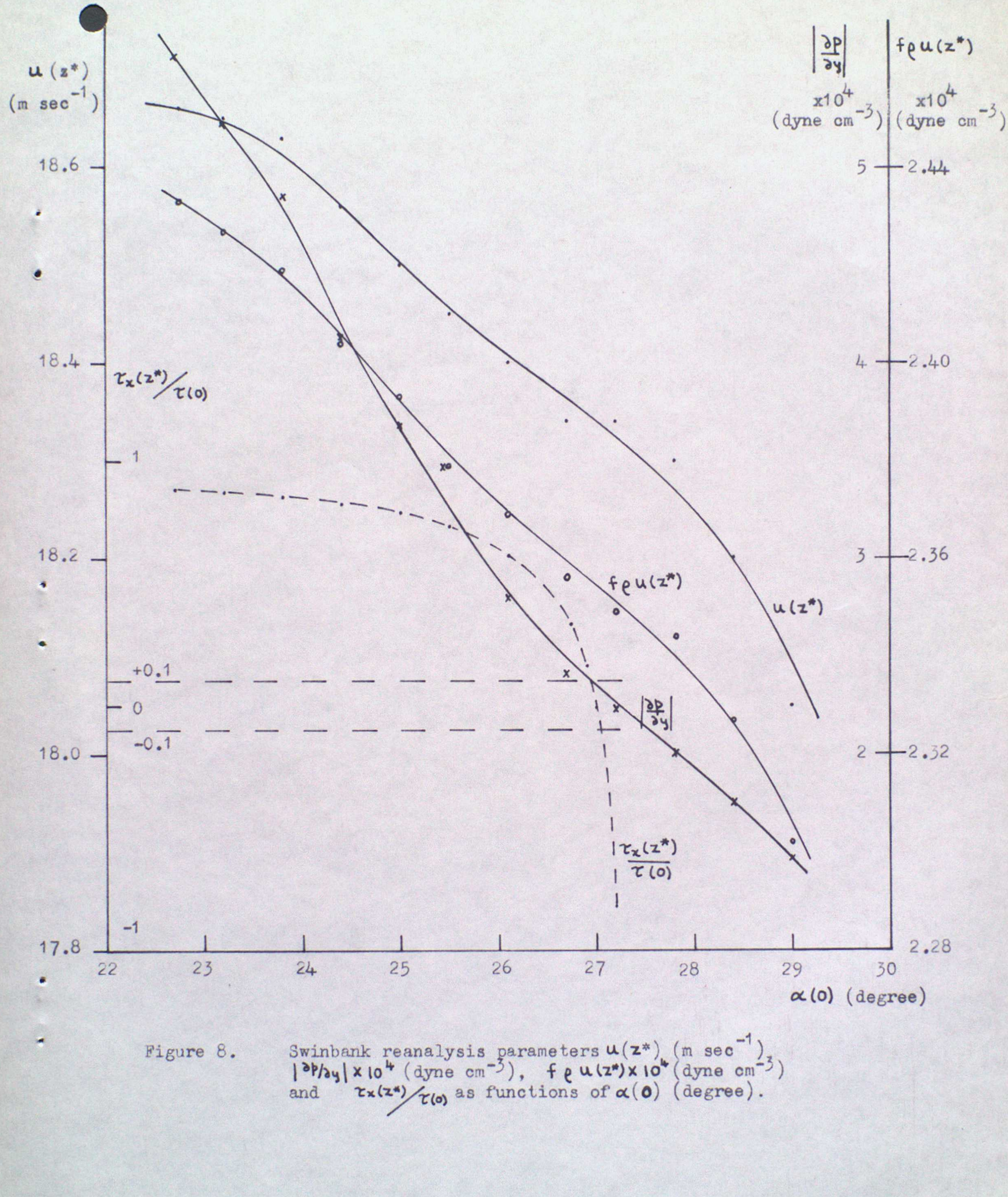

Figure 8. Swinbank reanalysis parameters $u(z^*)$ (m sec^{-1}), $|\partial p/\partial y| \times 10^4$ (dyne cm^{-3}), $f\rho\, u(z^*) \times 10^4$ (dyne cm^{-3}) and $\tau_x(z^*)/\tau(0)$ as functions of $\alpha(0)$ (degree).

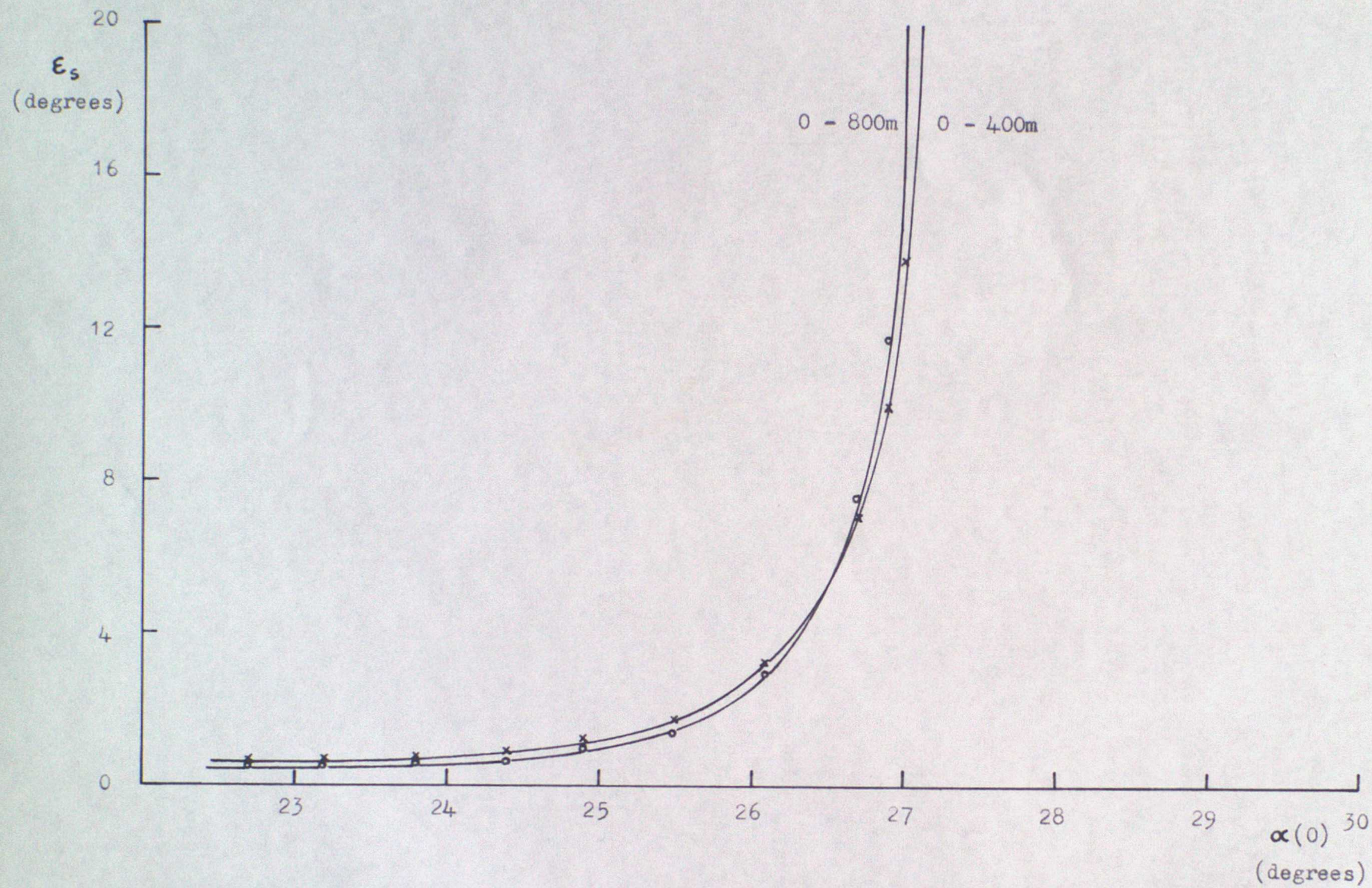

Figure 9. The root-mean-square angular deviation for Swinbank's hypothesis, ε_s (degrees), measured over the vertical ranges 0 - 400m and 0 - 800m, and presented as a function of $\alpha(0)$ (degrees).

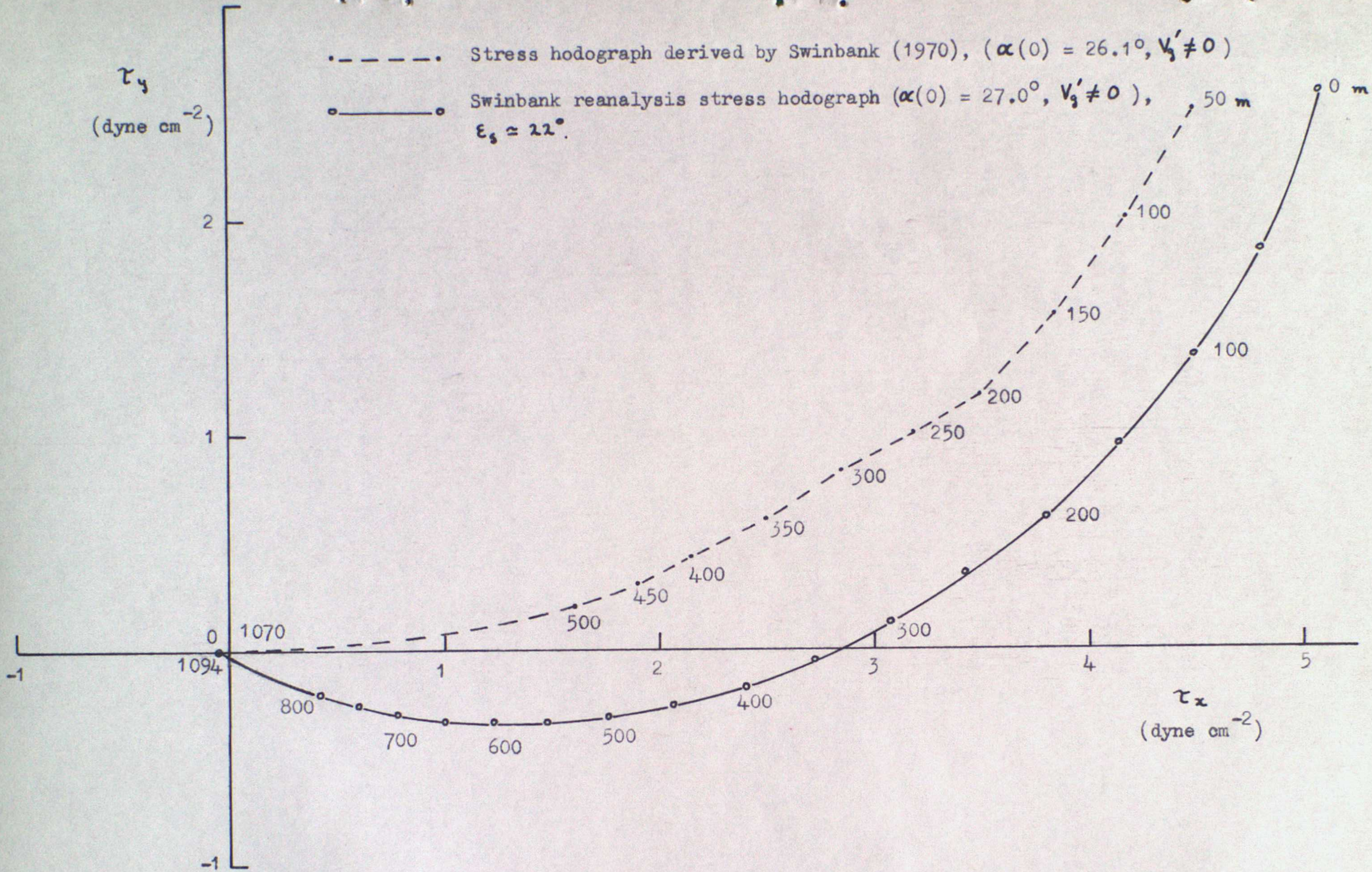

Figure 10. Shearing stress components computed on the basis of Swinbank's hypothesis.

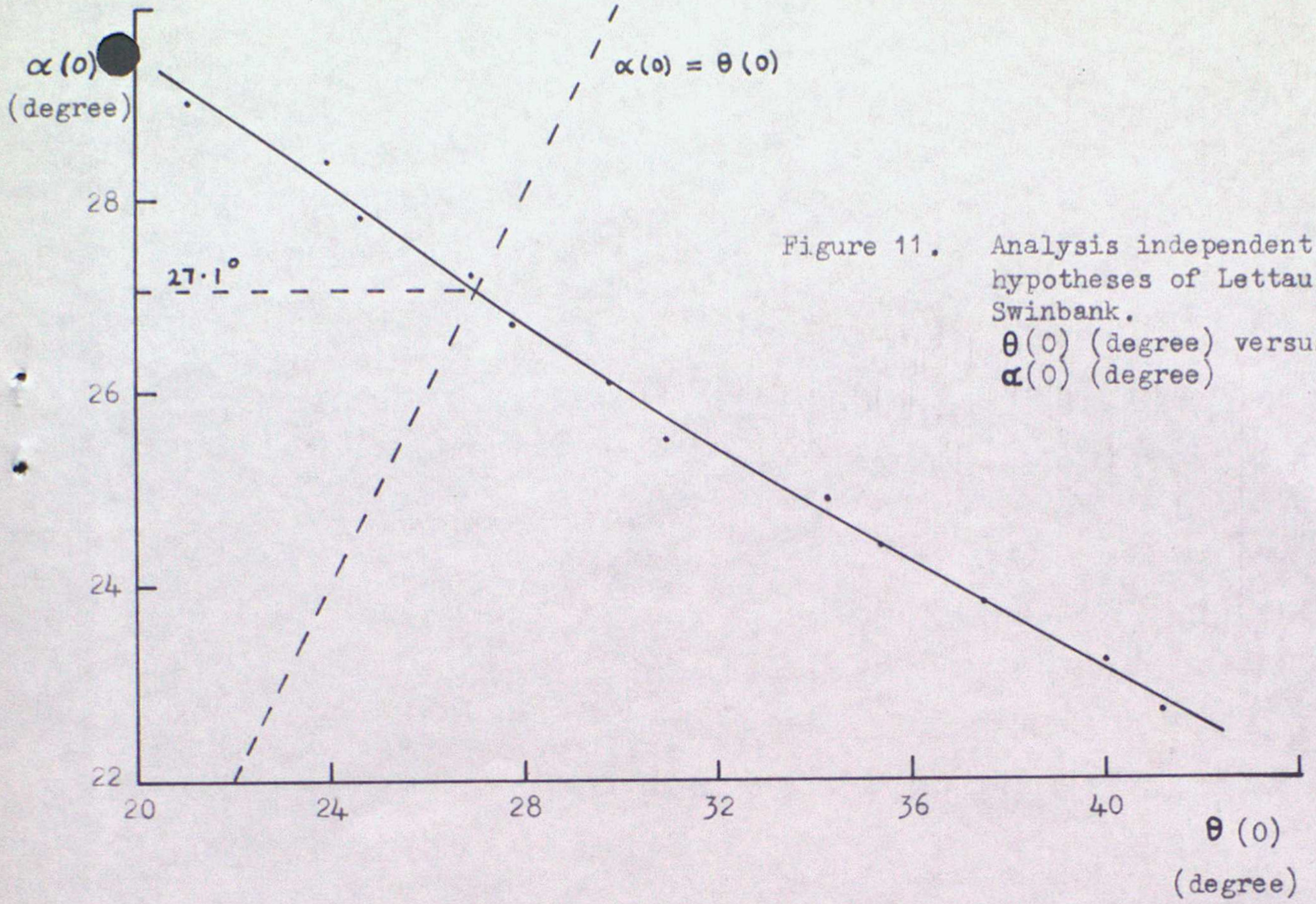

Figure 11. Analysis independent of hypotheses of Lettau and Swinbank. $\theta(0)$ (degree) versus $\alpha(0)$ (degree)

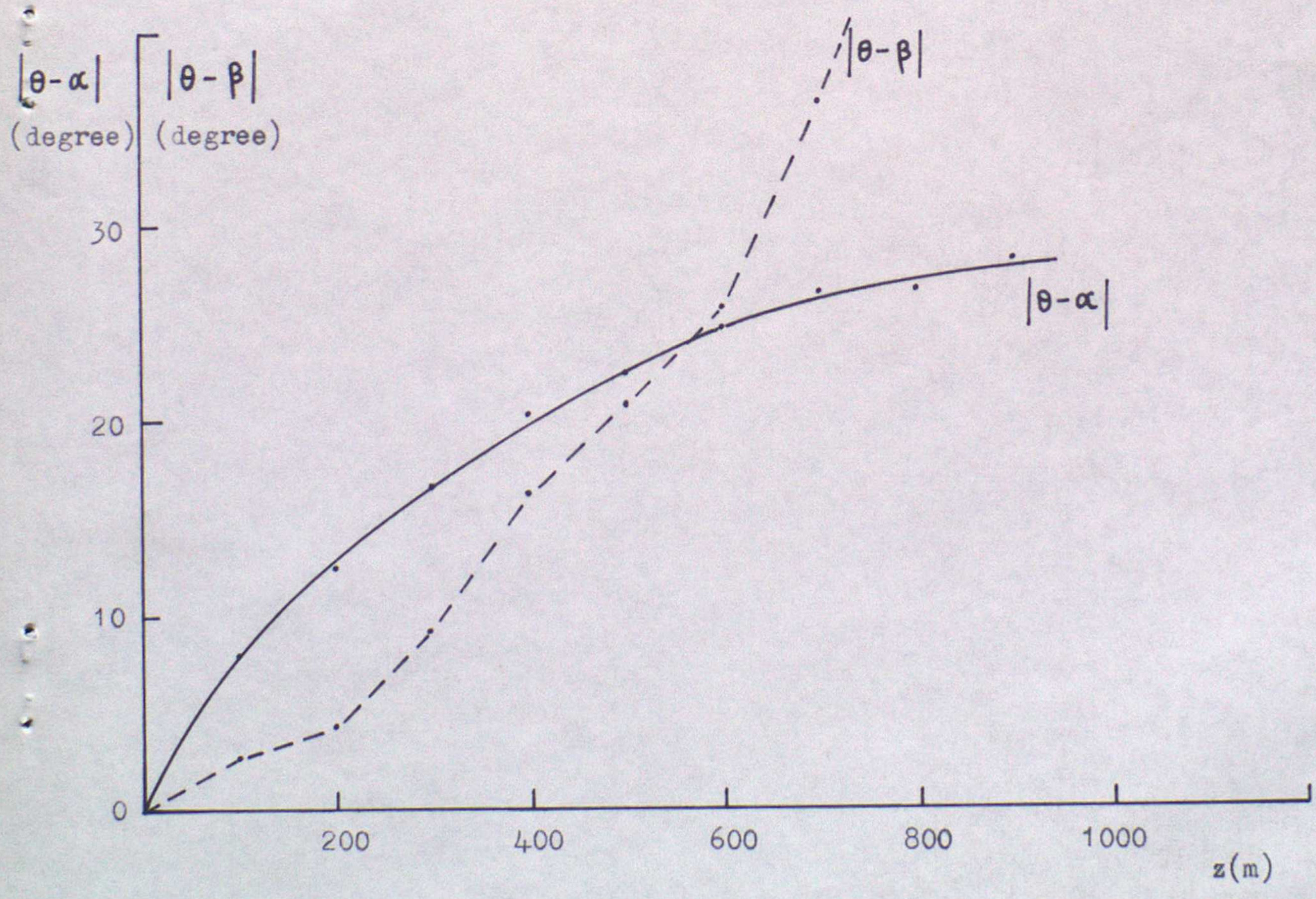

Figure 12. Analysis independent of hypotheses of Lettau and Swinbank. Vertical profiles of $|\theta-\beta|$ (degree) and $|\theta-\alpha|$ (degree) when $\alpha(0)$ = 27.1 degree

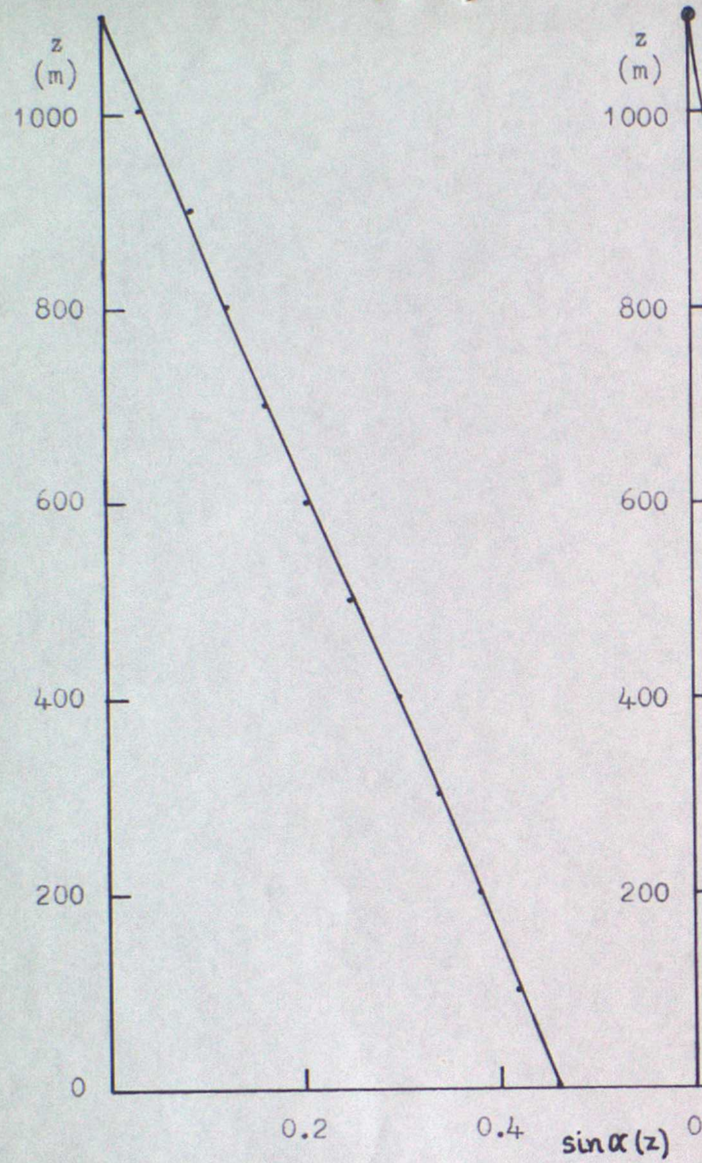

Figure 13. Analysis independent of the hypotheses of Lettau and Swinbank. Vertical profile of $\sin\alpha(z)$ when $\alpha(0) = 27.1$ (degree)

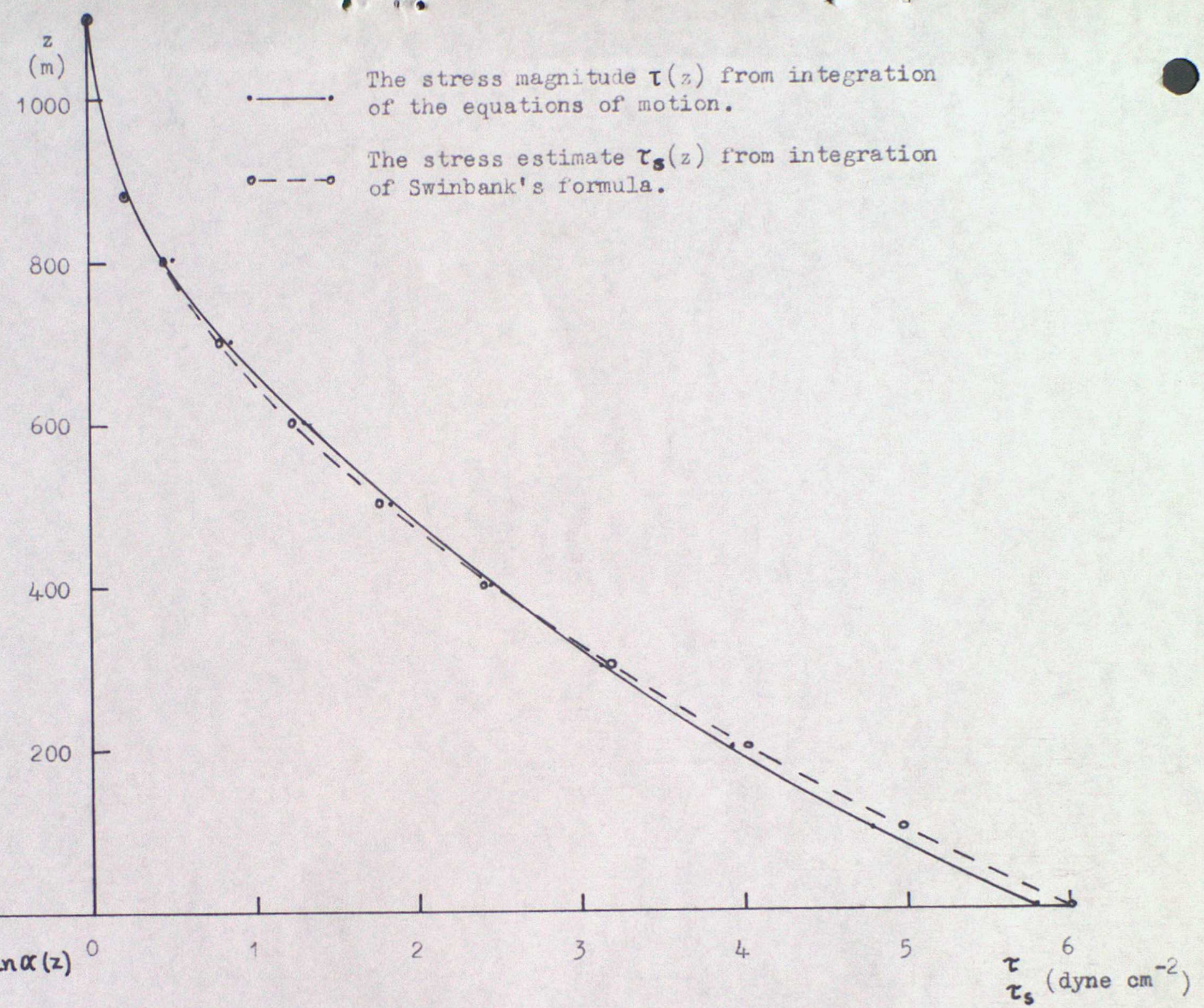

Figure 14. Analysis independent of the hypotheses of Lettau and Swinbank. Vertical profiles of $\tau(z)$ (dyne cm^{-2}) and $\tau_s(z)$ (dyne cm^{-2}) when $\alpha(0) = 27.1$ (degree).

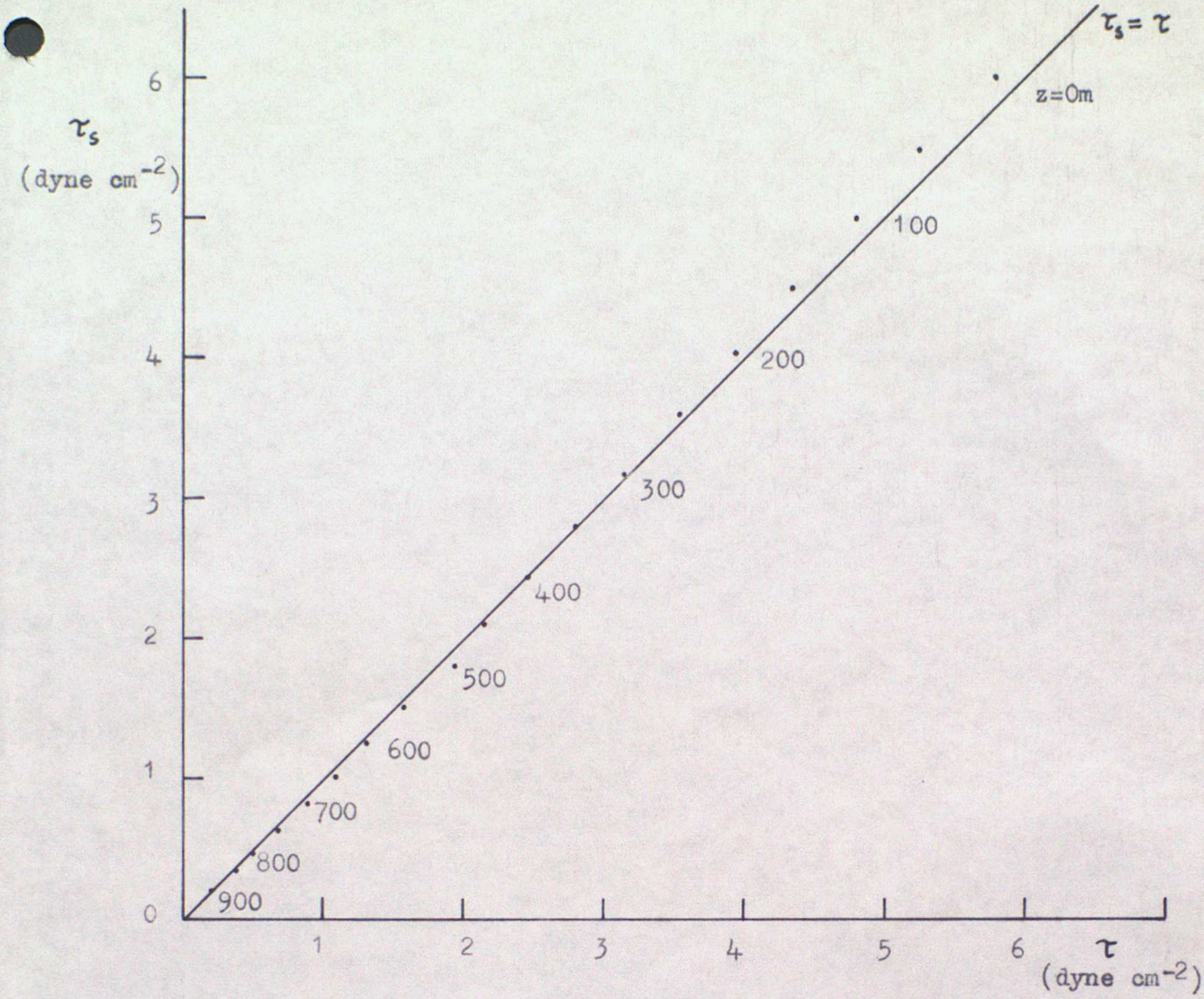

Figure 15. Analysis independent of hypotheses of Lettau and Swinbank. τ_s(dyne cm^{-2}) versus τ (dyne cm^{-2}), for values of z (m) at 50m intervals.

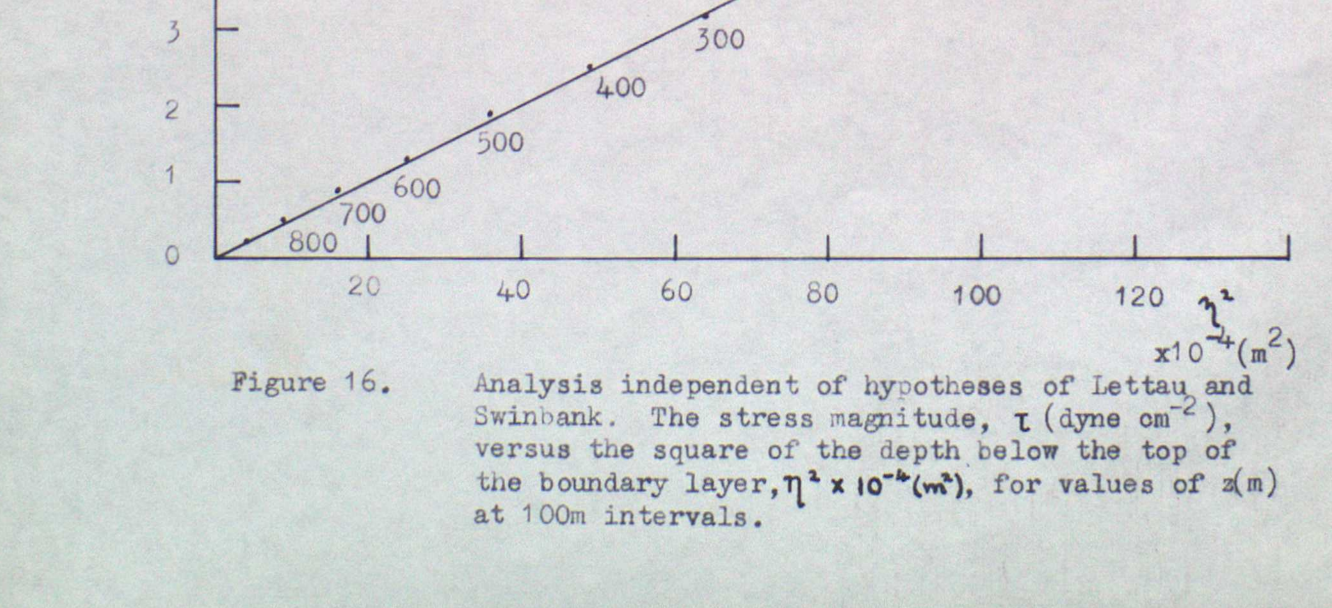

Figure 16. Analysis independent of hypotheses of Lettau and Swinbank. The stress magnitude, τ (dyne cm^{-2}), versus the square of the depth below the top of the boundary layer, $\eta^2 \times 10^{-4}$(m^2), for values of z(m) at 100m intervals.