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Met O (P) Turbulence and Diffusion Note No. 201

On the use of ∇^{2n} operators for smoothing

by

D. J. Thomson

10th March 1992

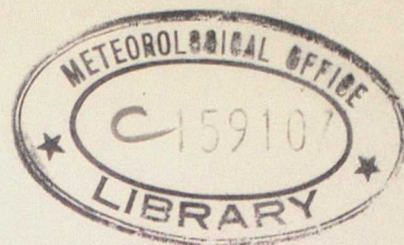
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On the use of ∇^{2n} operators for smoothing.

David J. Thomson, Meteorological Office, Bracknell, U.K.

10th March 1992

In general circulation models it is usual to apply diffusion or smoothing to prevent the build up of excessive fluctuations on small scales. In order to reduce the damping of the low wave number modes, it is quite common to use operators of the form ∇^{2n} (n an integer greater than 1) as an alternative to the 'eddy-diffusivity' approach based on ∇^2 :

$$\frac{\partial(-)}{\partial t} = \dots + K(-1)^{n-1} \nabla^{2n}(-). \quad (1)$$

In spectral terms, the operator $(-1)^{n-1} \nabla^{2n}$ corresponds to multiplication by wave number to the power $2n$ and so the relative damping of the large scales decreases as n increases. The purpose of this note is to record a problem with such operators which, although well-established, does not appear to be widely known. The emphasis here will be on the transport and diffusion of tracers.

For a tracer being advected and diffused in an arbitrary flow, the value of the tracer density c cannot become negative if it is initially non-negative everywhere. It is of course desirable that smoothing procedures preserve this property. However operators of the form ∇^{2n} fail to do this in general because there is no guarantee that $(-1)^{n-1} \nabla^{2n} c$ is non-negative when c is zero. As an example consider $n = 2$ and $c = x^4$. In this case $-\nabla^4 c$ equals -24 and c is zero at $x = 0$. Note that this failure is a property of the continuous equations and can't be blamed on any finite-difference considerations.

There is a sense in which ∇^2 (or its generalisation $\frac{\partial}{\partial x_i} K_{ij} \frac{\partial}{\partial x_j}$) is the most general *local* operator which has all the desired properties. Suppose c evolves according to

$$\left. \frac{\partial c}{\partial t} \right|_{(\mathbf{x}, t)} = [Q(c(-, t))]_{\mathbf{x}} \quad (2)$$

with Q a mapping from the space of functions $R^3 \rightarrow R$ to itself. The subscripts (x, t) and x indicate the points at which the expressions are evaluated. We will assume Q is sufficiently well behaved for (2) to yield unique solutions. Let us also assume (i) Q is such that $c \geq 0 \forall x, t$ if this is true at $t = 0$, (ii) Q is linear and (iii) $\int c dx$ is conserved by (2). The desirability of (i) was discussed above. (ii) and (iii) are of course desirable too, with (ii) allowing superposition of c -fields originating from different sources. Let $P(x, t|y, s)$ be the c -field resulting from a unit source at (y, s) . As a consequence of (i) and (iii), P is a probability density function and, from (ii), P satisfies the Chapman-Kolmogorov equation

$$P(x, t|y, s) = \int P(x, t|z, r)P(z, r|y, s) dz \quad (3)$$

for $t > r > s$. Hence P is the transition probability density for some Markov process. Physically the Markov process can be thought of as corresponding to the motion of 'particles' of tracer. We will now assume for the moment that Q is homogeneous in the sense that, for sources of tracer at (x_0, t_0) and (x_1, t_1) , the resulting c -fields are identical apart from a displacement in space and time. The Markov process now becomes a process with independent increments, with $P(x, t|y, s) = P'(x - y, t - s)$. The possible forms that such processes can take are well understood. In particular P' corresponds to an infinitely divisible distribution (see e.g. Feller 1971) and the characteristic function of P' (defined by $J(u, t) = \int e^{i x \cdot u} P'(x, t) dx$) must take the form $e^{g(u)t}$ with g having the form

$$g(u) = i a \cdot u - K_{ij} u_i u_j + \int_S (e^{i x \cdot u} - 1 - i x \cdot u) \Pi(dx) + \int_{R^3 \setminus S} (e^{i x \cdot u} - 1) \Pi(dx) \quad (4)$$

where a is a vector, K is a positive semi-definite matrix, S is a certain sphere centred on $x = 0$ and Π is a measure satisfying $\Pi(\{0\}) = 0$, $\int_S |x|^2 \Pi(dx) < \infty$ and $\Pi(R^3 \setminus S) < \infty$ (see Gihman and Skorohod 1980, Chapter 3, §1 and Gihman and Skorohod 1983, Chapter 1, §1 or, for an alternative representation of (4), Feller 1971). By dividing (3) by $t - r$, letting $t \rightarrow r$ and using (4) we have

$$\begin{aligned} \frac{\partial P}{\partial t} = & -\nabla \cdot (aP) + \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (K_{ij}P) + \\ & + \int_S (P(x - y) - P(x) + y \cdot \nabla P|_x) \Pi(dy) + \\ & + \int_{R^3 \setminus S} (P(x - y) - P(x)) \Pi(dy). \end{aligned} \quad (5)$$

Clearly Π must be zero if the process is to be local, and so Q consists of a combination of a term representing advection of tracer particles at velocity a and a diffusive term:

$$\frac{\partial P}{\partial t} = -\nabla \cdot (aP) + \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (K_{ij}P). \quad (6)$$

The diffusive term corresponds to particles undergoing random Gaussian displacements with the covariance matrix of the displacements over a time Δt equal to $2K\Delta t$. Note that if

$\Pi(R^3) < \infty$, the term $\int_S \mathbf{y} \cdot \nabla P|_{\mathbf{x}} \Pi(d\mathbf{y})$ in (5) can be absorbed in a redefinition of \mathbf{a} , and the term involving Π , which is then $\int_{R^3} (P(\mathbf{x} - \mathbf{y}) - P(\mathbf{x})) \Pi(d\mathbf{y})$, corresponds to a compound Poisson process, in which the particles move in jumps which occur at random times, the probability distribution of the jump sizes being proportional to Π and the probability of a jump occurring in any short interval Δt being $\Pi(R^3)\Delta t$. It is intuitively clear therefore that the assumption that Q is local could be replaced by the assumption that the particle trajectories are continuous with the same result that Q must have the form (6) (see Gihman and Skorohod 1980, Chapter 3, §5). The general case of (5) corresponds in essence to a continuous representation of 'transilient' diffusion models (see Ebert et al 1989) or, more precisely, of the transilient diffusion models which can be implemented in a time step independent manner and which are homogeneous in the sense described above. For $\mathbf{a} = 0$, $\mathbf{K} = 0$ and Π a finite measure, (5) forms the basis for the 'integral equation' approach to turbulent diffusion (Smith 1982), although here the underlying domain of the Markov process is position-velocity space.

We will now discuss the situation which occurs when we relax the requirement that Q be homogeneous (but maintain a requirement for results to depend smoothly on source position). Intuitively it seems clear that a similar result to (4) should hold with \mathbf{a} , \mathbf{K} and Π functions of position and time. This is in fact the case under fairly weak regularity conditions, namely under the assumption that the Markov process is weakly differentiable in the sense of Gihman and Skorohod (1983, Chapter 1, §1). If we define $J(\mathbf{u}, t|\mathbf{y}, s) = \int e^{i(\mathbf{x}-\mathbf{y}) \cdot \mathbf{u}} P(\mathbf{x}, t|\mathbf{y}, s) d\mathbf{x}$ then

$$\lim_{t \rightarrow s} \frac{J(\mathbf{u}, t|\mathbf{y}, s) - 1}{t - s} \quad (7)$$

has the form (4) with \mathbf{a} , \mathbf{K} and Π functions of \mathbf{y} and s . Together with (3) this determines the rate of change of P . The locality condition then gives the result that Π must be zero and the operator Q must be as in (6), where \mathbf{a} and \mathbf{K} are now functions of \mathbf{x} and t . As for the homogeneous case, the assumption that the particle trajectories are continuous can be adopted instead of the locality assumption (the argument given by Gihman and Skorohod for the homogeneous case can be extended to the more general case — Thomson 1987, Appendix B).

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