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Short Range Forecasting Division Technical Report No 1

ON THE TIME SAVING THAT CAN BE ACHIEVED BY THE USE OF AN OPTIMISED COURSE IN AN AREA OF VARIABLE FLOW

by

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July 1991

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1. Introduction

In planning routes between well-defined points of departure and arrival, both aircraft and ships can take into account forecast values of certain geophysical parameters so that the route chosen is in some sense optimised. For aircraft flying the North Atlantic the methods used are described in the papers by Attwooll, Bennett, and Monk (all 1982). There have been a number of papers on ship-routing: that by Motte and Calvert (1988) has the most comprehensive bibliography. However it should be noted that whereas for aircraft the dominant consideration is the wind, for which the maritime equivalent is the current, ship-routing on the trans-oceanic scale is dominated by considerations of waves. However on smaller scales currents can be the dominant consideration: for example see the paper by Fales (1991). That paper does not make use of the basic theory that was applied to the aeronautical problem in the 1940s. Although the work reported here is orientated to aeronautical applications, it clearly has ramifications for certain maritime problems.

The calculation of optimum routes for aircraft across the Atlantic has an interesting history. The fundamental theory was well understood in the 1940s but at that stage there were no computers on which it could be implemented directly. Consequently graphical methods were developed which could be applied to the meteorological charts which were available then. The methods and the basic theory are described in Sawyer (1949) and the basic theoretical results will be repeated in section 2 of this paper. In the 1970s wind forecasts started to be produced by computer and it was clearly possible to develop computer algorithms for determining the minimum-time tracks. However the algorithms in use today are not based on the basic theory: they are based on considerations of networks of possible routes. In the context of the organised track system over the North Atlantic this approach is reasonable, but for more general applications reference to the basic theory is desirable.

The motivation behind the present work was to answer the question of how much time could be saved by the use of optimal routes in other sectors of airspace. In discussing the economics of the North Atlantic Air Traffic System, Attwooll (1986) comments that "The North Atlantic is a very windy place". It is also true to say that the North Atlantic is a relatively easy place to implement a meteorologically dependent route structure. It is clearly beyond the scope of the current study to address the issue of the feasibility of meteorologically dependent route structures in other sectors of airspace, but it is recognised that this is a very significant issue. However it is noted that Area Navigation (RNAV) routes are now being set up across Europe: although they will not be able to support the flexibility that is available in the North Atlantic, in assessing their value meteorological factors should be taken into consideration.

It was considered desirable to obtain a result which was as general as possible, i.e. it would not be dependent on current route structures as the possible utilisation of optimised routes in the future would render these obsolete. Therefore of necessity the present work can reasonably be regarded as being somewhat academic, but it was considered a sensible preliminary to more practical studies to be done in the future. The original intention was to consider potential time-savings in European airspace: interest then arose in potential savings in trans-Asia routes. Because evaluating the latter required accessing data from a global model the work was extended to evaluate savings globally, omitting the portions of the globe poleward of 72°.

The theory as developed in the 1940s did not provide a simple algorithm for how much time would be saved by the use of an optimal route. In the present work such an algorithm is derived, although care has to be taken with its application. Section 2 of this paper revisits the the basic theory, and describes the algorithm to be used for calculating the time-saving. Section 3 discusses the results of applying the algorithm to atmospheric data, but of necessity the algorithm is something of an approximation. Section 4 describes a case study of time savings in the North Atlantic in which the results from the algorithm are compared with a direct, more accurate calculation.Section 5 draws conclusions and makes suggestions for further studies. The algorithm is derived in the appendix.

2. Theoretical aspects

The equation for the path of least time, quoted by Sawyer (1949) is

$$\frac{d\theta}{dt} = \frac{\partial u}{\partial n} \qquad (1)$$

where $d\theta/dt$ is the rate of change of the aircraft's heading and $\partial u/\partial n$ is the rate at which the wind component in the direction of the aircraft's heading (tail wind) varies in a perpendicular direction. This equation ignores the curvature of the earth but Sawyer quotes a correction term to allow for this.

In the following we are going to extend the above theory in a way which will enable us to gain understanding of the nature of optimum tracks in generalised wind fields. The extension to the theory makes use of the approximation that airspeeds are in general an order of magnitude greater than wind speeds, which is generally true for transatlantic transport aircraft of the 1990s, although it would not have been nearly so applicable in the 1940s. Although $d\theta/dt$ will be the same for aircraft now as it was 50 years ago, the rate of change of heading with distance along ground track will be much less.

Now, let us define γ to be the drift angle, χ to be the ground track angle, v to be the component of wind perpendicular to the aircraft's heading, and A to be the aircraft's airspeed.
From these definitions it follows that

$$\gamma + \theta = \chi \qquad (2)$$

and

$$\tan \gamma = v/A \qquad (3)$$

If $v \ll A$, then from (3)

$$\gamma = v/A \qquad (4)$$

Assuming constant airspeed, then from (4)

$$\frac{d\gamma}{dt} = \frac{1}{A}\frac{\partial v}{\partial t} = \frac{\partial v}{\partial s} \qquad (5)$$

where s is distance along a line parallel to the aircraft's heading.
Now, using (1), (2) and (5)

$$\frac{\partial u}{\partial n} + \frac{\partial v}{\partial s} = \frac{d\chi}{dt} \qquad (6)$$

A further relationship that we wish to use is

$$\frac{\partial u}{\partial n} + \frac{\partial v}{\partial s} \approx \nabla \wedge \mathbf{v} = \zeta, \quad \text{the vorticity.} \qquad (7)$$

(This is an approximation only because u,v,n,and s are defined relative to the aircraft's heading which will in general vary only slowly along the aircraft's track. Thus, combining (6) and (7), we have shown that the equation for the path of least time can be written

$$\frac{d\chi}{dt} = \zeta \qquad (8)$$

i.e. the rate of chnge of the ground track angle is equal to the vorticity, a quantity which is independent of the aircraft's orientation.

This might seem a curious result and it is worth considering a simple case to show that it is, qualitatively at least, plausible. Figure 1 shows a wind field in which the wind is everywhere northerly or southerly, with the wind speed being proportional to distance east-west from the centre line. We could imagine this to be a portion of a trough with it axis running north-south. It will be noted that the vorticity of this wind field is constant everywhere. Also shown are approximate minimum time tracks between two pairs of points, A and B are on the same north-south line while C and D are on the same east-west line. The minimum time tracks between A and B are flown by first pointing the aeroplane towards the stronger tail winds and then changing the heading steadily with time so that the aeroplane arrives at the desired point. In contrast, the tracks between C and D are flown at constant heading, either due east or due west, and the aircraft deviates from the straight line path as a result of the cross-winds. In figure 1 the optimum tracks were drawn as arcs of circles: this is a slight approximation because although the rate of change of track angle with time will be constant, due to variations in ground speed the rate of change of track angle with distance along ground track will vary slightly. Apart from that factor the ground tracks are realistic for an aircraft with an airspeed of 250 m/s (486 knots): the figure represents an area 1000km by 1000km (540nm by 540nm).

Figure 1 prompts one to ask what time will be saved in such an idealised case and the theoretical part of this study was aimed at answering the specific question as to what time would be saved when flying across an area where the vorticity is constant everywhere and the terms which contribute to the divergent part of the flow are both zero. It may be recalled that a horizontal wind field can be decomposed into a rotational part and a divergent part. The vorticity is a function only of the rotational part. Equation 8 strongly implies that terms giving rise to divergence can be ignored.

As the appendix shows, the answer to this question is $\zeta^2T^3/24$, where T is the time that would be taken to fly across the area if the flow was uniform. The formula is an approximation: there are terms in T^5 and higher powers of T which can normally be ignored. As is suggested qualitatively by figure 1, the result is independent of the orientation of the aircraft's route. For the case shown in figure 1, in which ζ is $10^{-4}s^{-1}$ and T is 4000s the time saved is 26.7s.

It is not immediately apparent that this formula can be applied to the real atmosphere in which, in general, the vorticity is varying everywhere. The approach lies in considering the variability of vorticity on different spatial scales. Let us consider two portions of the atmosphere, each having constant vorticity: an aircraft would take 1 hour to fly across the first in which the vorticity is ζ_1 and 2 hours to fly across the second in which the vorticity is ζ_2. For the percentage time saved across both to be the same, $\zeta_1 = 2\zeta_2$. Intuitive knowledge of the atmosphere suggests that this is unlikely, it is more likely that $\zeta_1 < 2\zeta_2$, and if this is true for a range of scales then the time saved in an area of variable vorticity can be estimated by considering only the vorticity on the largest scale within that range. This assertion will be tested in the next section.

The implication of this assumption is that time saved on a particular flight will be dominated by consideration of the vorticity on the largest scale relevant to that flight. This in turn implies that there will be proportionately bigger savings on longer flights than short. This statement is broadly correct, and is another reason why the north Atlantic is an attractive place to introduce a meteorologically dependent route structure.

On a philosophical note, it is commentworthy that the concept underpinning the choice of optimum route is that of resonance. Both equations (1) and (8) are expressions of equality between a frequency (i.e. a quantity having the dimensions t^{-1}) characteristic of the aircraft and a frequency characteristic of the atmosphere. This concept is exploited explicitly in the appendix, which uses the theory of forced simple harmonic motion.

3. Calculation of global statistics of ζ^2.

Following section 2, it was considered desirable to derive a "climatology" of squared averaged vorticity, here the averaging is performed on a variety of spatial scales. The basic source of data was archived analyses of 250mb wind fields obtained by the then Met Office global operational NWP model (Bell and Dickinson, 1987). This model had a resolution of $1\frac{1}{2}$ ^{o} latitude by $1\frac{7}{8}$ ^{o} longitude. Although it is possible to calculate vorticity on a grid with the same resolution it was considered that the values would be unreliable, so sets of 3*3 values were averaged. Statistics were thus calculated for areas of $4\frac{1}{2}$ ^{o} by $5\frac{5}{8}$ ^{o}, 9^o by $11\frac{1}{4}$ ^{o}, 18^0 by $22\frac{1}{2}$ ^{o} and 36^o by 45^o.
In general all the statistics for the smaller areas within one 36^o by 45^o block were lumped together but for the European sector (36^o to 72^o north and 10^o west to 35^o east) seperate statistics were maintained.

The quantities calculated are the the temporally averaged squared spatially averaged vorticities. The units used for vorticity^2 are (knots per degree latitude)^2. In these units the time saved in seconds = $(\zeta^2 T^3)/24$ where T is in hours.

Statistics were generated for winter 1990/1991, i.e. the months of December 1990, January 1991 and February 1991. In principal several years data (7 at least) should be use to generate a climatology, and seperate climatologies should be generated for different seasons. However we consider the present work to be a pilot study and that therefore generating statistics pertaining to a single season is worthwhile. In general the winter season is the windiest and as relatively few aircraft fly in the southern hemisphere, northern hemisphere winter was considered the most meaningful season to study.

First, however, results are shown which offer qualified support for the hypothesis that certain scales can be ignored in the calculation of time saved. In figure 2 the absissa is horizontal scale, ranging from 36° by 45° (scale 1) to $4\frac{1}{2}{}^{\circ}$ by $5\frac{5}{8}{}^{\circ}$ (scale 4), while the ordinate is the log of mean squared vorticity. Results for all 32 sectors are displayed. Also shown are straight line whose slope is such that if the real data had that slope, then all scales would contribute equally to the time saved. It is clear that the two smaller scales contribute much less than the two larger scales.

Figure 3(a) shows the mean squared vorticities for all the processed sectors of the world on the 36^{0} by 45° scale. Figure 3(b) shows the corresponding figures for the 18^{0} by $22\frac{1}{2}{}^{\circ}$scale, figure 3(c) shows the figures for the 9° by $11\frac{1}{4}{}^{\circ}$ scale and figure 3(d) shows figures for the $4\frac{1}{2}{}^{\circ}$ by $5\frac{5}{8}{}^{\circ}$scale.

These figures show that while the north Atlantic is an area where considerable savings can be made, so is Europe. In order to determine the likely time saving for a flight of a particular length, one should make use of the figures pertaining to the scale which is commensurate with the particular flight.

In the following, the literal interpretation of the numbers presented in figures 3(a) to 3(d) is given. However, the direct calculation of the time saving, described in section 4, implies that a more ad hoc interpretation of the data is appropriate.

For the north Atlantic, it can be shown that the 18^{0} by $22\frac{1}{2}{}^{\circ}$scale contributes most to the time saved. A sector of this scale would typically take approximately 3 hours to cross, so the time saved would be $(22.26 * 3^{3})/24 = 25$ seconds. Thus the total saving on a six hour flight would be of the order of 50 seconds. Other scales of course contribute to the saving, so the total saving will be greater than this. However the method as developed cannot accurately quantify the savings in a field of variable vorticity, as exemplified by the superimposition of two or more scales. Note also that in the north Atlantic the zonal variability of the wind field is rather less than the meridional, so it would have been more accurate (given that most flights are roughly parallel to lines of latitude) to use sectors having different proportions: this would almost certainly give an increase in the estimate of the time saved.

Another cause for underestimation is the fact that the theory neglects the curvature of the earth. If this is taken into account, the values shown in figure 3(a) should be increased by approximately 16%, while for the smaller scales the increase is much less. If scales greater than the 36^{0} by 45^{o} scale are addressed then this factor is of course much bigger.

Figures 4(b), (c), and (d) are analagous to the corresponding figure 3s but show the breakdown for the various areas of Europe. These are presented because there are considerable variations in meteorology over Europe, and also considerable variations in air traffic density and length of routes. Note that figure 4(a) is redundant because there is no breakdown on this scale and therefore is not shown.

4. The direct calculation of the time saved through use of optimum routes

In this section we will describe a method of obtaining the optimum route by integrating equation (1), the time saved by using the optimum route calculated this way and a comparison with the time saved calculated using the algorithm $\Delta T = \zeta^2 T^3/24$. In principle, equation (1) is not adequate for calculating the optimum route, since it defines the rate of change of heading but does not tell us what the *initial* heading should be. However, an iterative approach is perfectly possible, in which routes are calculated for a sequence of initial headings, the choice of heading being dependent on the results of previous calculations. If one defines x and y axes (perpendicular) such that the route to be followed is approximately parallel to the x axis, one can then integrate equation (1) along the route until the x coordinate is equal to that of one's desired point of arrival. If one notes the y coordinate at that point on the route, it is in general found that such y values are a monotonic function of the initial heading. The choice of heading for the next iteration is made assuming that the dependency of the final y on the initial heading is a linear function. There is, of course, no theoretical justification for such an assumption, but it allows the iterations to converge rapidly on the required solution.

In fact, rather than use equation (1), equation (8) was used: it was shown in section 2 that these are essentially equivalent. The use of this equation is not quite as straightforward as might appear because the conversion from rate of change of ground track angle to curvature of ground track depends on ground speed, which is a function of wind speed and direction, airspeed and of course ground track angle itself. In the procedure adopted, the ground speed used in each increment was determined using the *initial* ground track angle: this is a slight approximation, but with increments in x of 50km the error is almost certainly negligible.

In the calculation, x and y are coordinates in a polar stereographic projection and this projection is used in the figures that follow. It will be recalled that in a polar stereographic projection a great circle route is the arc of a circle. The radius of curvature of this circle is calculated before simulations for the particular route start, and this curvature increment is added to the curvature calculated from the vorticity before each increment of the route is calculated. In this way the fact that equation (8) ignores the curvature of the earth is corrected.

Figure 5(a) shows the optimum routes between three plausible entry points into the North Atlantic organised track system and three plausible exit points for eastbound routes. Figure 5(b) is analagous but for westbound routes, and also shown in both figures are a subset of the wind data used in the calculation. The wind data pertained to a case during November 1991: this case was deliberately chosen as one with sufficient wind for the optimum routes to depart markedly from great circle routes. In the calculation it was assumed that the aircraft had a constant airspeed of 500 knots. The wind data were for 200 mb (approximately 39000 feet), and were taken from the assimilated field of the new operational limited area model (Cullen, 1991), which has a horizontal resolution of approximately 50km. Although this model has better horizontal resolution than any of its predecessors, it still underestimates the magnitudes of vorticities near jet streams, so any calculated time-savings using data from the model will be slight underestimates.

However certain features of the optimum routes are worthy of comment. The eastbound routes concentrate themselves into a relatively short north-south distance, in order to gain maximum benefit from the jet stream. Of the westbound routes, the southernmost shows a characteristic "S" shaped pattern which is the optimum course for an aircraft which wishes to fly obliquely across a prevailing jet stream.

The times taken for both optimum and great circle routes, both east and west bound are shown in table 1.

Table 1.

	mean time following great circle routes	mean time following optimum routes	time saving
eastbound	2hrs 54 mins	2hrs 51 mins	3 mins
westbound	4hrs 14 mins	4hrs 5mins	9 mins

Superficially it might seem surprising that the eastbound savings are so much smaller than the westbound savings on the same routes, but it arises from the fact that the times generally on the eastbound routes are so much less than the westbound times, because of the strong westerly flow. If one uses the formula $\Delta T = \zeta^2 T^3/24$ inversely to infer ζ^2 from the time saved, one obtains 171.36 (knots per degree latitude)^2 for the westbound routes and 141.84 for the eastbound routes, which are tolerably close, given the geographical distance between the sets of routes.

When the mean squared values of vorticity are calculated from the model fields for the scales used in section 3, values of 181.08, 97.56 and 22.32 are obtained for scales $4\frac{1}{2}^{o}$ by $5\frac{5}{8}^{o}$, 9^{o} by $11\frac{1}{4}^{o}$, and 18^{o} by $22\frac{1}{2}^{o}$

respectively. Thus it appears that the values for the technically appropriate scale are too small, although if one uses the values for the smallest scale one gets essentially the correct answer. This arises because the conceptual model of an area of constant vorticity is clearly not applicable to a domain straddled by a jet stream which will in general comprise an area of strong cyclonic vorticity on the poleward side and an area of strong anticyclonic vorticity on the equatorward side.

5. Conclusions and suggestions for further studies

The main conclusion from the case study is that in order to obtain approximately correct results one should use the values of squared vorticity pertaining to the smallest scale. Thus to calculate the average time saving for four-hour flight sectors over the North Atlantic, on would take the figure of 83.19 from figure 3(d), multiply by 4^3 and divide by 24, giving 221.84 seconds, or 3 minutes 41 seconds. Although this might seem rather small, multiplied byt eh number of aircraft flying the North Atlantic, it represents a significant fuel saving. It should be borne in mind that the savings on longer flight sectors will be greater, although it is difficult to quantify this from the information presented. Thus it is considered that if one wants to assess how great the savings for a particular set of routes would be, one should first examine figure 3(d) to see whether this is an area of the world where significant savings should be broadly expected, and then perform a set of calculations for the routes envisaged, as suggested in section 4.

A comparison with the time taken to fly a great circle route would be of interest, but given that, at present, great circle routes are not flown because of air traffic control constraints, this is still a somewhat academic exercise. Thus any future study should probably consider two or more plausible air traffic management scenarios.

In addition the operation of the aircraft should be considered: the optimisation considered in the present work was that of minimising the time taken to fly a particular route at constant airspeed and constant flight level. This is sometimes the relevant criterion: on other occasions aircraft minimise cost or fuel. Also it is desirable to consider all phases of flight, rather than just the cruise phase as is addressed by the present study.

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Note. It is hoped to submit this paper to the Journal of Navigation so in the following references to "This Journal" mean that publication.

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Appendix. Derivation of the time saving algorithm

In the following we are using a coordinate system fixed relative to the ground, as opposed to the system used in section 2, which was defined relative to the aircraft's heading. Thus u and v do not have the meaning they did in that section.

Assume that the aircraft is flying at an airspeed A, that it is changing its heading at a constant rate ω ($\equiv d\vartheta/dt$ of section 2) and the initial heading is ε. Given the spatial invariance of the vorticity and equation (1), a contant rate of heading change is a plausible strategy. Then if U,V are the components of the aircraft velocity (parallel to the x and y axes respectively) relative to the air

$$\left.\begin{aligned} U &= A\sin(\omega t+\varepsilon) \\ V &= A\cos(\omega t+\varepsilon) \end{aligned}\right\} \qquad (A1)$$

Assume that the corresponding components of the wind field are given by

$$\left.\begin{aligned} u &= -\alpha y \\ v &= +\beta x \end{aligned}\right\} \qquad (A2)$$

From (A1) and (A2) it follows that

$$U_{ground} = A\sin(\omega t+\varepsilon) - \alpha y \qquad (A3)$$

$$V_{ground} = A\cos(\omega t+\varepsilon) + \beta x \qquad (A4)$$

We wish to set up differential equations governing the motion of the aircraft, so in the following, we use x and y to represent the position of the aircraft. Hence from (A3) and (A4)

$$\ddot{y} = \frac{dV_{ground}}{dt} = -A\omega\sin(\omega t+\varepsilon) + \beta A\sin(\omega t+\varepsilon) - \alpha\beta y$$

$$\ddot{y} = A(\beta-\omega)\sin(\omega t+\varepsilon) - \alpha\beta y$$

This is the equation of forced simple harmonic motion (see, for example, Synge and Griffith (1959) p152), of which the general solutions are :

$$y = y_1 + y_2$$

where y_1 is the solution of the equation

$$\ddot{y} + \alpha\beta y = 0$$

The general solution of this equation is

$$y = a\sin(\rho t+\phi) \tag{A5}$$

where $\rho = \sqrt{\alpha\beta}$ and a and ϕ are arbitrary.

y_2 is given by

$$y_2 = A^*(\beta-\omega)\sin(\omega t+\varepsilon) \tag{A6}$$

$$\text{where } A^* = \frac{A}{\rho^2 - \omega^2}$$

The consistent solutions for x are

$$x_1 = \frac{a\rho}{\beta}\cos(\rho t+\phi) \tag{A7}$$

$$x_2 = -A^*(\alpha-\omega)\cos(\omega t+\varepsilon) \tag{A8}$$

From (A4), (A7) and (A8)

$$V_{ground} = A\cos(\omega t+\varepsilon) + a\rho\cos(\rho t+\phi) - A^*\beta(\alpha-\omega)\cos(\omega t+\varepsilon)$$

$$= A^*\omega(\beta-\omega)\cos(\omega t+\varepsilon) + a\rho\cos(\rho t+\phi) \tag{A9}$$

In the following we will assume that the aircraft wishes to fly between two points having the same x coordinate so that we wish to minimise some integral of V_{ground}.

Now we assume that when $t = -\Delta T$, $x = x_0$

$$t = +\Delta T,\ x = x_0$$

Using (A7) and (A8) we get

$$\frac{a\rho}{\beta}\cos(\phi-\rho\Delta T) - A^*(\alpha-\omega)\cos(\varepsilon-\omega\Delta T) = x_0$$

$$\frac{a\rho}{\beta}\cos(\phi+\rho\Delta T) - A^*(\alpha-\omega)\cos(\varepsilon+\omega\Delta T) = x_0$$

Adding,

$$\frac{a\rho}{\beta}\cos\phi\cos\rho\Delta T - A^*(\alpha-\omega)\cos\varepsilon\cos\omega\Delta T = x_0 \tag{A10}$$

We wish to determine

$$V_{mean} = \frac{1}{2\Delta T}\int_{-\Delta T}^{\Delta T} V_{ground}\, dt$$

Using (A9)

$$V_{mean} = \frac{1}{2\Delta T}\left\{A^*(\beta-\omega)\Big(\sin(\omega\Delta T+\varepsilon)+\sin(\omega\Delta T-\varepsilon)\Big)+a\Big(\sin(\rho\Delta T+\phi)+\sin(\rho\Delta T-\phi)\Big)\right\}$$

$$= \frac{1}{2\Delta T}\left\{A^*(\beta-\omega)\cos\varepsilon\ \sin\omega\Delta T + a\cos\phi\ \sin\rho\Delta T\right\}$$

Substituting for a from (A10)

$$V_{mean} = \frac{1}{\Delta T}\left\{A^*(\beta-\omega)\cos\varepsilon\ \sin\omega\Delta T + \frac{\rho}{\alpha}\tan\rho\Delta T\left(A^*(\alpha-\omega)\ \cos\varepsilon\ \cos\omega\Delta T + x_0\right)\right\}$$

$$= \frac{A^*}{\Delta T}\cos\varepsilon\left\{(\beta-\omega)\sin\omega\Delta T + \frac{\rho}{\alpha}(\alpha-\omega)\tan\rho\Delta T\ \cos\omega\Delta T\right\} + \frac{x_0}{\Delta T}\ \frac{\rho}{\alpha}\tan\rho\Delta T$$

write as

$$V_{mean} = \psi\ \cos\varepsilon + \frac{x_0\rho}{\Delta T\alpha}\ \tan\rho\Delta T \qquad \text{(A11)}$$

$$\text{where } \psi = \frac{A^*}{\Delta T}\left\{(\beta-\omega)\sin\omega\Delta T + \frac{\rho}{\alpha}(\alpha-\omega)\tan\rho\Delta T\ \cos\omega\Delta T\right\}$$

The x_o term in (A11) is independent of ω so it does not affect the optimisation of route choice and we will disregard it for now.

Expand to third order in ΔT

$$\psi \approx \frac{A^*}{\Delta T}\left\{(\beta-\omega)\left(\omega\Delta T - \frac{(\omega\Delta T)^3}{6}\right) + (\alpha-\omega)\beta\Delta T\left(1 - \frac{(\omega\Delta T)^2}{2}\right)\left(1 + \frac{1}{3}\alpha\beta\Delta T^2\right)\right\}$$

(here we have used (A5) to substitute for ρ)

$$= A^*\left\{(\alpha\beta-\omega^2)\left(1 + \frac{\Delta T^2}{6}\left(2\alpha\beta - 2\beta\omega - \omega^2\right) + 0(\Delta T^4)\right)\right\}$$

Recalling the definition of A^*

$$\psi \approx A\left(1 + \frac{\Delta T^2}{6}\left(2\alpha\beta - 2\beta\omega - \omega^2\right)\right)$$

ψ_{max} occurs when $\omega = -\beta$, which is exactly what we would expect from (1).

i.e. $\psi_{max} = A\left(1 + \frac{\Delta T^2}{6}\left(2\alpha\beta + \beta^2\right)\right)$

If aircraft is to follow straight line, $\omega = \alpha$ (this follows from (2) and (5), with χ set constant).

Under these circumstances

$$\psi = A\left(1 - \alpha^2\ \frac{\Delta T^2}{6}\right)$$

$\therefore$ Improvement in speed by using optimum ω compared with straight line

$$= A\frac{\Delta T^2}{6}\left(2\alpha\beta+\beta^2+\alpha^2\right)\ \cos\ \varepsilon$$

$$= A\frac{\Delta T^2}{6}\left(\alpha+\beta\right)^2 \cos\,\varepsilon = A\frac{\Delta T^2}{6}\zeta^2 \cos\,\varepsilon$$

Now define T as the time taken to fly across the region at the initial heading ε in still air, so that to a very good approximation $T = 2\Delta T$, and we derive the algorithm for the time saved as being $\zeta^2 T^3/24$. This T depends on the strength of the mean cross-wind (through the ε term) but not on the strength of the mean following wind (which is the x_o term in equation (A11)) which was disregarded. Assuming once again that airspeeds are an order of magnitude greater than windspeeds, ε will be quite small and to a very good approximation T can be taken to be the time taken to fly the route in still air.

Figure 1. Shows a hypothetical wind field and approximate minimum time tracks between points A and B and between points C and D.

Figure 2. Shows the logarithm of squared vorticity for 4 scales, where scale 1 is 36° by 45° and scale 4 is $4\frac{1}{2}^\circ$ by $5\frac{5}{8}^\circ$.

Figure 3(a). Shows temporally averaged squared spatially averaged vorticities, where the temporal averaging is for winter 1990-91 and the spatial averaging is over areas of $36^\circ$by 45°. Units are (knots per degree latitude)^2.

Figure 3(b). As figure 3(a) but over areas of 18° by $22\frac{1}{2}^\circ$.

Figure 3(c). As figure 3(a) but over areas of 9° by $11\frac{1}{4}^\circ$.

Figure 3(d). As figure 3(a) but over areas of $4\frac{1}{2}^\circ$ by $5\frac{5}{8}^\circ$.

Figure 4(b). As figure 3(b) but giving detailed breakdown for Europe.

Figure 4(c). As figure 3(c) but giving detailed breakdown for Europe.

Figure 4(d). As figure 3(d) but giving detailed breakdown for Europe.

Figure 5(a). Shows minimum time tracks between three entry points into North Atlantic airspace and three exit points, for eastbound flights. Also shows a subset of the wind data used in the calculation.

Figure 5(b). As figure 5(a) but for westbound flights

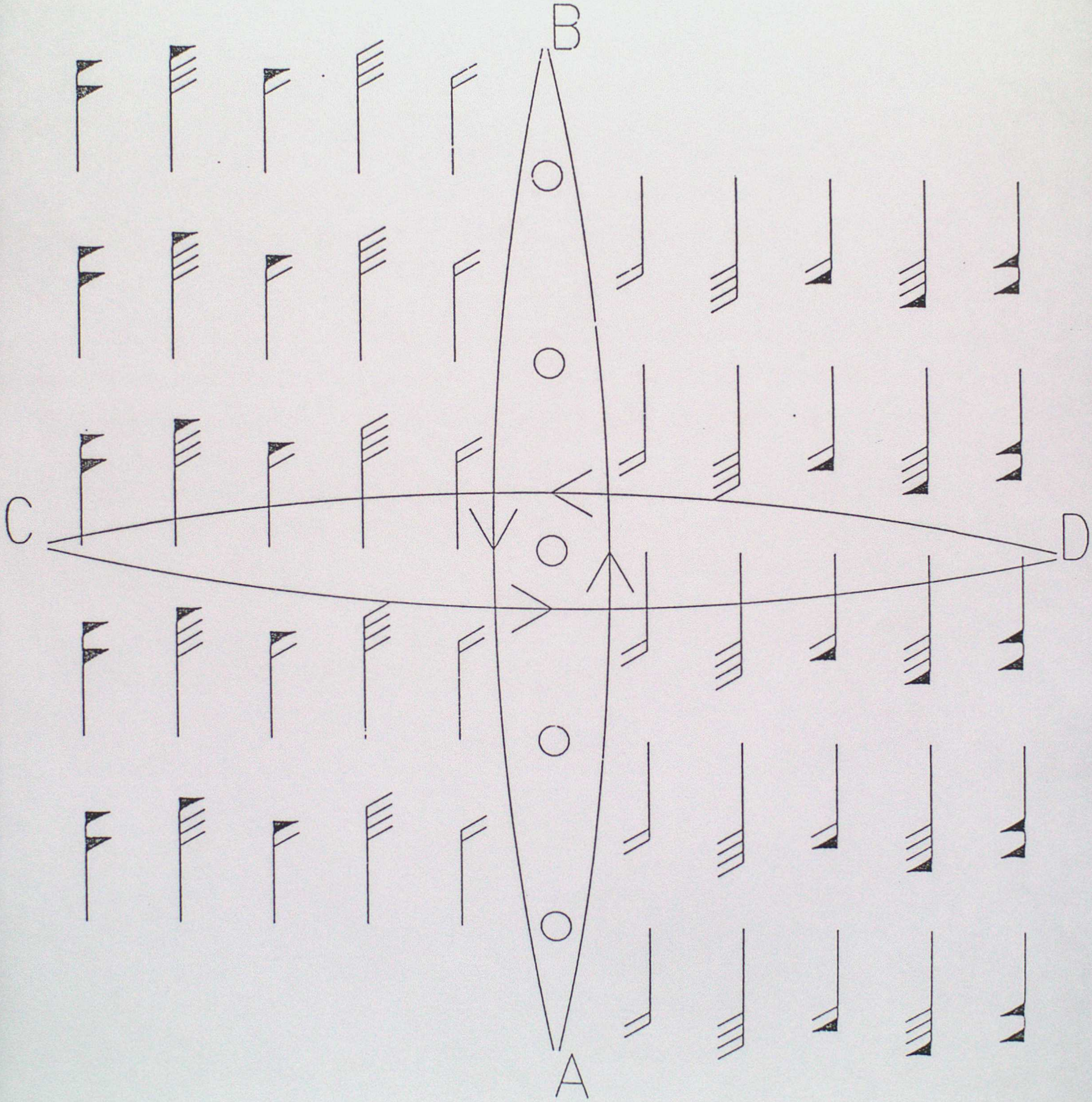


Figure 1. Shows a hypothetical wind field and approximate minimum time tracks between points A and B and between points C and D.

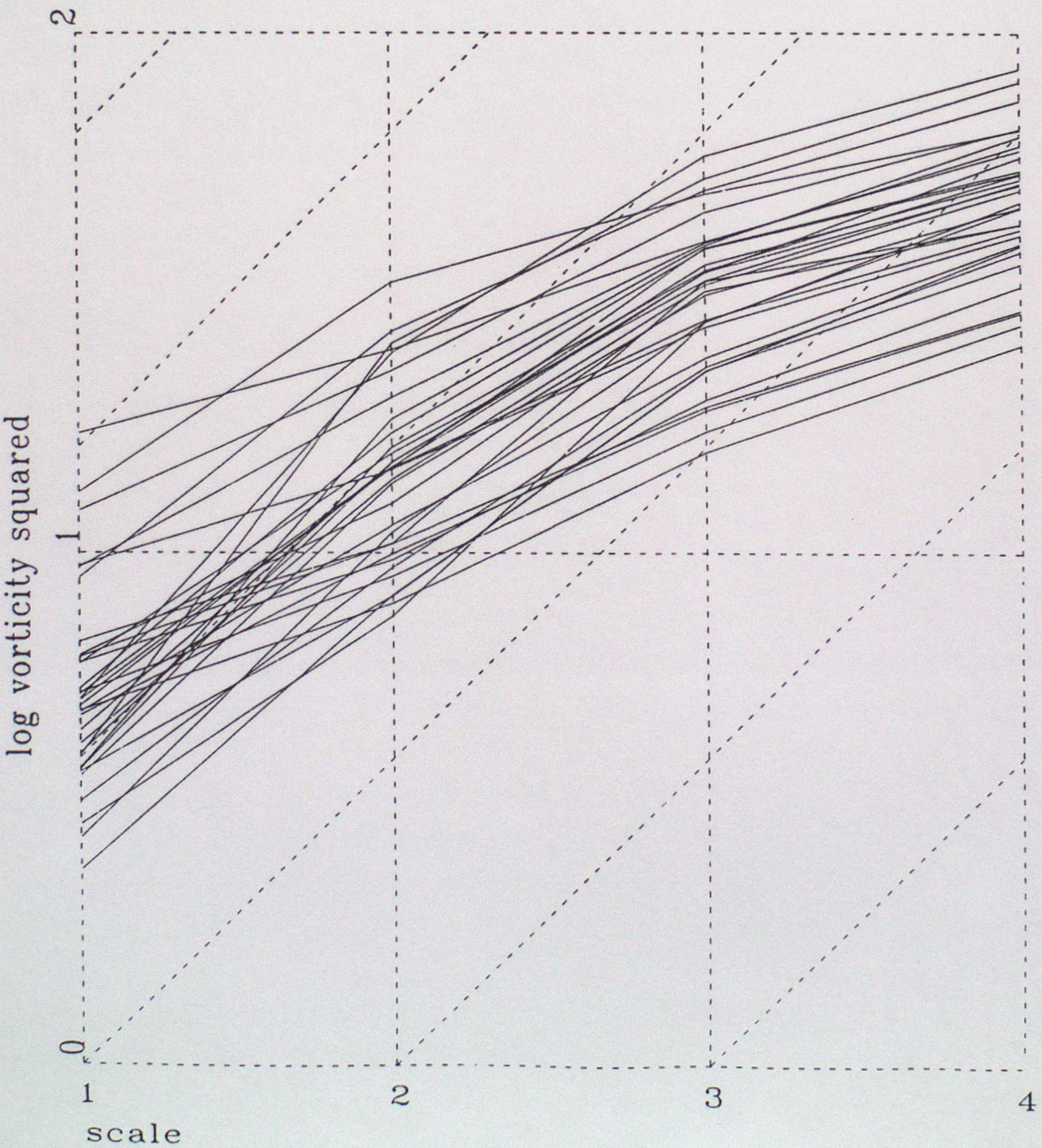


Figure 2. Shows the logarithm of squared vorticity for 4 scales, where scale 1 is 36° by 45° and scale 4 is $4\frac{1}{2}^{\circ}$ by $5\frac{5}{8}^{\circ}$.

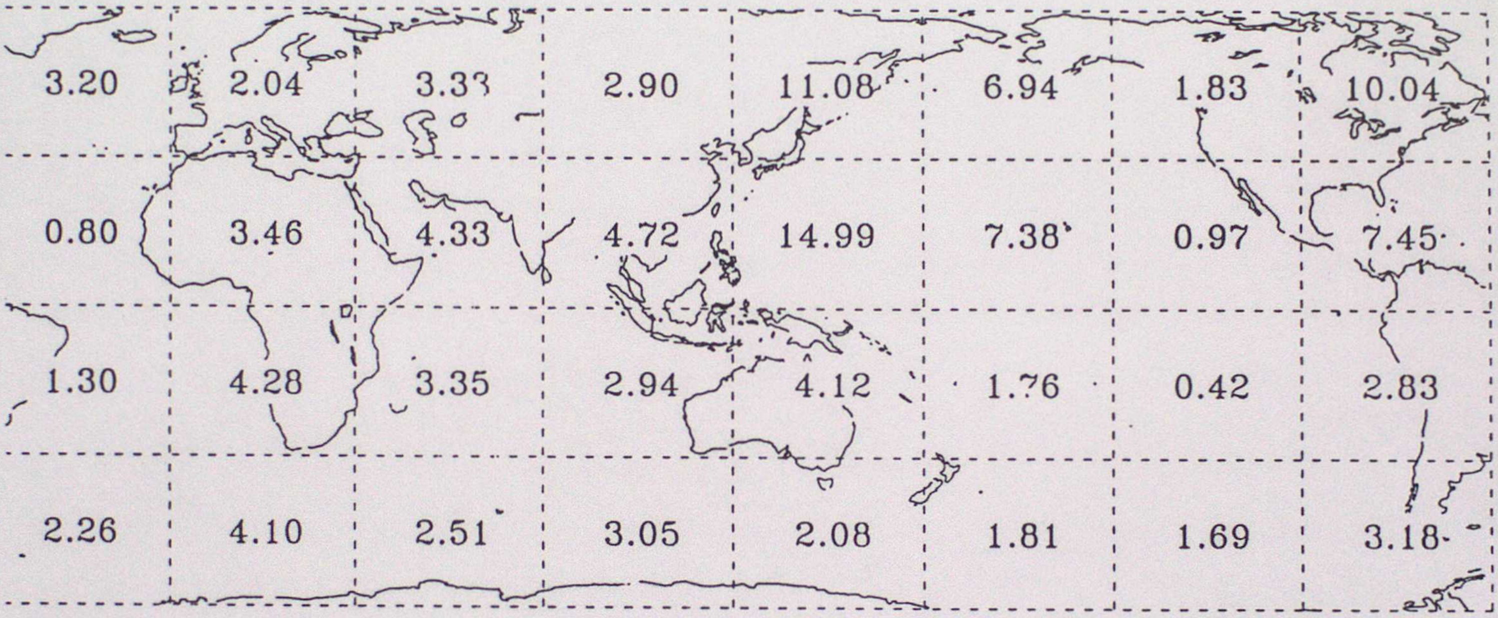


Figure 3(a). Shows temporally averaged squared spatially averaged vorticities, where the temporal averaging is for winter 1990-91 and the spatial averaging is over areas of 36°by 45°. Units are (knots per degree latitude)^2.

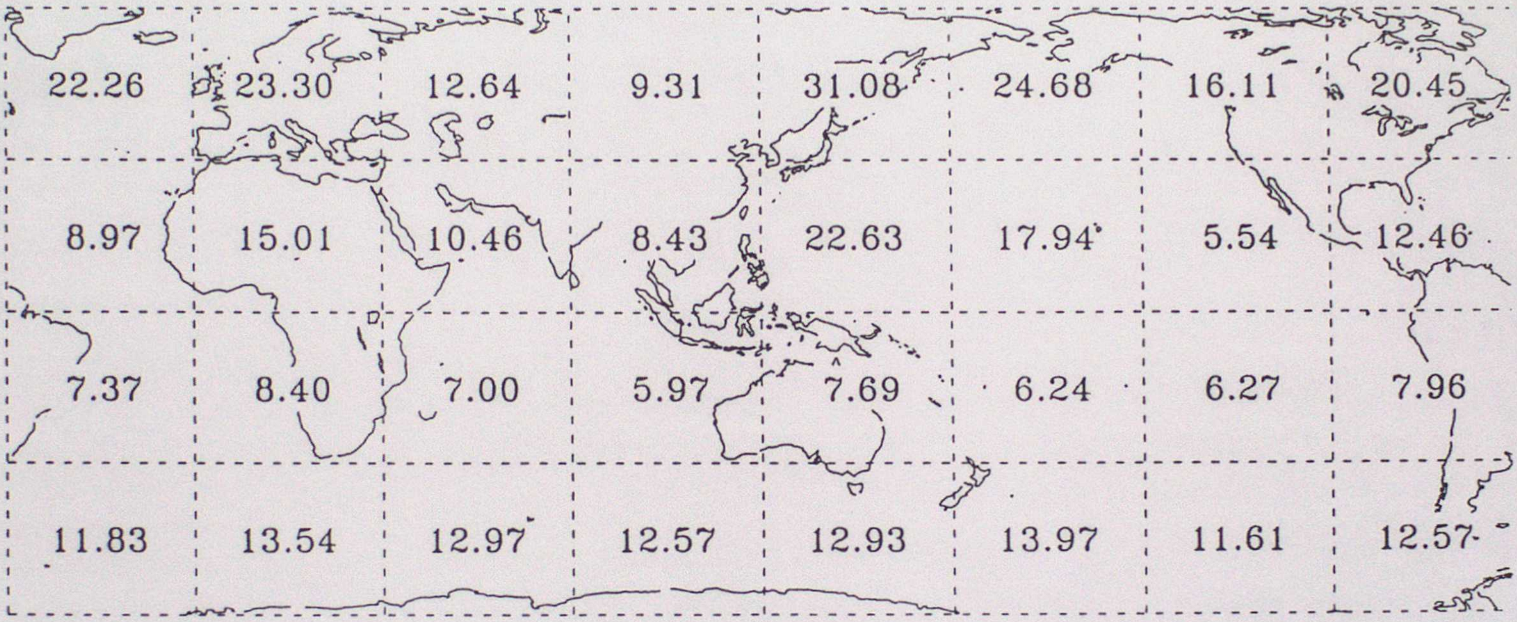


Figure 3(b). As figure 3(a) but over areas of 18° by $22\frac{1}{2}^{\circ}$.

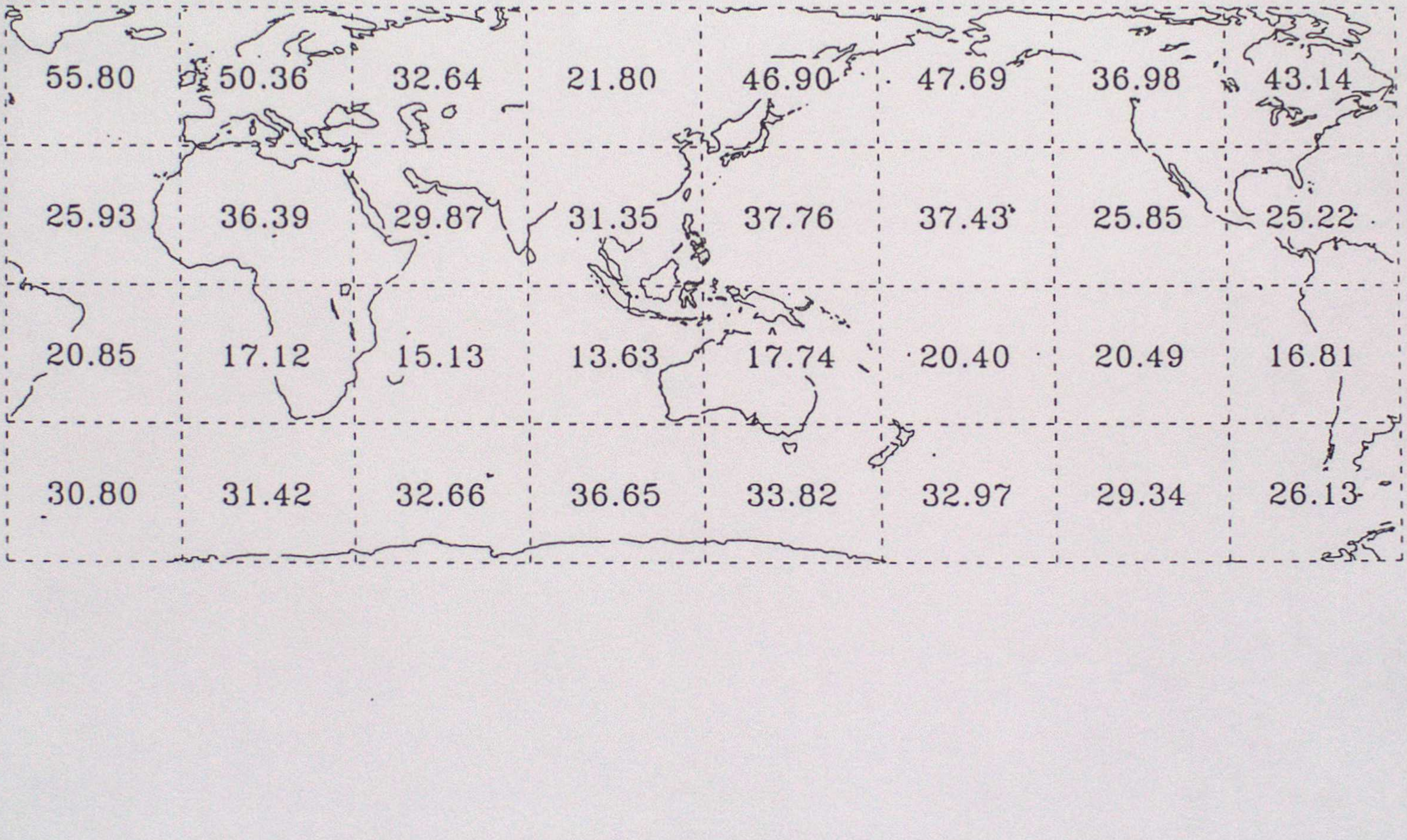


Figure 3(c). As figure 3(a) but over areas of 9° by $11\frac{1}{4}$°.

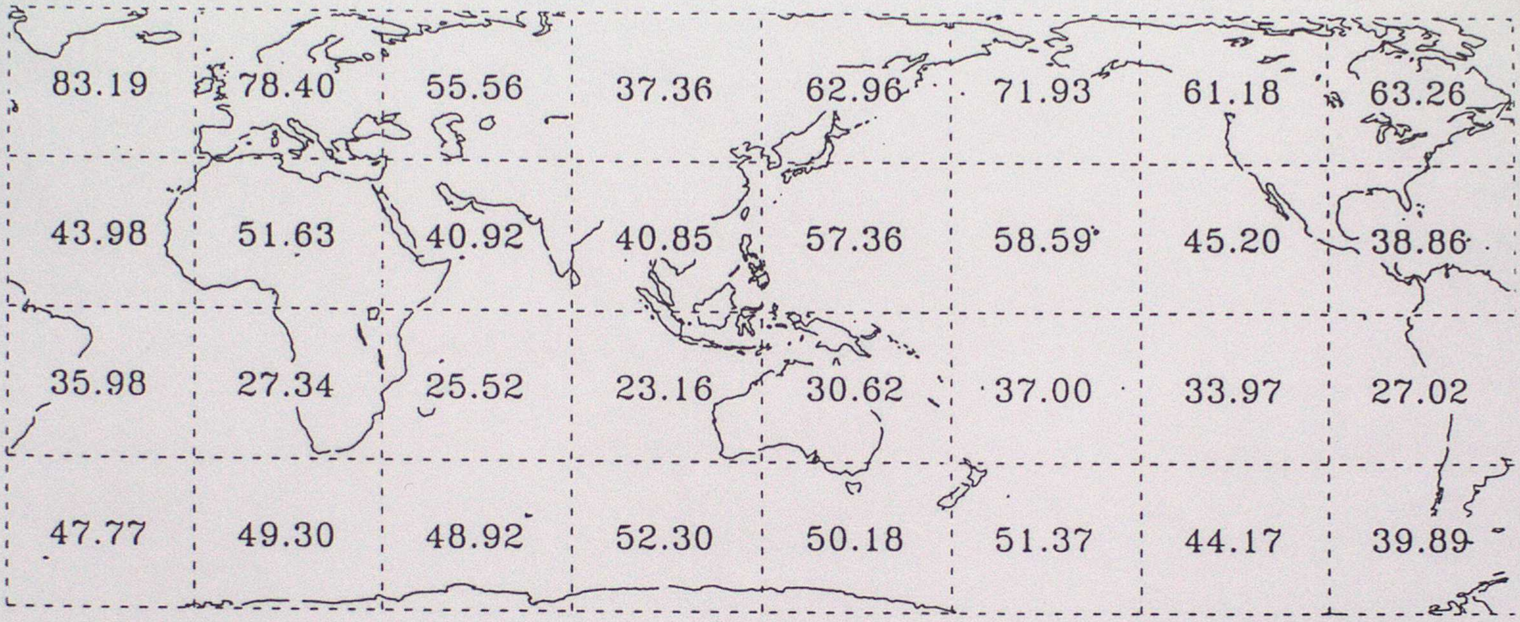


Figure 3(d). As figure 3(a) but over areas of $4\frac{1}{2}^{o}$ by $5\frac{5}{8}^{o}$.

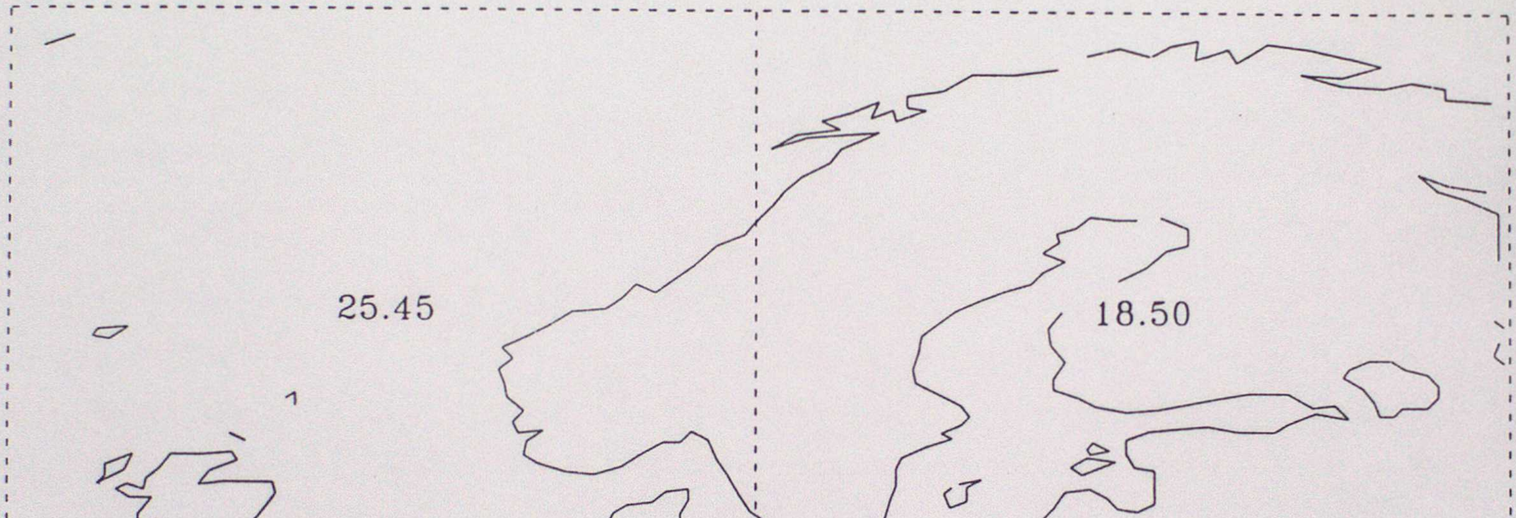


Figure 4(b). As figure 3(b) but giving detailed breakdown for Europe.

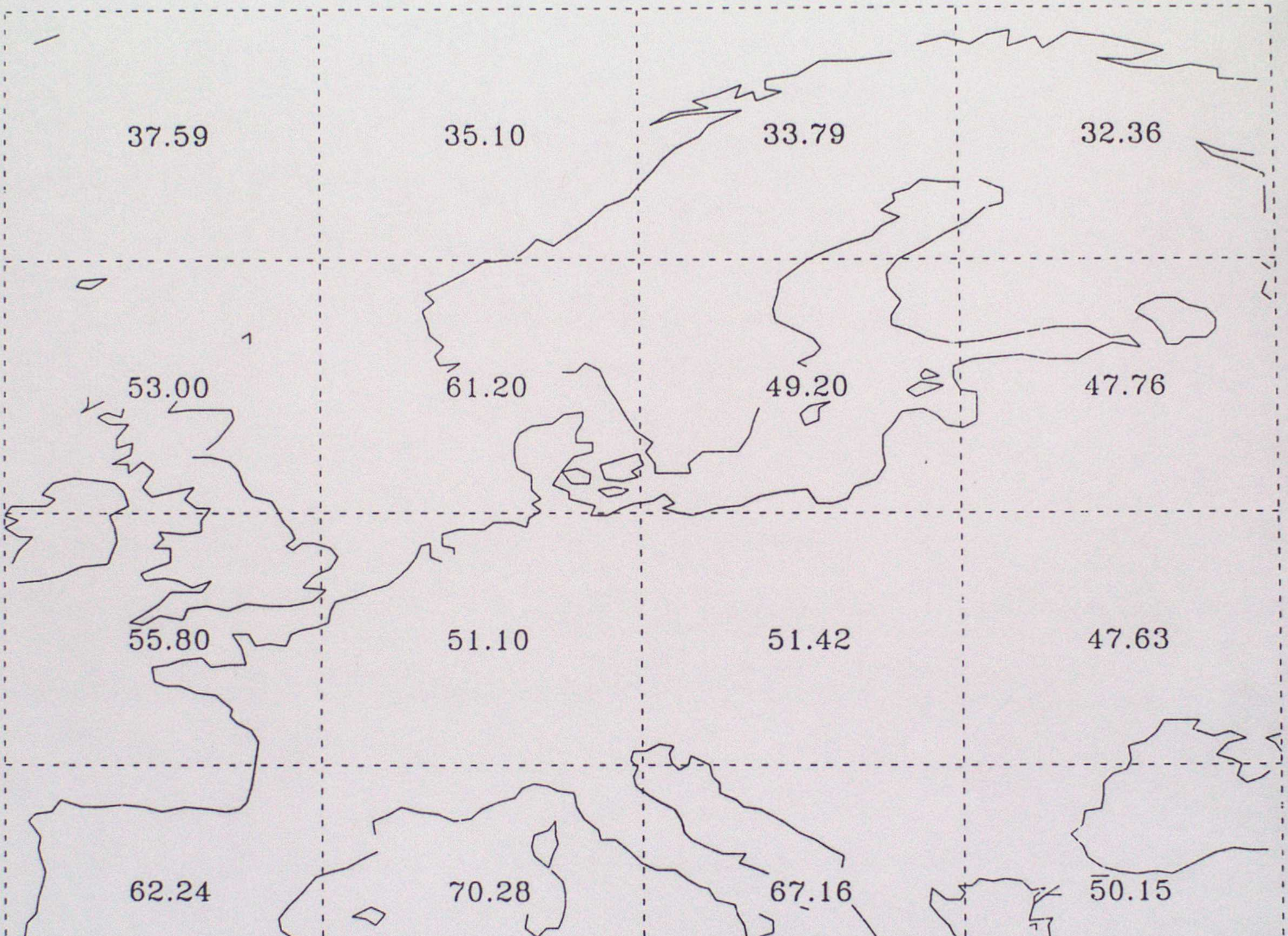


Figure 4(c). As figure 3(c) but giving detailed breakdown for Europe.

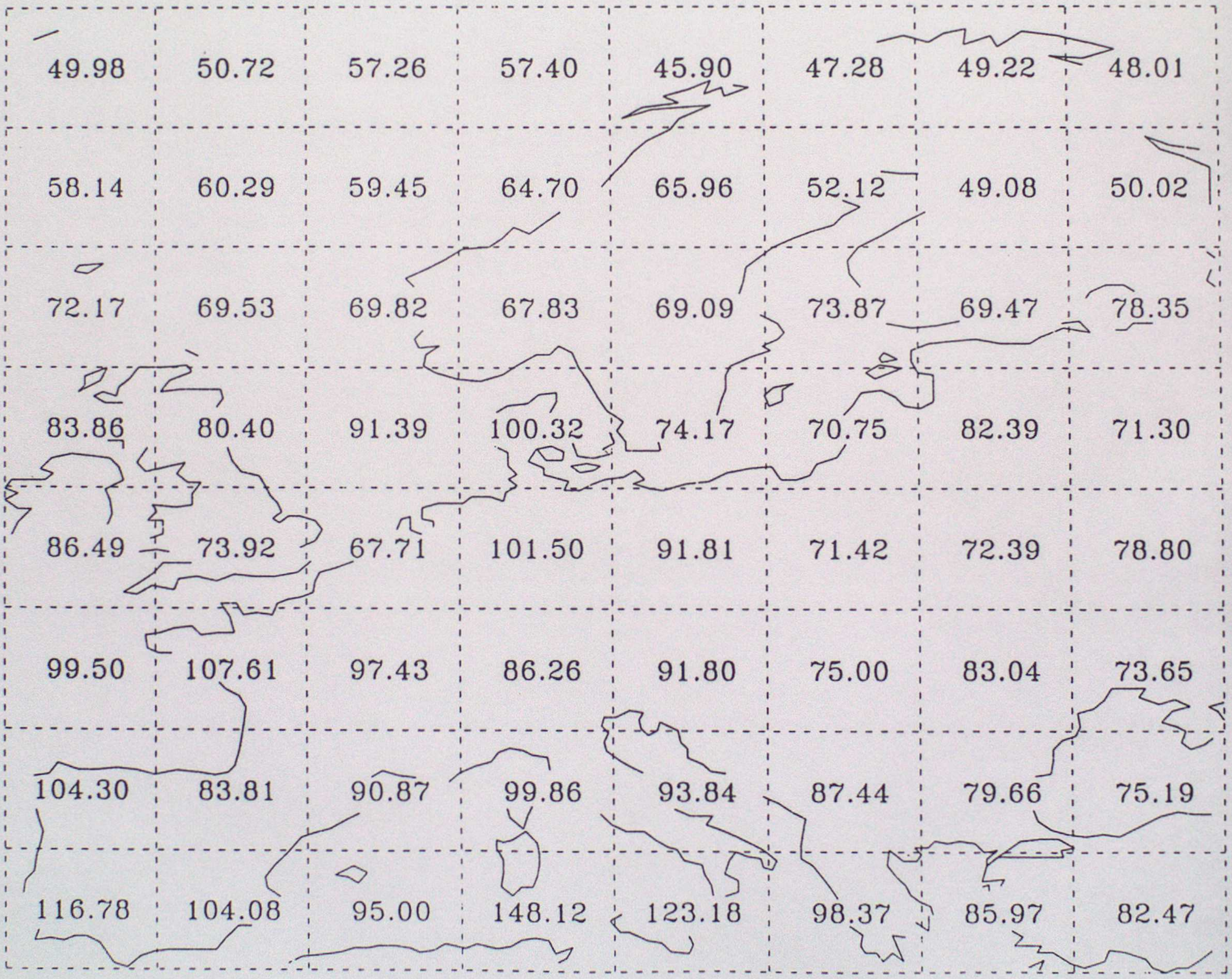
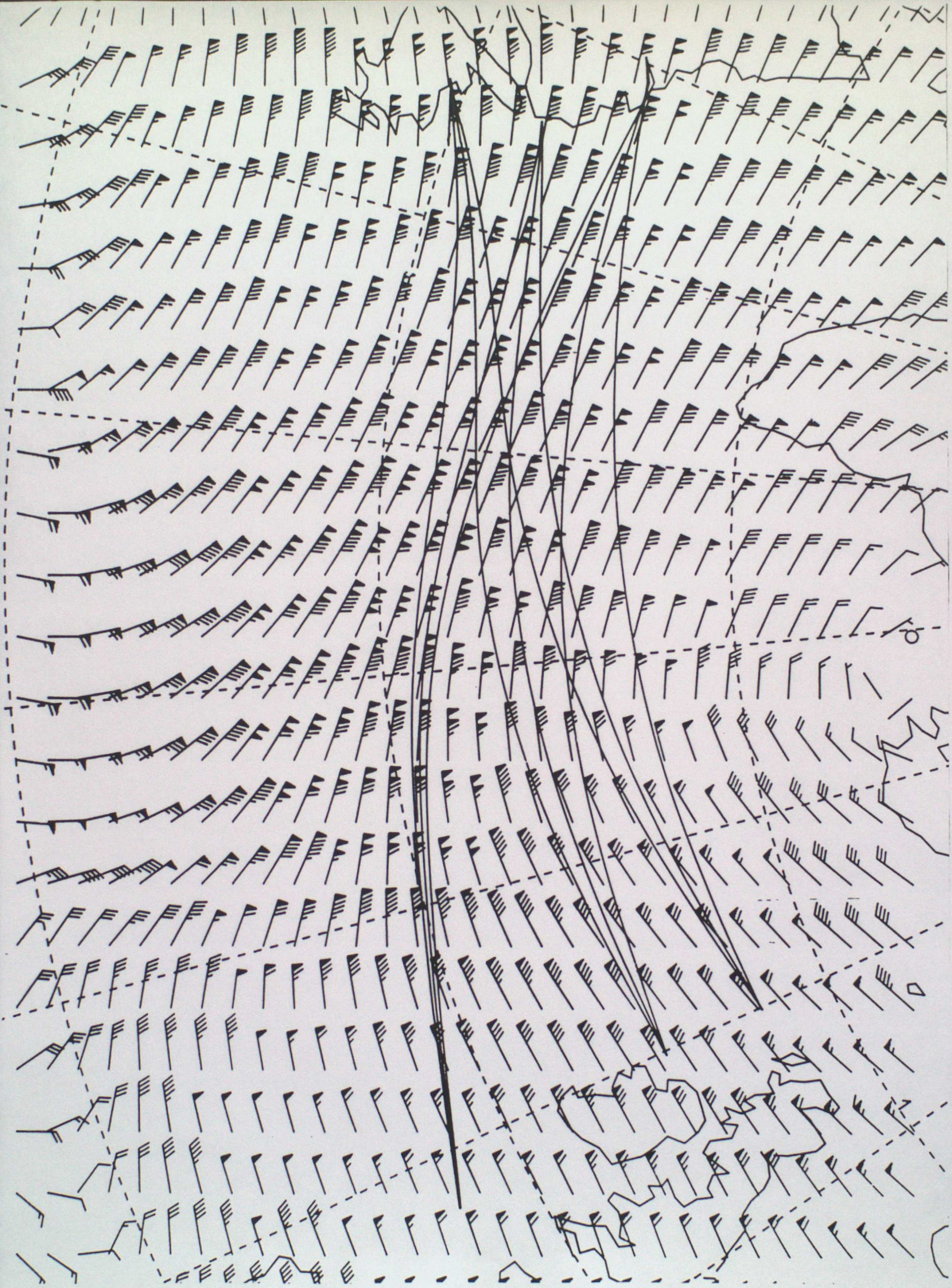


Figure 4(d). As figure 3(d) but giving detailed breakdown for Europe.



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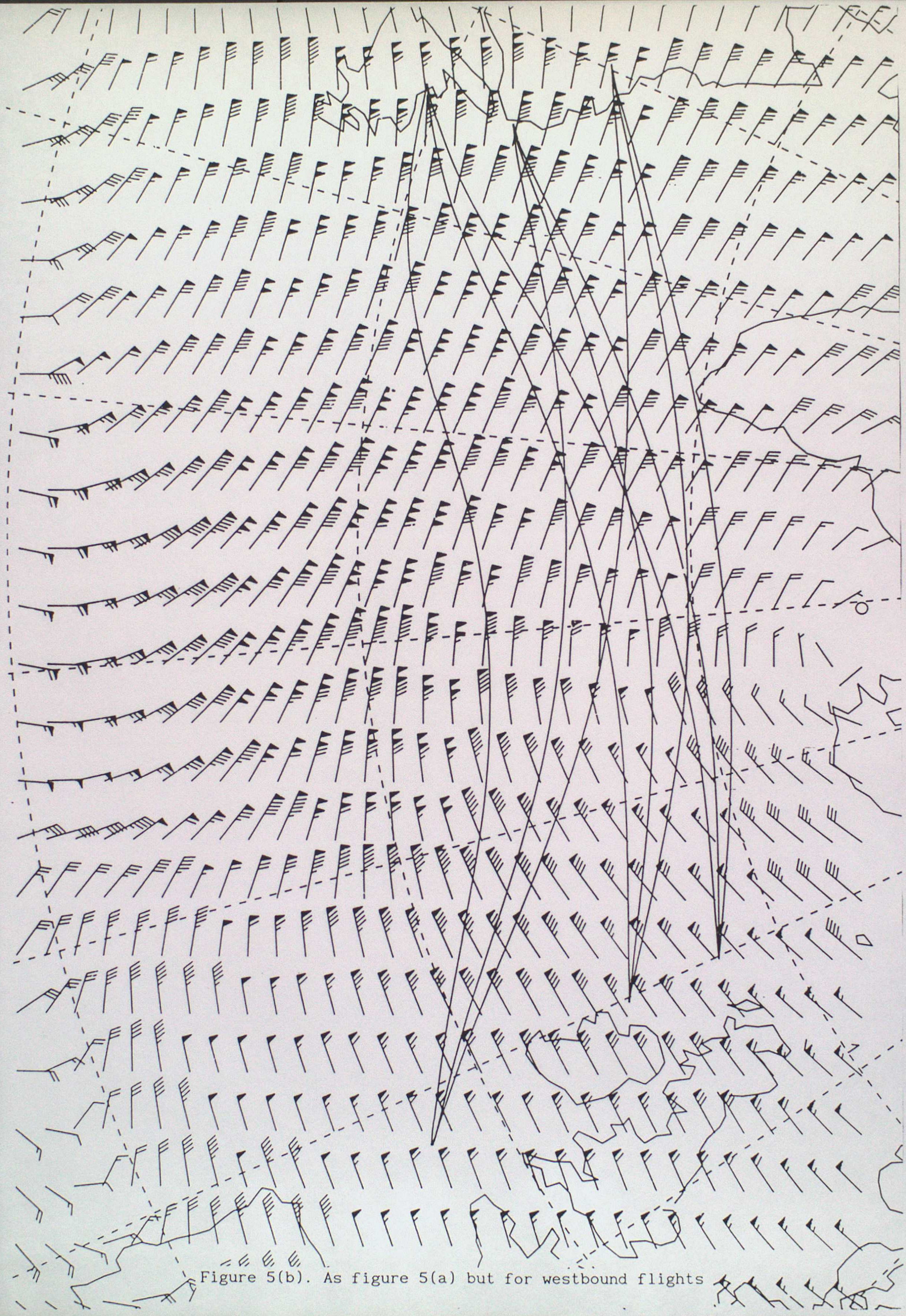

Figure 5(b). As figure 5(a) but for westbound flights